Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 6 (2) (2016) 134 – 149

Results on Approximation Properties in Intuitionistic Fuzzy Normed Linear Spaces

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Abstract

In this paper we introduce the notions approximation properties (APs) and bounded approximation properties (BAPs) in the setting of intuitionistic fuzzy normed linear spaces (IFNLSs). Further, we define strong intuitionistic fuzzy continuous and strong intuitionistic fuzzy bounded operators and using them we prove the existence of an IFNLS which does not have the approximation property. In addition, we give example of an IFNLS with the AP which fails to have the BAP.

Keywords: Intuitionistic fuzzy normed linear space, approximation property, bounded approximation property.
2010 MSC: 55M20, 54H25, 47H09.

1. Introduction

In analysis many problems we study are concerned with large classes of objects most of which turn out to be vector spaces or linear spaces. Since limit process is indispensable in such problems, a metric or topology may be induced in those classes. If the induced metric satisfies the translation invariance property, a norm can be defined in that linear space and we get a structure of the space which is compatible with that metric or topology. The resulting structure is a normed linear space. There are situations where crisp norm can not measure the length of a vector accurately and in such cases the notion of fuzzy norm happens to be useful. There has been a systematic development of fuzzy normed linear spaces (FNLSs) and one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS). The study of analytic propertis of IFNLSs, their topological structure and generalizations, therefore, remain well motivated areas of research.

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The idea of a fuzzy norm on a linear space was introduced by Katsaras (Katsaras, 1984). Felbin (Felbin, 1992) introduced the idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala (Kaleva & Seikkala, 1984) type. Cheng and Mordeson (Cheng & Mordeson, 1994) introduced another notion of fuzzy norm on a linear space whose associated metric is Kramosil and Michalek (Kramosil & Michalek, 1975) type. Again, following Cheng and Mordeson, one more notion of fuzzy normed linear space was given by Bag and Samanta (Bag & Samanta, 2003*a*).

The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov (Atanassov, 1986) has triggered some debate (for details, see (Cattaneo & Ciucci, 2006; Dubois *et al.*, 2005; Grzegorzewski & Mrowka, 2005)) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs in not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of L-fuzzy sets. In response to this debate, Atanassov justified the terminology in (Atanassov, 2005). Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations which can not be handled with fuzzy set theory alone and also IFS found its application in various areas of science and engineering.

With the help of arbitrary continuous t-norm and continuous t-conorm, Saadati and Park (Saadati & Park, 2006) introduced the concept of IFNLS. There has been further development over IFNLS, e.g., the topological structure of an intuitionistic fuzzy 2-normed space has been studied by Mursaleen and Lohani in (Mursaleen & Lohani, 2009). Recently, a number of interesting properties of IFNLS have been studied by Mursaleen and Mohiuddine (Mursaleen & Mohiuddine, 2009*a*,*b*,*c*,*d*). Further, generalizing the idea of Saadati and Park, an intuitionistic fuzzy n-normed linear space (IFnNLS) has been defined by Vijayabalaji et al. (Vijayabalaji *et al.*, 2007*b*). More properties of IFnNLS have been studied by N. Thillaigovindan, S. Anita Shanti and Y. B. Jun in (Vijayabalaji *et al.*, 2007*a*). Some more recent work in similar context can be found in (Debnath, 2015; Debnath & Sen, 2014*a*,*b*; Esi & Hazarika, 2012; Mursaleen *et al.*, 2010*a*; Sen & Debnath, 2011).

In classical Banach space theory, some most important properties are "Approximation properties" which were investigated by Grothendick (Grothendiek, 1955). We say that a Banach space X has the approximation property (AP) if, for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ such that $||T(x) - x|| < \epsilon$, for all $x \in K$, i.e. $I(x)$-the identity operator on X- can be approximated by finite rank operators uniformly on compact sets. Also X has the bounded approximation property (BAP) if for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ with $||T|| \leqslant \lambda$ such that $||T(x) - x|| < \epsilon$ for all $x \in K$ for some $\lambda > 0$. The APs play very crucial role in the study of infinite dimensional Banach space theory and also in the investigation of Schauder bases. Some of the important references from related works being (Choi *et al.*, 2009; Enflo, 1973; Kim, 2008; Mursaleen *et al.*, 2010*b*; Szarek, 1987).

Yilmaz (Yilmaz, 2010*a*) introduced the notion of the AP in fuzzy normed spaces and established some interesting results on it. Very recently Keun Young Lee (Lee, 2015) identified some limitations in Yilmaz's definitions regarding the continuity of fuzzy operators. He modified Yilmaz's definitions and studied approximation property (AP) and bounded approximation property (BAP) on fuzzy normed spaces.

In this article we address the questions raised by Keun Young Lee (Lee, 2015) and also generalize the work of Figel and Johnson (Figel & Johnson, 1973) in the context of AP and BAP in the new setting of IFNLS.

First we recall some basic definitions and results which will be used subsequently.

Definition 1.1. (Saadati & Park, 2006) The 5-tuple $(X, \mu, \nu, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leqslant 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(e) $\mu(x, t) * \mu(y, s) \leqslant \mu(x + y, t + s)$,
(f) $\mu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \circ \nu(y, s) \geqslant \nu(x + y, t + s)$,
(l) $\nu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. When no confusion arises, an IFNLS will be denoted simply by X.

Definition 1.2. (Debnath, 2012) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \varepsilon$ and $\nu(x_k - \xi, t) < \varepsilon$ for all $k \geqslant k_0$. It is denoted by $(\mu, \nu) - \lim x_k = \xi$.

Definition 1.3. (Saadati & Park, 2006) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\alpha \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \alpha$ and $\nu(x_k - x_m, t) < \alpha$ for all $k, m \geqslant k_0$.

Definition 1.4. (Debnath & Sen, 2014*a*) Let X be an IFNLS. Then X is said to be complete if and only if every Cauchy sequence of X is convergent.

Definition 1.5. (Lael & Nourouzi, 2007) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A subset S in X is said to be compact if each sequence of elements of S has a convergent subsequence.

Definition 1.6. (Debnath, 2012) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with center at $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}.$$

Proof of the following lemma is similar to its analogue in case of fuzzy normed spaces (Bag & Samanta, 2003*b*).

Lemma 1.1. *Let* $(X, \mu, \nu, *, \circ)$ *be an IFNLS with the condition*

$$\mu(x, t) > 0 \text{ and } \nu(x, t) < 1 \text{ implies } x = 0, \text{ for all } t \in \mathbb{R}^+. \tag{1.1}$$

Let $\|x\|_\alpha = inf\{t \in \mathbb{R}^+ : \mu(x, t) > \alpha \text{ and } \nu(x, t) < 1 - \alpha\}$ *for each* $\alpha \in (0, 1)$. *Then* $\{\| \cdot \|_\alpha : \alpha \in (0, 1)\}$ *is an ascending class of norms on X. These norms are called* α*- norms on the intuitionistic fuzzy norm* (μ, ν).

Definition 1.7. (Mursaleen *et al.*, 2010*a*) Let (x_n) be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. It is said to be basis of X if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that

$$(\mu, \nu) - \lim \textstyle\sum_{k=1}^{n} a_k x_k = x.$$

that is, for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists $n_0 = n_0(\alpha, \epsilon) \in N$ such that $n \geqslant n_0$ implies,

$$\mu(x - \textstyle\sum_{k=1}^{n} a_k x_k, \epsilon) > 1 - \alpha \text{ and } \nu(x - \sum_{k=1}^{n} a_k x_k, \epsilon) < \alpha, \text{ where } x = \sum_{k=1}^{\infty} a_k x_k.$$

2. Main Results

Now we are ready to discuss our main results. First we define some important notions in connection with approximation property in IFNLS.

Definition 2.1. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A complete IFNLS is said to have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha > (0, 1)$ and $\epsilon > 0$, there exists an operator T of finite rank such that

$$\mu\left(T_\alpha\left(x\right) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T_\alpha\left(x\right) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.2. Let λ be a real number. An IFNLS $(X, \mu, \nu, *, \circ)$ is said to have the λ-bounded approximation property, briefly λ-BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in F\left(X, X, \lambda\right)$ such that

$$\mu\left(T\left(x\right) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T\left(x\right) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.3. Suppose that an IFNLS $(X, \mu, \nu, *, \circ)$ has a basis (x_n). For each positive integer m, the m^{th} natural projection P_m for x_m is the map

$$\sum_{n=1}^{\infty} a_n x_n \longrightarrow \sum_{n=1}^{m} a_n x_n \text{ from } (X, \mu, \nu, *, \circ) \text{ to } (X, \mu, \nu, *, \circ).$$

Definition 2.4. Let $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLS and $T : X \longrightarrow Y$ be a linear operator where (μ, ν) and (μ', ν') are intuitionistic fuzzy normed. Then

1. The operator T is called strongly intuitionistic fuzzy (shortly sif) continuous at $a \in X$ if, for given $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in X$,
$$\mu'(T(x) - T(a), \epsilon) \geqslant \mu(x - a, \delta) \text{ and } \nu'(T(x) - T(a), \epsilon) \leqslant \nu(x - a, \delta).$$
If T is sif-continuous at each point of X, then T is said to be sif-continuous on X.
2. The operator T is called strongly intuitionistic fuzzy bounded on X if there exists a positive real number M such that $\mu'(T(x), t) \geqslant \mu(x, \frac{t}{M})$ and $\nu'(T(x), t) \leqslant \nu(x, \frac{t}{M})$ for all $x \in X$ and $t \in \mathbb{R}$. We will denote the set of all strongly intuitionistic fuzzy (shortly sif) bounded operators form X to Y by $F(X, Y)$. Then $F(X, Y)$ is a vector space. For all $M > 0$, $F(X, Y, M)$ is denoted by
$$\left\{T \in F(X, Y) : \mu'(T(x), t) \geqslant \mu(x, \tfrac{t}{M}), \nu'(T(x), t) \leqslant \nu(x, \tfrac{t}{M}), \forall x \in X, \forall t \in \mathbb{R}\right\},$$
where M is a positive real number.
For some $M > 0$ if $\mathbb{S} = F(X, Y, M)$ then $\mathbb{S}$ is called a bounded subset of $F(X, Y)$. Again the set of all finite rank sif-bounded operators from X to Y is denoted by $\bar{F}(X, Y)$. Then $\bar{F}(X, Y)$ is subspace of $F(X, Y)$. Similarly, we can say that $\bar{F}(X, Y, M)$ is also a subspace of $F(X, Y, M)$ for some $M > 0$.

Proof of the following is similar to its fuzzy analogue in (Bag & Samanta, 2005).

Lemma 2.1. *Let $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLSs satisfying condition 1.1 and $T : X \longrightarrow Y$ be a linear operator. Then T is sif-bounded if and only if it is uniformly bounded with respect to α- norms of (μ, ν) and (μ', ν'). That is, there exists some $M > 0$, independent of α, such that $||T(x)||_\alpha \leqslant M||x||_\alpha$, for all $\alpha \in (0, 1)$.*

Remark. If $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLSs satisfying the conditions:
$\mu(x, t) > 0$ and $\nu(x, t) < 1$ implies $x = 0$ for all $t \in \mathbb{R}^+$ and
for $x \neq 0$, $\mu(x, t)$ is continuous and strictly increasing on $\{t : 0 < \mu(x, t) < 1\}$, while $\nu(x, t)$ is continuous and strictly decreasing on $\{t : 0 < \mu(x, t) < 1\}$ and $M > 0$. Then we obtain

$$F(X, Y, M) = \{T \in F(X, Y) : ||T(x)||_\alpha \leqslant M||x||_\alpha, \forall x \in X, \forall \alpha \in (0, 1)\}.$$

Hence $F(X, Y, M)$ and $\bar{F}(X, Y, M)$ are bounded convex subsets of $F(X, Y)$.

Theorem 2.1. *Let X be a Banach space and (x_n) be a Schauder basis in X. Then (x_n) is a basis for an IFNLS $(X, \mu, \nu, *, \circ)$ where*

$$\mu(x, t) = \begin{cases} \frac{t - ||x||}{t + ||x||}, & \text{if } t > ||x|| \\ 0, & \text{if } t \leqslant ||x||, \end{cases}$$

$$\nu(x, t) = \begin{cases} 1 - \frac{t - ||x||}{t + ||x||}, & \text{if } t > ||x|| \\ 1, & \text{if } t \leqslant ||x||, \end{cases}$$

and every natural projection is sif-continuous.

Proof. Given that (x_n) is a basis for an IFNLS $(X, \mu, \nu, *, \circ)$.

It is enough to show that -

natural projection $P_n : (X, \mu, \nu, *, \circ) \to (X, \mu, \nu, *, \circ)$ is sif-bounded for each $x \in N$.

Let $n \in N$, $t \in R$, $x \in X$.

Consider $M = ||P_n||$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

The proof of $\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right)$ can be established in a similar manner as in Proposition 3.4 of (Lee, 2015).

Now considering for ν, we have

$$t > M||x||,$$

then

$$\nu\left(x, \tfrac{t}{M}\right) = 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

By the assumption,

$$t > M||x|| = ||P_n||||x|| \geqslant ||P_n\left(x\right)||$$

and

$$\frac{t - ||P_n(x)||}{t + ||P_n(x)||} \geqslant \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

Therefore, we have

$$\nu\left(P_n\left(x\right), t\right) = 1 - \frac{t - ||P_n(x)||}{t + ||P_n(x)||} \leqslant 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||} = \nu\left(x, \tfrac{t}{M}\right).$$

Hence

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

Secondly,

$$t \leqslant ||Mx||,$$

then

$$\nu\left(Mx, t\right) = 1.$$

Thus,

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

□

So, we have the existence of an IFNLS having a basis such that every natural projection is sif-continuous. Now provide modified definitions of APs and BAPs in IFNLSs by incorporating the continuity of approximating operators.

Definition 2.5. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then X is said to be have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}\,(X, X)$ such that

$$\mu\,(T\,(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu\,(T\,(x) - x, \epsilon) < \alpha$$

for every $x \in K$.

Definition 2.6. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and λ be a positive real number. Then X is said to be have the λ- bounded approximation property, briefly λ - BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}\,(X, X, \lambda)$ such that

$$\mu\,(T\,(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu(T\,(x) - x, \epsilon) < \alpha$$

for every $x \in K$. We can also say that X has the BAP if X has the λ-BAP for some $\lambda > 0$.

Theorem 2.2. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then the following are equivalent.*

1. $(X, \mu, \nu, *, \circ)$ *has the AP.*
2. *If $(Y, \mu^{'}, \nu^{'}, *, \circ)$ is an IFNLS, then for every $T \in F(X, Y)$, every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(X, Y)$ such that*

$$\mu^{'}(S\,(x) - T(x), t) > 1 - \alpha \text{ and } \nu^{'}(S\,(x) - T(x), t) < \alpha$$

for each $x \in K$.

3. *If $(Y, \mu^{'}, \nu^{'}, *, \circ)$ is an IFNLS, then for every $T \in F(Y, X)$, every compact set K in $(Y, \mu^{'}, \nu^{'}, *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(Y, X)$ such that*

$$\mu(S\,(y) - T(y), t) > 1 - \alpha \text{ and } \nu(S\,(y) - T(y), t) < \alpha$$

for each $y \in K$.

Proof. $(i) \Rightarrow (ii)$

Let $T \in F\,(X, Y)$ and K be a compact set in $(X, \mu, \nu, *, \circ)$ and $\alpha \in (0, 1)$ and $t > 0$ and $t \in \mathbb{R}$. Then there exists a positive real number M such that

$$\mu^{'}\,(T\,(x)\,, t) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu^{'}\,(T\,(x)\,, t) \leqslant \nu\left(x, \tfrac{t}{M}\right)$$

for all $x \in X$.

Since $(X, \mu, \nu, *, \circ)$ has the AP, there exists an operator $R \in F\,(X, X)$ such that

$$\mu\left(R\,(x) - x, \tfrac{t}{M}\right) > 1 - \alpha \text{ and } \nu\left(R\,(x) - x, \tfrac{t}{M}\right) < \alpha$$

for every $x \in K$.

Now we put $S = TR$. Since T and R both are sif-bounded operators, therefore S is also a sif-bounded operator.

$$\begin{aligned}\mu^{'}\left(S\left(x\right)-T\left(x\right),t\right) &= \mu^{'}\left(TR\left(x\right)-T\left(x\right),t\right)\\ &\geqslant \mu\left(R\left(x\right)-x,\frac{t}{M}\right)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu^{'}\left(S\left(x\right)-T\left(x\right),t\right) &= \nu^{'}\left(TR\left(x\right)-T\left(x\right),t\right)\\ &\leqslant \nu\left(R\left(x\right)-x,\frac{t}{M}\right)\\ &< \alpha.\end{aligned}$$

for every $x \in K$.

$(i) \Rightarrow (iii)$

Let $T \in F\left(Y,X\right)$ and K be a compact set in $\left(Y,\mu^{'},\nu^{'},*,\circ\right)$ and $\alpha \in (0,1)$ and $t > 0$ and $t \in R$.

Since $(X,\mu,\nu,*,\circ)$ has the AP and $T\left(K\right)$ is compact set in $(X,\mu,\nu,*,\circ)$, there exists an operator $R \in \bar{F}\left(X,X\right)$ such that

$$\mu\left(R\left(x\right)-x,t\right) > 1-\alpha \text{ and } \nu\left(R\left(x\right)-x,t\right) < \alpha$$

for every $x \in T\left(K\right)$.

Now we put, $S = RT \in \bar{F}\left(Y,X\right)$. Then we have,

$$\begin{aligned}\mu\left(S\left(y\right)-T\left(y\right),t\right) &= \mu\left(RT\left(y\right)-T\left(y\right),t\right)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu\left(S\left(y\right)-T\left(y\right),t\right) &= \nu\left(RT\left(y\right)-T\left(y\right),t\right)\\ &< \alpha,\end{aligned}$$

for each $y \in K$.

Since (i) implies both (ii) and (iii), hence (i), (ii) and (iii) are equivalent.

Hence proposition is proved.

□

Proof of the following Lemma is similar to Lemma 4.2 of (Lee, 2015).

Lemma 2.2. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS and K be a subset in X. If K is a compact set in $(X,\mu,\nu,*,\circ)$, then for every $\alpha \in (0,1)$ and $t > 0$, there exists a finite set $\{x_1,x_2,\ldots,x_n\}$ in K such that for every $x \in K$ we have $x \in B\left(x_i,\alpha,t\right)$ for some x_i.*

Theorem 2.3. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS with intuitionistic fuzzy norm (μ,ν) and $M > 0$. Suppose that there exists a sequence $\left(T_n\right) \in \bar{F}\left(X,X,M\right)$ such that $T_n\left(x\right) \longrightarrow x$ for every $x \in X$, then $(X,\mu,\nu,*,\circ)$ has the AP.*

Proof. Let (T_n) be a sequence in $\bar{F}(X, X, M)$ such that

$$T_n(x) \longrightarrow x \text{ for every } x \in X.$$

Let $\alpha \in (0,1)$ and $t > 0$, and K be a compact set in $(X, \mu, \nu, *, \circ)$.

By the above Lemma, there exists a finite set $\{x_1, x_2, ..., x_n\} \subset K$ such that for $x \in K$ we have $x \in B(x_i, \alpha, t)$ for some x_i.

Then there exists $N_1, N_2 \in \mathbb{N}$ such that if $n \geqslant N_1, N_2$ we have,

$$\mu(T_n(x_i) - x_i, t) > 1 - \alpha \text{ and } \nu(T_n(x_i) - x_i, t) < \alpha$$

for each i.

Let $x \in K$ and choose i such that $x \in B(x_i, \alpha, t)$, that is,

$$\mu(x_i - x, t) > 1 - \alpha \text{ and } \nu(x_i - x, t) < \alpha.$$

Then for $n \geqslant N_1, N_2$,

$$\begin{aligned}
\mu(T_n(x) - x, t) &= \mu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\geqslant min\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\geqslant min\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&> 1 - \alpha.
\end{aligned}$$

And

$$\begin{aligned}
\nu(T_n(x) - x, t) &= \nu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\leqslant max\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\leqslant max\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&< \alpha.
\end{aligned}$$

Therefore, $\mu(T_n(x) - x, t) > 1 - \alpha$ and $\nu(T_n(x) - x, t) < \alpha$.

Hence $(X, \mu, \nu, *, \circ)$ has the AP. □

By using the above result we derive the following.

Theorem 2.4. *Suppose $(X, \mu, \nu, *, \circ)$ has a basis $\{x_n\}$ and every natural projection*

$$P_n : (X, (\mu, \nu)) \longrightarrow (X, (\mu, \nu))$$

is sif-continuous. Then $(X, \mu, \nu, *, \circ)$ has the AP but the converse is not necessarily true.

Theorem 2.5. *An IFNLS $(X, \mu, \nu, *, \circ)$ satisfying condition 1.1 has the AP if and only if for every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0,1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X)$ such that*

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Theorem 2.6. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS satisfying condition 1.1 and $\lambda > 0$. Then $(X, \mu, \nu, *, \circ)$ has λ-BAP if and only if for every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X, \lambda)$ such that*

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Proof of the above two results follow from (Yilmaz, 2010*b*).

3. Examples

In this section, we give answers to the following interesting questions with proper examples:

1. Does every IFNLS have the AP?
2. Does in an IFNLSs the AP imply the BAP?

Now we are going to solve (in negative sense) the problem (i) and (ii) with the help of following two examples.

Example 3.1. As we know that there exists a Banach space $(X, || \cdot ||)$ which fails to have the approximation property, similarly there exists an IFNLS $(X, \mu, \nu, *, \circ)$ which fails to have the AP.

Let us define a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x, t) = \begin{cases} 1, & \text{if } t > ||x|| \\ 0, & \text{if } t \leqslant ||x||. \end{cases}$$

and

$$\nu(x, t) = \begin{cases} 0, & \text{if } t > ||x|| \\ 1, & \text{if } t \leqslant ||x||. \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm and $||x||_\alpha = ||x||$, for every $\alpha \in (0, 1)$.

Now suppose that $(X, \mu, \nu, *, \circ)$ has the AP.

Let $\alpha \in (0, 1)$ and $\epsilon > 0$ and K be a compact set in X. Since $||x||_\alpha = ||x||$ for each $\alpha \in (0, 1)$, K is compact in $(X, \mu, \nu, *, \circ)$. Then by Theorem 2.5, there exists an operator $T_\alpha \in \bar{F}(X, X)$ such that

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Hence we have,

$$||T(x) - x|| = ||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$, which is a contradiction as $(X, ||\cdot||)$ fails to have the approximation property.

$(X, \mu, \nu, *, \circ)$ has fails to have the AP.

As in Example 4.9 of (Lee, 2015), we give below an example of the existence of an IFNLS which has the AP but fails to have BAP.

Example 3.2. Enflo and Lidenstrauss (Enflo, 1973; Lindenstrauss, 1971) has proved the existence of a Banach Space X_0 which has the metric approximation property but its dual space X_0^* fails to have the approximation property. There is a sequence $(||\cdot||_n)$ of equivalent norms on X_0 so that $(X_0, ||\cdot||_n)$ fails to have the n- BAP. Consider $X_n = (X_0, ||\cdot||_n)$. Thus $\left(\sum \oplus X_n\right)_{l_2}$ fails to have the BAP where $\left(\sum \oplus X_n\right)_{l_2}$ is a Banach space whose elements are sequence of the form $(x_1, x_2, ...)$, where $\sum_{n=1}^{\infty} ||x_n||_n^2 < \infty$ and $x_n \in X_n$.

Now we consider, $X = \left(\sum \oplus X_n\right)_{l_2}$, and define $||x|| = \left(\sum_{n=1}^{\infty} ||x_n||_n^2\right)^{\frac{1}{2}}$ and $||x||_1 = sup_n ||x_n||$ for all $x = (x_1, x_2, ...) \in X$.

Let us defined a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x, t) = \begin{cases} 1, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 0, & \text{if } t \leqslant ||x||_1, \end{cases}$$

and

$$\nu(x, t) = \begin{cases} 0, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 1, & \text{if } t \leqslant ||x||_1, \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm.

Consider the α-norms as-

$$||x||_\alpha = \begin{cases} ||x||, & \text{if } 1 > \alpha > \frac{1}{2} \\ ||x||_1, & \text{if } 0 < \alpha \leqslant \frac{1}{2}. \end{cases}$$

Suppose that $(X, \mu, \nu, *, \circ)$ has the BAP. Let us assume that K be a compact set in $(X, ||\cdot||)$. Then we have to show that K is a compact set in $(X, \mu, \nu, *, \circ)$.

Let $\epsilon > 0$ and (x_n) be a sequence in K. As K is compact subset in $(X, ||\cdot||)$, there exists subsequence (x_{n_k}) in $(X, ||\cdot||)$. Therefore there exists an $x \in X$ and integers $\mu, \nu > 0$ such that for $k \geqslant \mu, \nu$

$$||x_{n_k} - x|| < \epsilon.$$

Since $||x||_1 \leqslant ||x||$ for all $x \in X$, therefore for $k \geqslant \mu, \nu$

$$||x_{n_k} - x||_\alpha < \epsilon$$

for all $\alpha \in (0,1)$.

Hence K is a compact set in $(X,\mu,\nu,*,\circ)$.

Next consider $\alpha \in \left(\frac{1}{2},1\right)$ and $\epsilon > 0$. As K is a compact set in $(X,\mu,\nu,*,\circ)$ and using $\alpha \in \left(\frac{1}{2},1\right)$ and $\epsilon > 0$ we have $\lambda > 0$ and $T_{\alpha,\epsilon} \in \bar{F}(X,X,\lambda)$ such that

$$||T_{\alpha,\epsilon}(x) - x||_\alpha < \epsilon \text{ for every } x \in K.$$

Then we have $||T_{\alpha,\epsilon}(x) - x|| < \epsilon$ and $||T_{\alpha,\epsilon}(x)|| \leqslant \lambda||x||$, which is a contradiction as $(X,||\cdot||)$ fails to have the BAP.

Hence $(X,\mu,\nu,*,\circ)$ has fails to have the BAP.

Finally, we have to show that $(X,\mu,\nu,*,\circ)$ has the AP. Let $\epsilon > 0$ and K be a compact subset in $(X,\mu,\nu,*,\circ)$. Again let $P_j : X \longrightarrow \left(\sum_{n=1}^{j} \oplus X_n\right)_{l_2}$ be the projection given by

$$P((x)) = (x_1, x_2, ..., x_j).$$

Since K is a compact set in X, therefore by Theorem 2.4 of (Choi *et al.*, 2009) there exists a natural number $m \in N$ and a finite rank operator $T' : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow (\sum_{n=1}^{m} \oplus X_n)_{l_2}$ such that

$$||kT'P_m(x) - x|| < \epsilon$$

for every $x \in K$, where k is the map defined as $k : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow X$ such that

$$k(x_1, x_2, ..., x_m) = (x_1, x_2, ..., x_m, 0, ...).$$

Now we put $T = kT'P_m$. As T is a finite rank operator defined as $T : X \longrightarrow X$ and $||x||_1 \leqslant ||x||$ for all $x \in X$, we have

$$||T(x) - x||_1 < \epsilon$$

that is, for every $\alpha \in (0,1)$, we have

$$||T(x) - x||_\alpha < \epsilon.$$

Next we have to show that T is sif-bounded on X. Since $(\sum_{n=1}^{m} \oplus X_n)_{l_2}$ and $(\sum_{n=1}^{m} \oplus X_n)_{l_\infty}$ are equivalent, there exists $M' > 1$ such that

$$(\textstyle\sum_{n=1}^{m} ||x_n||_n^2)^{\frac{1}{2}} \leqslant M' sup_{1 \leqslant n \leqslant m} ||x_n||_n.$$

Then,

$$\begin{aligned}\|T(x)\|_1 \leqslant ||T(x)|| &= \|kT'P_m(x)\| \\ &\leqslant ||kT'||(\sum_{n=1}^{m} \|x_n\|_n^2)^{\frac{1}{2}} \\ &\leqslant ||kT'||M' \sup_{1 \leqslant n \leqslant m} ||x_n||_n \\ &\leqslant ||kT'||M'||x||_1.\end{aligned}$$

Taking $M = max\left\{||T||, ||kT'||, M'\right\}$, we have to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(x,\frac{t}{M}\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(x,\frac{t}{M}\right)$$

for all $x \in X$ and $t \in \mathbb{R}$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right), \text{ for all } x \in X \text{ and } t \in \mathbb{R}.$$

Now first consider for μ:

For the first condition:

$$t > M||x||$$

then

$$\mu\left(Mx,t\right) = 1$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T\left(x\right)||$$

we have

$$\mu\left(T\left(x\right),t\right) = 1.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\mu\left(Mx,t\right) = \frac{1}{2}.$$

By the assumption

$$t > M||x||_1 \geqslant ||kT^{'}||M^{'}||x||_1 \geqslant ||T\left(x\right)||_1$$

we have

$$\mu\left(T\left(x\right),t\right) \geqslant \frac{1}{2}.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the third condition :

$$t \leqslant M||x||$$

we have

$$\mu\left(Mx, t\right) = 0.$$

Then by the assumption trivially we obtain,

$$\mu\left(T\left(x\right), t\right) \geqslant \mu\left(Mx, t\right).$$

Next considering for ν :
For the first condition :

$$t > M||x||$$

then

$$\nu\left(Mx, t\right) = 0.$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T\left(x\right)||$$

we have

$$\nu\left(T\left(x\right), t\right) = 0.$$

Hence

$$\nu\left(T\left(x\right), t\right) \leqslant \nu\left(Mx, t\right).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\nu\left(Mx, t\right) = \tfrac{1}{2}$$

By the assumption

$$t > M||x||_1 \geqslant ||kT^{'}||M^{'}||x||_1 \geqslant ||T\left(x\right)||_1,$$

thus

$$\nu\left(T\left(x\right), t\right) \leqslant \tfrac{1}{2}$$

Hence

$$\nu\left(T\left(x\right), t\right) \leqslant \nu\left(Mx, t\right)$$

For the third condition :

$$t \leqslant M||x||_1.$$

Then

$$\nu\left(Mx, t\right) = 1.$$

By the assumption trivially we have,

$$\nu\left(T\left(x\right), t\right) \leqslant \nu\left(Mx, t\right).$$

Hence $(X, \mu, \nu, *, \circ)$ has the AP.

4. Conclusion

In this paper we introduced and investigated the concepts of AP and BAP in the context of an IFNLS. We have shown that there are IFNLSs which fail to have the AP and also there are IFNLSs with AP but not the BAP. The current results give us a better understanding of the analytical structure of an IFNLS.

Acknowledgement

The authors are enormously thankful to the reviewers for their comments and suggestions towards improvement of the paper. The authors also express their gratitude to the Managing Editor for his support.

Research of Nabanita Konwar is supported by North Eastern Regional Institute of Science and Technology through GATE fellowship (Ministry of Human Resource Development, Govt. of India).

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