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On a New BV_σ I-Convergent Double Sequence Spaces

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Abstract

In this article we study ${}_2({}_0BV^I_\sigma(M))$, ${}_2BV^I_\sigma(M)$, ${}_2({}_\infty BV^I_\sigma(M))$ double sequence spaces with the help of BV_σ space and an Orlicz function M. The BV_σ space was introduced and studied by (Mursaleen, 1983). We study some of its properties and prove some inclusion relations.

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1. Introduction

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote

$$ {}_2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \ or \ \mathbb{C}\}, $$

showing the space of all real or complex sequences.

Definition 1.1. A double sequence of complex numbers is defined as a function $X : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other. A number a$\in \mathbb{C}$ is called double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that,

$$ |(x_{ij}) - a| < \epsilon, \text{ for all } i, j \geqslant N, \tag{1.1} $$

(see (Habil, 2006)). Let l_∞ and c denote the Banach space bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let v be denote the space of sequences of bounded variation. That is,

$$ v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\} \tag{1.2} $$

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where v is a Banach space normed by $\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$ (see (Mursaleen, 1983)). Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if:
1. $\phi(x) \geqslant 0$ where the sequence $x = (x_k)$ has $x_k \geqslant 0$ for all k,
2. $\phi(e) = 1$ where e={1,1,1,1,......},
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x\} \tag{1.3}$$

where m$\geqslant 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + \dots\dots + x_\sigma^m(k)}{m+1} \text{ and } t_{-1,k} = 0, \tag{1.4}$$

where $\sigma^m(k)$ denote the m^{th}-iterate of $\sigma(k)$ at k . In this case σ is the translation mapping,that is, $\sigma(k) = k + 1, \sigma$-mean is called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.4) in which $\sigma(k) = k + 1$ was given by (Lorentz, 1948), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{1.5}$$

Theorem 1.1. *A σ-mean extends the limit functional on* c *in the sense that $\phi(x) = \lim x$ for all $x \in$ c if and only if σ has no finite orbits. That is, if and only if for all $k \geqslant 0, j \geqslant 1, \sigma^j(k) \neq k$, (see (Khan, 2008))*

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{1.6}$$

assuming that $t_{-1,k}(x) = 0$. A straight forward calculation shows that (Mursaleen, 1983),

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^m J(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if } m \geqslant 1 \\ x_k, \quad \text{if } m = 0. \end{cases}$$

For any sequence x,y and scalar λ, we have $\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$ and $\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x)$.

Definition 1.2. A sequence $x \in l_\infty$ is of σ-bounded variation if and only if:
(i)$\sum |\phi_{m,k}(x)|$ converges uniformly in k,
(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variation (see (Khan, 2008)):

$$BV_\sigma = \{x \in l_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{ uniformly in } k\}.$$

Theorem 1.2. *BV_σ is a Banach space normed by*

$$\|x\| = \sup_k \sum_{m=0}^{\infty} |\phi_{m,k}(x)|, \tag{1.7}$$

(see (Khan & Ebadullah, 2012)).
Subsequently invariant mean studied by (Mursaleen, 1983), (Ahmad & Mursaleen, 1988), (Raimi & A., 1963), (Khan & Ebadullah, 2011), (Khan & Ebadullah, 2012), (Schaefer, 1972) and many others.

Definition 1.3. A function $M : [0, \infty) \longrightarrow [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions;
(i)M is continuous ,convex and non-decreasing,
(ii)$M(0) = 0, M(x) > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Remark. (see (Tripathy & Hazarika, 2011)). **(i)** If the convexity of an Orlicz function is replaced by $M(x + y) \leqslant M(x) + M(y)$, then this function is called Modulus function.
(ii) If M is an Orlicz function , then $M(\lambda X) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy $\triangle_2$-condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leqslant KLM(u)$ for all values of L> 1(see (Tripathy & Hazarika, 2011)). (Lindenstrauss & Tzafriri, 1971) used the idea of an Orlicz function to construct the sequence space $l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty$ for some $\rho > 0\}$. The space l_∞ becomes a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leqslant 1\Big\}, \tag{1.8}$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leqslant p < \infty$. Later on, some Orlicz sequence spaces were investigated by (Hazarika & Esi, 2013), (Maddox, 1970), (Parshar & Choudhary, 1994), (Bhardwaj & Singh, 2000), (Et, 2001), (Tripathy & Hazarika, 2011) and many others. Initially, as a generalization of statistical convergence, the notation of I-convergence was introduced and studied by P. Kostyrko and Wilczynski(Kostyrko *et al.*, 2000). Later on, it was studied by Hazarika and Esi (Hazarika & Esi, 2013) and many others.

Definition 1.4. A double sequence $x = x_{ij} \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geqslant \epsilon\} \in I. \tag{1.9}$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 1.5. Let X be a non empty set. Then, a family of sets $I \subseteq 2^X$ is said to be an Ideal in X if
(i)$\phi \in I$;
(ii)I is additive; that is,$A, B \in I \Rightarrow A \cup B \in I$;
(iii)I is hereditary that is,$A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.
A non trivial ideal I is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset.

Definition 1.6. A non empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if
(i) $\emptyset \notin \mathcal{F}$;
(ii) for, A, B $\in$ $\mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii) for each A$\in$ $\mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$$\mathcal{F}(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where} K^c = N - K. \tag{1.10}$$

Definition 1.7. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I - null if L=0. In this case, we write

$$I - \lim x_{ij} = 0. \tag{1.11}$$

Definition 1.8. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geqslant \epsilon\} \in I. \tag{1.12}$$

Definition 1.9. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}. \tag{1.13}$$

Definition 1.10. A double sequence space E is said to be solid or normal if $x_{ij} \in E$ implies that $(\alpha_{ij}x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.11. A double sequence space E is said to be symmetric if $(x_{\pi(i)\pi(j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i)$ and $\pi(j)$ is a permutation on $\mathbb{N}$.

Definition 1.12. A double sequence space E is said to be sequence algebra if $(x_{ij}y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.13. A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.14. Let K=$\{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 < ....and\ \ k_1 < k_2 < k_3 < ....\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space.A K-step space of E is a sequence space

$$\lambda_k^E = \{(\alpha_{ij}x_{ij}) : (x_{ij}) \in E\}.$$

Definition 1.15. A cannonical preimage of a sequence $(x_{n_ik_j}) \in E$ is a sequence $(b_{nk}) \in E$ defined as follows

$$b_{n,k} = \begin{cases} a_{n,k}, & \text{for n , k } \in K \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.16. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Remark. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 1.17. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$ and we call the I_δ -convergence as the logarithmic statistical convergence.

Definition 1.18. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then,I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as asymptotic statistical covergence.

Lemma 1.1. *((Tripathy & Hazarika, 2011)). Every solid space is monotone.*

Lemma 1.2. *Let $\mathcal{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 1.3. *If $I \subseteq 2^N$ and $M \subseteq N$.If $M \notin I$, then $M \cap N \notin I$.*

2. Main Results

Recently (Khan & Khan, 2013) introduced and studied the following sequence space. For m,n$\geqslant 0$

$$ {}_2BV_\sigma^I = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x) - L| \geqslant \epsilon\} \in I, for\ some\ L \in \mathbb{C}\}. \quad (2.1) $$

In this article we introduce the following double sequence spaces. For m,n$\geqslant 0$

$$ {}_2BV_\sigma^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0,\ for\ some\ L \in \mathbb{C}, \rho > 0\} \quad (2.2) $$

$$ {}_2({}_0BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \rho > 0\}, \quad (2.3) $$

$$ {}_2({}_\infty BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N}\ \ \exists\, k > 0\ s.t\ M(\frac{|\phi_{mnij}(x)|}{\rho}) \geqslant k\} \in I, \rho > 0\} \quad (2.4) $$

$$ {}_2({}_\infty BV_\sigma(M)) = \{x = (x_{ij}) \in {}_2\omega : \sup M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty, \rho > 0\}. \quad (2.5) $$

We also denote

$$ {}_2M_{BV_\sigma}^I(M) = {}_2BV_\sigma^I(M) \cap {}_2({}_\infty BV_\sigma(M)) $$

and

$$ {}_2({}_0M_{BV_\sigma}^I(M)) = {}_2({}_0BV_\sigma^I(M)) \cap {}_2({}_\infty BV_\sigma(M)). $$

Theorem 2.1. *For any Orlicz function M, the classes of double sequence ${}_2({}_0BV_\sigma^I(M))$,${}_2BV_\sigma^I(M)$, ${}_2({}_0M_{BV_\sigma}^I(M))$, and ${}_2M_{BV_\sigma}^I(M)$ are linear spaces.*

Proof. Let $x = (x_{ij}), (y_{ij}) \in {}_2BV_\sigma^I(M)$ be any two arbitrary elements, and let α, β are scalars. Now, since $(x_{ij}), (y_{ij}) \in {}_2BV_\sigma^I(M)$. Then this implies that $\exists$ some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1, \rho_2 > 0$ such that,

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) = 0, \tag{2.6}$$

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) = 0. \tag{2.7}$$

$\Rightarrow$ for any given $\epsilon > 0$, the sets

$$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) \geqslant \frac{\epsilon}{2}\} \in I, \tag{2.8}$$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) \geqslant \frac{\epsilon}{2}\} \in I. \tag{2.9}$$

Now let

$$A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(x) - L_1|}{\rho_1}\Big) < \frac{\epsilon}{2}\} \in I, \tag{2.10}$$

$$A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(y) - L_2|}{\rho_2}\Big) < \frac{\epsilon}{2}\} \in I. \tag{2.11}$$

be such that $A_1^c, A_2^c \in I$. Let $\rho_3 = max\{2|\alpha|\rho_1\ ,\ 2|\beta|\rho_2\}$
Since M is non decreasing and convex function,we have

$$M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = M(\frac{|(\alpha\phi_{mnij}(x) + \beta\phi_{mnij}(y)) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\begin{aligned} &= M(\tfrac{|\alpha(\phi_{mnij}(x) - L_1) + \beta(\phi_{mnij}(y) - L_2)|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_3}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_1}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_2}) \\ &\leqslant \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \in I$
implies that , $I - \lim_{i,j} M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = 0$.
Thus $\alpha(x_{ij}) + \beta(y_{ij}) \in {}_2BV_\sigma^I(M)$. As (x_{ij}) and (y_{ij}) are two arbitrary element then $\alpha x_{ij} + \beta y_{ij} \in {}_2BV_\sigma^I(M)$ for all $x_{ij},\ y_{ij} \in {}_2BV_\sigma^I(M)$, for all scalars α, β . Hence ${}_2BV_\sigma^I(M)$ is linear space. The proof for other spaces will follow similarly. □

Theorem 2.2. *Let M_1, M_2 be two Orlicz functions and statisfying $\triangle_2$ condition ,then*
(a)$X(M_2) \subseteq X(M_1M_2)$
(b)$X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for $X = {}_2BV_\sigma^I,\ {}_2({}_0BV_\sigma^I),\ {}_2M_{BV_\sigma}^I,\ {}_2({}_0M_{BV_\sigma}^I)$.

Proof. (a)Let $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_2))$ be an arbitrary element
$\Rightarrow \rho > 0$ such that

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.12}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 < t \leqslant \delta$.
Write $y_{ij} = M_2(\frac{|\phi_{mnij}(x)|}{\rho})$ and
consider,

$$\lim_{ij} M_1(y_{ij}) = \lim_{y_{ij}\leqslant\delta, i,j\in\mathbb{N}} M_1(y_{ij}) + \lim_{y_{ij}>\delta, i,j\in\mathbb{N}} M_1(y_{ij}). \tag{2.13}$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x) \leqslant \lambda M_1(x), 0 < \lambda < 1$.
Therefore we have,

$$\lim_{y_{ij}\leqslant\delta,\ i,j\in\mathbb{N}} M_1(y_{ij}) \leqslant M_1(2) \lim_{y_{ij}\leqslant\delta,\ i,j\in\mathbb{N}} (y_{ij}). \tag{2.14}$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that,

$$M_1(y_{ij}) < M_1(1 + \frac{y_{ij}}{\delta}) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_{ij}}{\delta}). \tag{2.15}$$

Since M_1 satisfies the $\triangle_2$- condition we have,

$$\begin{aligned} M_1(y_{ij}) &< \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) + \frac{1}{2}KM_1(\frac{2y_{ij}}{\delta}) \\ &< \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) + \frac{1}{2}K\frac{y_{ij}}{\delta}M_1(2) \\ &= K\frac{y_{ij}}{\delta}M_1(2). \end{aligned} \tag{2.16}$$

This implies that,

$$M_1(y_{ij}) < K\frac{y_{ij}}{\delta}M_1(2). \tag{2.17}$$

Hence,we have

$$\lim_{y_{ij}>\delta,\ i,j\in\mathbb{N}} M_1(y_{ij}) \leqslant max\{1, K\delta^{-1}M_1(2) \lim_{y_{ij}>\delta, i,j\in\mathbb{N}} (y_{ij})\}. \tag{2.18}$$

Therefore from (2.12),and (2.13) we have

$$I - \lim_{ij} M_1(y_{ij}) = 0.$$

$$\Rightarrow I - \lim_{ij} M_1M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

This implies that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1M_2))$.Hence $X(M_2) \subseteq X(M_1M_2)$ for $X = {}_2({}_0BV_\sigma^I)$. The other cases can be proved in similar way.

(b) Let $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2))$. Let $\epsilon > 0$ be given.Then $\exists\rho > 0$ such that,

$$I - \lim M_1(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \tag{2.19}$$

and

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.20}$$

Therefore

$$I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = I - \lim_{ij} M_1(\frac{|\phi_{mnij}(x)|}{\rho}) + I - \lim_{ij} M_2(\frac{|\phi_{mnij}(x)|}{\rho}),$$

from eqs (2.19) and (2.20)

$$\Rightarrow I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

we get

$$x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1 + M_2)).$$

Hence we get ${}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2)) \subseteq {}_2({}_0BV_\sigma^I(M_1 + M_2))$.
For $X = {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I), {}_2(M_{BV_\sigma}^I)$ the inclusion are similar. □

Corollary 2.1. *$X \subseteq X(M)$ for $X = {}_2(BV_\sigma^I), {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I)$ and ${}_2M_{BV_\sigma}^I$.*

Proof. For this let $M(x) = x$, for all $x = (x_{ij}) \in X$. Let us suppose that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I)$. Then for any given $\epsilon > 0$ we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| \geqslant \epsilon\} \in I.$$

Now let

$$A_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| < \epsilon\} \in I,$$

be such that $A_1^c \in I$. Now consider , for $\rho > 0$,

$$M(\frac{|\phi_{mnij}(x)|}{\rho}) = \frac{|\phi_{mnij}(x)|}{\rho}$$

$$< \frac{\epsilon}{\rho} < \epsilon.$$

$\Rightarrow I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0$, which implies that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M))$. Hence we have

$${}_2({}_0BV_\sigma^I) \subseteq {}_2({}_0BV_\sigma^I(M)).$$

$$\Rightarrow X \subseteq X(M)$$

and the other cases will be proved similarly. □

Theorem 2.3. *For any Orlicz function M, the spaces* ${}_2({}_0BV_\sigma^I(M))$ *and* ${}_2({}_0M_{BV_\sigma}^I)$ *are solid and monotone.*

Proof. Here we consider ${}_2({}_0BV_\sigma^I)$ and for ${}_2({}_0BV_\sigma^I(M))$ the proof shall be similar.
Let $x = x_{ij} \in {}_2({}_0BV_\sigma^I(M))$ be an arbitrary element ,$\Rightarrow \exists\rho > 0$ such that

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

Let α_{ij} be a sequence of scalars with $|\alpha_{ij}| \leqslant 1$ for $i, j \in \mathbb{N}$.
Now, M is an Orlicz function. Therefore

$$M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = M(\frac{|\alpha_{ij}||\phi_{mnij}(x)|}{\rho})$$

$$\leqslant |\alpha_{ij}|M(\frac{|\phi_{mnij}(x)|}{\rho})$$

$\Rightarrow M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho})$ for all $i, j \in \mathbb{N}$.

$$\Rightarrow I - \lim_{ij} M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = 0.$$

Thus we have $(\alpha_{ij}x_{ij}) \in {}_2({}_0BV_\sigma^I(M))$. Hence ${}_2({}_0BV_\sigma^I(M))$ is solid. Therefore ${}_2({}_0BV_\sigma^I(M))$ is monotone. Since every solid sequence space is monotone. □

Theorem 2.4. *For any Orlicz function M,the space* ${}_2BV_\sigma^I(M)$ *and* ${}_2(M_{BV_\sigma^I}(M))$ *are neither solid nor monotone in general.*

Proof. Here we give counter example for establishment of this result. Let $X = {}_2BV_\sigma^I$ and ${}_2(M_{BV_\sigma^I})$.
Let us consider $I = I_f$ and $M(x) = x$, for all $x = x_{ij} \in [0, \infty)$. Consider,the K-step space $X_K(M)$ of $X(M)$ defined as follows:
Let $x = (x_{ij}) \in X(M)$ and $y = (y_{ij}) \in X_K(M)$ be such that $(y_{ij}) = (x_{ij})$, if i,j is even and $(y_{ij}) = 0$, otherwise.
Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $i, j \in \mathbb{N}$. Then $x = (x_{ij}) \in {}_2BV_\sigma^I(M)$ and ${}_2M_{BV_\sigma^I}(M)$, but K-step space preimage does not belong to $BV_\sigma^I(M)$ and ${}_2M_{BV_\sigma}^I(M)$. Thus ${}_2BV_\sigma^I(M)$ and ${}_2M_{BV_\sigma}^I(M)$ are not monotone and hence they are not solid. □

Theorem 2.5. *For an Orlicz function M, the spaces* ${}_2BV_\sigma^I(M)$ *and* ${}_2BV_\sigma^I(M)$ *are sequence algebra.*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2({}_0(BV_\sigma^I(M)))$ be any two arbitrary elements. $\Rightarrow \rho_1,\rho_2 > 0$ such that,

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho_1}) = 0,$$

and

$$I - lim_{ij} M(\frac{|\phi_{mnij}(y)|}{\rho_2}) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$. Then

$$M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho_1\rho_2})$$

$$\Rightarrow I - \lim_{ij} M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = 0.$$

Therefore we have $(x_{ij}y_{ij}) \in\ _2(_0BV_\sigma^I(M))$. Hence $_2(_0BV_\sigma^I(M))$ is sequence algebra. □

Theorem 2.6. *For any Orlicz function M, the spaces* $_2(_0BV_\sigma^I(M))$ *and* $_2BV_\sigma^I(M)$ *are not convergence free.*

Proof. To show this let $I = I_f$ and $M(x) = x$, for all $x = [0, \infty)$. Now consider the double sequence $(x_{ij}), (y_{ij})$ which defined as follows:

$$x_{ij} = \frac{1}{i+j} and \;\; y_{ij} = i + j, \forall i, j \in \mathbb{N}.$$

Then we have (x_{ij}) belong to both $_2(_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$, but (y_{ij}) does not belong to $_2(_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$. Hence, the spaces $_2(_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$ are not convergence free. □

Theorem 2.7. *Let M be an Orlicz function. Then*

$$_2(_0BV_\sigma^I(M)) \subseteq\ _2BV_\sigma^I(M) \subseteq\ _2(_\infty BV_\sigma^I(M)).$$

Proof. For this let us consider $x = (x_{ij}) \in\ _2(_0BV_\sigma^I(M))$. It is obvious that it must belong to $_2BV_\sigma^I(M)$. Now consider

$$M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho}) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the limit on both sides we get

$$I - lim_{ij} M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0.$$

Hence $x = (x_{ij}) \in\ _2BV_\sigma^I(M)$.
Now it remains to show that $_2(BV_\sigma^I(M)) \subseteq\ _2(_\infty BV_\sigma^I(M))$. For this let us consider $x = (x_{ij}) \in\ _2BV_\sigma^I(M) \Rightarrow \exists\rho > 0$ s.t

$$I - \lim_{ij} M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) = 0.$$

Now consider

$$M\Big(\frac{|\phi_{mnij}(x)|}{\rho}\Big) \leqslant M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the supremum on both sides we get

$$\sup_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty.$$

Hence $x = (x_{ij}) \in\ _2(_\infty BV_\sigma^I(M))$. □ □

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