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Fixed Points of Generalized Kannan Type α-admissible Mappings in Cone Metric Spaces with Banach Algebra

S.K. Malhotra[a], J.B. Sharma[b], Satish Shukla[c,*]

[a]*Department of Mathematics, Govt. S.G.S.P.G. College, Ganj Basoda, Distt Vidisha M.P., India*
[b]*Department of Mathematics, Choithram College of Professional Studies, Dhar Road, Indore, India*
[c]*Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Gram Baroli, Sanwer Road, Indore (M.P.) 453331 India*

Abstract

In this paper, we introduce the generalized Kannan type α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra. Our results generalize and extend the fixed point result for Kannan type mappings in metric and cone metric spaces. An example is presented which illustrates our main result.

Keywords: Cone metric space, α-admissible mapping, Kannan's contraction, Solid cone, Banach algebra, Fixed point.
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1. Introduction

Huang and Zhang (Huang & Zhang, 2007) introduced the notion of cone metric spaces as a generalization of metric spaces. They replaced the set of nonnegative real numbers by a subset of a Banach space called the cone; and defined the metric as a vector-valued function. They obtained some fixed point results in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani (Rezapour & Hamlbarani, 2008). Liu and Xu (Liu & Xu, 2013*a*) defined the cone metric spaces with Banach algebra and defined the vector-valued metric into a subset of a Banach algebra. The motivation for the work of Liu and Xu (Liu & Xu, 2013*a*) can be found in (Cakalli *et al.*, 2012; Kadelburg *et al.*, 2011; Du, 2010; Feng & Mao, 2010). The results proved by Liu and Xu (Liu & Xu, 2013*a*) demands the normality of the underlying cone. Later on, Xu and Radenović (Xu

*Corresponding author
Email addresses: `jagdishmathematics@gmail.com` (J.B. Sharma), `satishmathematics@yahoo.co.in` (Satish Shukla)

& Radenović, 2014) showed that the condition of normality of cone can be removed, and so, the results of Liu and Xu (Liu & Xu, 2013*a*) are also true in case of a non-normal cone.

Let (X, d) be a metric space and $T: X \to X$ be a mapping satisfying the following condition: there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Then the mapping T is called a Banach contraction. The Banach's contraction principle states that a Banach contraction on a complete metric space has a unique fixed point, i.e., there exists a unique point $x^* \in X$ such that $x^* = Tx^*$. Kannan (Kannan, 1968, 1969) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leqslant \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1.2}$$

Kannan (Kannan, 1968, 1969) showed that the conditions (1.1) and (1.2) are independent of each other, and proved a fixed point result for the mapping satisfying the condition (1.2) instead the condition (1.1).

Samet et al. (Samet *et al.*, 2012) introduced a new type of mappings called α-admissible mappings, and with the help of this new class of mappings they generalized several known results of metric spaces. Very recently, Malhotra et al. (Malhotra *et al.*, 2015) introduced the α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra and solid cones. They generalized and extended several known results of metric and cone metric spaces by proving a fixed point result for generalized Lipschitz contraction over cone metric spaces. The main result of (Malhotra *et al.*, 2015) was a generalization of Banach's fixed point theorem. In this paper, we introduce the notion of generalized Kannan type α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra which extend the concept introduced in (Malhotra *et al.*, 2015) and generalize the result of Kannan (Kannan, 1968, 1969) in cone metric spaces equipped with Banach algebra.

2. Preliminaries

First, we state some known definitions and results which will be used in the sequel.

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in A, a \in \mathbb{R}$

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $a(xy) = (ax)y = x(ay)$;
4. $\|xy\| \leqslant \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details we refer to (Rudin, 1991).

The following proposition is well known (Rudin, 1991).

Proposition 2.1. *Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than one, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if

1. P is non-empty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;
2. $a_1P + a_2P \subset P$ for all non-negative real numbers a_1, a_2;
3. $P^2 = PP \subset P$
4. $P \bigcap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in A$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leqslant K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see (Huang & Zhang, 2007)). Note that, for any normal cone P we have $K \geqslant 1$ (see (Rezapour & Hamlbarani, 2008)). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \phi$ (i.e., the cone P is a solid cone) and $\preceq$ is the partial ordering with respect to P.

The following lemmas and remark will be useful in the sequel.

Lemma 2.1 (See (Kadelburg *et al.*, 2010))**.** *If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leqslant \lambda < 1$, then $a = \theta$.*

Lemma 2.2 (See (Radenović & Rhoades, 2009))**.** *If E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.3 (See (Radenović & Rhoades, 2009))**.** *If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.*

Remark (See (Xu & Radenović, 2014)). If $\rho(x) < 1$ then $\|x^n\| \to 0$ as $n \to \infty$.

Definition 2.1 (See (Liu & Xu, 2013*a*,*b*; Huang & Zhang, 2007))**.** Let X be a non-empty set. Suppose that the mapping $d\colon X \times X \to A$ satisfies:

1. $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$.
2. $d(x,y) = d(y,x)$ for all $x, y \in X$.
3. $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over the Banach algebra A.

Definition 2.2 (See (Huang & Zhang, 2007))**.** Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

1. The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. The sequence $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X.

It is obvious that the limit of a convergent sequence in a cone metric space is unique. A mapping $T : X \to X$ is called continuous at $x \in X$, if for every sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.3 (See (Samet *et al.*, 2012))**.** Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is α-admissible if $(x, y) \in X, \alpha(x, y) \geqslant 1 \implies \alpha(Tx, Ty) \geqslant 1$.

Now, we define the generalized Lipschitz contractions on the cone metric spaces with a Banach algebra (see also, (Liu & Xu, 2013*a*)).

Definition 2.4. (Malhotra *et al.*, 2015) Let (X, d) be a complete cone metric space over a Banach algebra A, P the underlying solid cone and $\alpha : X \times X \to [0, \infty)$ be a function. Then the mapping $T : X \to X$ is said to be generalized Lipschitz contraction if there exists $k \in P$ such that $\rho(k) < 1$ and,

$$d(Tx, Ty) \preceq kd(x, y)$$

for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$. Here, the vector k is called the Lipschitz vector of T.

Malhotra et al. (Malhotra *et al.*, 2015) proved a fixed point result for such generalized contraction. Here, we prove a Kannan's version of the result of Malhotra et al. (Malhotra *et al.*, 2015).

Now we can state our main results.

3. Main results

First, we define generalized Kannan type contractions in cone metric spaces with Banach algebra.

Definition 3.1. Let (X, d) be a complete cone metric space over a Banach algebra A, P the underlying solid cone and $\alpha : X \times X \to [0, \infty)$ be a function. Then the mapping $T : X \to X$ is said to be generalized Kannan type contraction if there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and,

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \tag{3.1}$$

for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$. Here, the vector k is called the Kannan-Lipschitz vector of T.

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra A, P be the underlying solid cone and $\alpha\colon X \times X \to [0, \infty)$ be a function. Suppose, $T\colon X \to X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*
(iii) *T is continuous.*

Then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant 1 \implies \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geqslant 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.2}$$

Since T is generalized Kannan type contraction with Kannan-Lipschitz vector k, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &= k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

i.e.,

$$(e - k)d(x_n, x_{n+1}) \preceq kd(x_{n-1}, x_n).$$

Since $\rho(k) < \frac{1}{2} < 1$, $e - k$ is invertible, therefore it follows from the above inequality that

$$d(x_n, x_{n+1}) \preceq k(e - k)^{-1}d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n) \preceq \lambda^n d(x_0, x_1) \tag{3.3}$$

where $\lambda = k(e - k)^{-1}$. Since $(e - k)^{-1} = \sum_{i=0}^{\infty} k^i$ we have

$$\rho((e - k)^{-1}) = \rho\left(\sum_{i=0}^{\infty} k^i\right) \leqslant \sum_{i=0}^{\infty} \rho(k^i) \leqslant \sum_{i=0}^{\infty} [\rho(k)]^i = \frac{1}{1 - \rho(k)}.$$

Therefore,

$$\begin{aligned} \rho(\lambda) &= \rho\left(k(e - k)^{-1}\right) \leqslant \rho(k)\rho\left((e - k)^{-1}\right) \\ &\leqslant \frac{\rho(k)}{1 - \rho(k)} < 1 \quad (\text{since } \rho(k) < \frac{1}{2}). \end{aligned}$$

Thus, for $n < m$ it follows from the inequality (3.3) that

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &= (e + \lambda + \cdots + \lambda^{m-n-1})\lambda^n d(x_0, x_1) \\ &\preceq \left(\sum_{i=0}^{\infty} \lambda^i\right) \lambda^n d(x_0, x_1) \\ &= (e - \lambda)^{-1} \lambda^n d(x_0, x_1). \end{aligned}$$

Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \to 0$ as $n \to \infty$. Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \preceq (e - \lambda)^{-1} \lambda^n d(x_0, x_1) \ll c$$

for all $n > n_0$. It implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is continuous, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By the uniqueness of limit we get $x^* = Tx^*$, that is x^* is a fixed point of T. □

In the above theorem, we use the continuity of the mapping T. We now show that the assumption of continuity can be replaced by another condition.

Theorem 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra A, P be the underlying solid cone and $\alpha\colon X \times X \to [0, \infty)$ be a function. Suppose, $T\colon X \to X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*

(iii) *if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$.

Proof. By proof of theorem 3.1, we know that the sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ is a Cauchy sequence in complete cone metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, from (3.2) and hypothesis (iii), we have

$$\alpha(x_n, x^*) \geqslant 1, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.4}$$

Since T is a generalized Kannan type contraction, using (3.4) we obtain

$$\begin{aligned} d(x^*, Tx^*) &\preceq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\preceq d(x^*, x_{n+1}) + k[d(x_n, Tx_n) + d(x^*, Tx^*)] \end{aligned}$$

i.e.,

$$\begin{aligned} d(x^*, Tx^*) &\preceq (e-k)^{-1}[d(x^*, x_{n+1}) + kd(x_n, Tx_n)] \\ &= (e-k)^{-1}d(x^*, x_{n+1}) + \lambda d(x_n, Tx_n). \end{aligned}$$

By (3.3) we have $d(x_n, Tx_n) = d(x_n, x_{n+1}) \preceq \lambda^n d(x_0, x_1)$, therefore

$$d(x^*, Tx^*) \preceq (e-k)^{-1}d(x^*, x_{n+1}) + \lambda^{n+1}d(x_0, x_1).$$

As $x_n \to x^*$ as $n \to \infty$ and $\rho(\lambda) < 1$, for every $c \in P$ with $\theta \ll c$ and for every $m \in \mathbb{N}$ there exists $n(m)$ such that $d(x_{n+1}, x^*) \ll \frac{(e-k)c}{2m}$ and $\lambda^{n+1}d(x_0, x_1) \ll \frac{c}{2m}$ for all $n > n(m)$. Therefore, it follows from the above inequality that

$$d(x^*, Tx^*) \preceq \frac{c}{2m} + \frac{c}{2m} = \frac{c}{m} \text{ for all } n > n(m), m \in \mathbb{N}.$$

It implies that $\frac{c}{m} - d(x^*, Tx^*) \in P$ for all $m \in \mathbb{N}$. Since P is closed, letting $m \to \infty$ we obtain $\theta - d(x^*, Tx^*) \in P$. By definition, we must have $d(x^*, Tx^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T. □

Next, we give an example which illustrates the above result.

Example 3.1. Let $A = \mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Define the multiplication on A by

$$xy = (x_1y_1, x_1y_2 + x_2y_1) \quad \text{for all} \quad x = (x_1, x_2), y = (y_1, y_2) \in A.$$

Then, A is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1, x_2 \geqslant 0\}$. Then P is a positive cone.

Let $X = [0, 1] \times [0, 1]$ and define the cone metric $d \colon X \times X \to P$ by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then, (X, d) is a complete cone metric space. Let $\mathbb{Q} \cap [0, 1) = \mathbb{Q}_1$ and define the mappings $T \colon X \to X$ and $\alpha \colon X \times X \to [0, \infty)$ by:

$$T(x_1, x_2) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), & \text{if } x_1, x_2 \in \mathbb{Q}_1; \\ \left(\frac{1}{4}, \frac{1}{4}\right), & \text{if } x_1 = x_2 = 1; \\ (x_1, x_2), & \text{otherwise.} \end{cases}$$

and

$$\alpha((x_1, x_2), (y_1, y_2)) = \begin{cases} 1, & \text{if } (x_1, x_2, y_1, y_2 \in \mathbb{Q}_1) \text{ or } (x_1, x_2 \in \mathbb{Q}_1, y_1 = y_2 = 1); \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is a generalized Kannan type contraction with Kannan-Lipschitz vector $k = \left(\frac{1}{3}, 0\right)$, where $\rho(k) = \frac{1}{3} < \frac{1}{2}$. Indeed, $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ then (3.1) is satisfied trivially. If $x_1, x_2 \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$ then we have

$$\begin{aligned} d(T(x_1, x_2), T(y_1, y_2)) &= d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) \\ &= \left(\frac{1}{4}, \frac{1}{4}\right) \\ &\preceq \left(\frac{1}{3}, 0\right) [d((x_1, x_2), T(x_1, x_2)) + d((y_1, y_2), T(y_1, y_2))]. \end{aligned}$$

T is obviously an α-admissible mapping, and for every $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ we have

$$\alpha((x_1, x_2), T(x_1, x_2)) = 1.$$

Therefore, the conditions (i) and (ii) of Theorem 3.2 are satisfied. Finally, one can see that the condition (iii) of Theorem 3.2 is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied and we conclude the existence of at least one fixed point of T. Indeed, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and all the points $(x, 1), x \in \mathbb{Q}_1$ and $(1, x), x \in \mathbb{Q}_1$ are fixed points of T.

Remark. Notice that, in the above example the results of Malhotra et al. (Malhotra *et al.*, 2015) are not applicable. Indeed, if $x_1 = x_2 = \frac{3}{4} \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$, then $\alpha((x_1, x_2), (y_1, y_2)) = 1$ and

$$d(T(x_1, x_2), T(y_1, y_2)) = d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Now

$$d((x_1, x_2), (y_1, y_2)) = d\left(\left(\frac{3}{4}, \frac{3}{4}\right), (1, 1)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Therefore, there exists no $k \in P$ such that $\rho(k) < 1$ and the following inequality is satisfied:

$$d(T(x_1, x_2), (y_1, y_2)) \preceq kd((x_1, x_2), (y_1, y_2)).$$

This shows that T is not a generalized Lipschitz contraction, and so, the results of Malhotra et al. (Malhotra *et al.*, 2015) are not applicable here.

In the Example 3.1 we can see that the mapping T may have more than one fixed points. Let us denote the set of all fixed points of T by $\mathrm{Fix}(T)$.

Next, to assure the uniqueness of fixed point of a generalized Kannan type contraction we use the following property (see (Samet *et al.*, 2012)):

$$\forall\ x, y \in \mathrm{Fix}(T)\ \exists\ z \in X: \quad \alpha(x, z) \geqslant 1, \alpha(y, z) \geqslant 1. \tag{H}$$

Theorem 3.3. *Adding condition (H) to the hypothesis of Theorem 3.1 (resp. Theorem 3.2) we obtain the uniqueness of the fixed point of T.*

Proof. Following similar arguments to those in the proof of Theorem 3.1 (resp. Theorem 3.2) we obtain the existence of fixed point. Let the condition (H) is satisfied and $x^*, y^* \in \mathrm{Fix}(T)$ and $x^* \neq y^*$. By (H) there exists $z \in X$ such that

$$\alpha(x^*, z) \geqslant 1 \quad \text{and} \quad \alpha(y^*, z) \geqslant 1. \tag{3.5}$$

Since T is α-admissible and $x^*, y^* \in \mathrm{Fix}(T)$, therefore from (3.5) we obtain

$$\alpha(x^*, T^n z) \geqslant 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geqslant 1. \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.6}$$

Since T is generalized Kannan type contraction, using (3.6), we have

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, T(T^{n-1}z)) \\ &\preceq k[d(x^*, Tx^*) + d(T^{n-1}z, T(T^{n-1}z))] \\ &= kd(T^{n-1}z, T^n z) \\ &\preceq k[d(x^*, T^{n-1}z) + d(x^*, T^n z)] \end{aligned}$$

i.e.,

$$d(x^*, T^n z) \preceq k(e-k)^{-1} d(x^*, T^{n-1}z) = \lambda d(x^*, T^{n-1}z) \ \text{ for all } \ n \in \mathbb{N}.$$

Repetition of this process we obtain

$$d(x^*, T^n z) \preceq \lambda^n d(x^*, Tz) \ \text{ for all } \ n \in \mathbb{N}.$$

where $\lambda = k(e-k)^{-1}$ and $\rho(\lambda) < 1$. Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \to 0$ as $n \to \infty$, and so,

$$\|\lambda^n d(x^*, Tz)\| \leqslant \|\lambda^n\| \|d(x^*, Tz)\| \to 0 \text{ as } n \to \infty.$$

Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, T^n z) \preceq \lambda^n d(x^*, Tz) \ll c.$$

it implies that

$$T^n z \to x^* \quad \text{as} \quad n \to \infty.$$

Similarly we get

$$T^n z \to y^* \quad \text{as} \quad n \to \infty.$$

Therefore, by uniqueness of the limit we obtain $x^* = y^*$. This finishes the proof. □

4. Some consequences

In this section, we give some consequences of the results of previous section. The following corollary is Theorem 3.3 of Xu and Radenović (Liu & Xu, 2013*a*).

Corollary 4.1 (Theorem 3.3, Xu and Radenović (Liu & Xu, 2013*a*))**.** *Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone with $k \in P$ where $\rho(k) < \frac{1}{2}$. Suppose the mapping $T: X \to X$ satisfies the following condition :*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \quad \textit{for all} \quad x, y \in X.$$

Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point of X.

Proof. Define the function $\alpha: X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then, all the conditions of Theorem 3.3 are satisfied, and so, the mapping T has a unique fixed point in X. □

Next, we derive the ordered and cyclic versions of Kannan's contraction principle. In the next theorems, we prove results of Ran and Reurings (Ran & Reurings, 2003), Liu and Xu (Liu & Xu, 2013*a*) and Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) and Kirk et al. (Kirk *et al.*, 2003) for Kannan's mappings.

The following theorem is the Kannan's version of the result of Ran and Reurings (Ran & Reurings, 2003) in cone metric spaces when the cone metric is endowed with a Banach algebra.

Theorem 4.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a continuous nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assumptions hold:*

(i) *there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \textit{ for all } x, y \in X \textit{ with } x \sqsubseteq y;$$

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.*

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r: X \times X \to [0, \infty)$ by

$$\alpha_r(x, y) = \begin{cases} 1, & \text{if } x \sqsubseteq y; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, the condition (i) implies that the mapping T a generalized Kannan type contraction with Kannan-Lipschitz vector k, where $\rho(k) < \frac{1}{2}$. Since T is nondecreasing it is an α_r-admissible mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $\alpha_r(x_0, Tx_0) = 1$. Therefore, all the conditions of Theorem 3.1 are satisfied, and so, the mapping T has a fixed point in X. □

The following theorem is the Kannan's version of the result of Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) when the cone metric is endowed with a Banach algebra.

Theorem 4.2. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assumptions hold:*

(i) *there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } x \sqsubseteq y;$$

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*

(iii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r : X \times X \to [0, \infty)$ similar to that as in the proof of Theorem 4.1. Now, the proof follows from the Theorem 3.2. □

Next, we define the cyclic contractions (see (Kirk *et al.*, 2003)) in cone metric spaces.

Let X be a nonempty set, $T: X \to X$ a mapping and $A_1, A_2, \ldots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if

1. A_i, $i = 1, 2, \ldots, m$ are nonempty sets;
2. $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset T(A_1)$.

Remark. (See (Kirk *et al.*, 2003)) If $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, then $\mathrm{Fix}(T) \subset \bigcap_{i=1}^{m} A_i$.

A cyclic contraction on a cone metric space is defined as follows.

Definition 4.1. Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. A mapping $T: Y \to Y$ is called a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k if following conditions hold:

1. $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;
2. there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \tag{4.1}$$

for any $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$).

The following theorem is the Kannan's version of the result Kirk et al. (Kirk *et al.*, 2003) when the cone metric is endowed with a Banach algebra.

Theorem 4.3. *Let* $(X, \sqsubseteq)$ *be a partially ordered set and suppose that* (X, d) *be a complete cone metric space over a Banach algebra* A *with* P *the underlying solid cone. Suppose,* $A_1, A_2, \ldots, A_m$ *be closed subsets of* X *and* $Y = \bigcup_{i=1}^{m} A_i$ *and* $T: Y \to Y$ *be a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector* k*. Then,* T *has a unique fixed point in* X*.*

Proof. Define the mapping $\alpha_c: X \times X \to [0, \infty)$ by:

$$\alpha_c(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_i \times A_{i+1} \ (i = 1, 2, \ldots, m \text{ where } A_{m+1} = A_1); \\ 0, & \text{otherwise.} \end{cases}$$

First, by definition of the function α and the cyclic representation, T is α_c-admissible. Again, by definition of the function α_c, T is a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k. Suppose, for a sequence $\{x_n\}$ we have $\alpha_c(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then, as $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation with respect to T, we must have $x \in \bigcap_{i=1}^{m} A_i$. Therefore, $\alpha_c(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$. Now, the proof of existence of fixed point of T follows from Theorem 3.2. For uniqueness, if $x^*, y^* \in \text{Fix}(T)$, then by Remark 4 we have $x^*, y^* \in \bigcap_{i=1}^{m} A_i$. Since each A_i, $i \in \{1, 2, \ldots, m\}$ is nonempty, there exists $z \in Y$ such that $x^*, y^* \in A_i, z \in A_{i+1}$ for some $i \in \{1, 2, \ldots, m\}$, and so $\alpha_c(x^*, z) = \alpha_c(y^*, z) = 1$. Thus, the condition (H) is satisfied and the uniqueness of fixed point follows from Theorem 3.3. □

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