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Matrix Representations of Fuzzy Quaternion Numbers

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Abstract

In this paper, firstly we discuss basic arithmetic operations with fuzzy quaternion numbers. Then, we introduce a noncommutative field, formed with 2×2 fuzzy complex matrices which is used for the matrix representation of fuzzy quaternion numbers as elements within this field. Finally, another way of representing fuzzy quaternion numbers is obtained by using 4×4 fuzzy real matrices.

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1. Introduction

The fuzzy quaternion numbers were defined in many ways by many authors. For example, in paper (Moura *et al.*, 2013) the authors proposed an extension for the set of fuzzy real numbers to the set of fuzzy quaternion numbers, defining the notion of fuzzy quaternion numbers by an application $h' : H \to [0,1]$ such that $h'(a+bi+cj+dk) = min\{\widetilde{A}(a), \widetilde{B}(b), \widetilde{C}(c), \widetilde{D}(d)\}$ for some $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ real fuzzy numbers. In paper (Moura *et al.*, 2014), the authors defined the fuzzy quaternion numbers using triangular fuzzy numbers.

Recently, in paper (Sida *et al.*, 2016) a new approach is proposed in order to introduce the fuzzy quaternion numbers concept, similar to the way that Fu and Shen (see (Fu & Shen, 2011)) have introduced the fuzzy complex numbers.

The study of fuzzy quaternion numbers is continued in this paper. More precisely, another approach for multiplication and division operation is presented. Then, following the ideas in (Moţ & Popa, 2014), we will represent fuzzy quaternion numbers as a 2×2 fuzzy complex matrices, as well as 4×4 fuzzy real matrices.

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2. Preliminaries

For the concept of fuzzy real numbers and their arithmetic operations we make reference to the following papers: (Das & Mandal, 2002), (Dubois & Prade, 1980), (Dzitac, 2015), (Felbin, 1992), (Janfada *et al.*, 2011), (Kaleva & Seikkala, 1984), (Mizumoto & Tanaka, 1979), (Xiao & Zhu, 2002).

Definition 2.1. (Dzitac, 2015) A fuzzy set in $\mathbb{R}$, namely a mapping

$$\widetilde{A} : \mathbb{R} \to [0,1],$$

with the following properties:

(i) $\widetilde{A}$ is convex, i.e. $\widetilde{A}(y) \geq \min\left(\widetilde{A}(x), \widetilde{A}(z)\right)$, for $x \leq y \leq z$;
(ii) $\widetilde{A}$ is normal, i.e. $(\exists)x_0 \in \mathbb{R}$; $\widetilde{A}(x_0) = 1$;
(iii) $\widetilde{A}$ is upper semicontinuous, i.e.

$$(\forall)x \in \mathbb{R},\ (\forall)\alpha \in (0,1] : \ \widetilde{A}(x) < \alpha,$$

$$(\exists)\delta > 0 \text{ such that } \mid y - x \mid < \delta \Rightarrow \widetilde{A}(y) < \alpha$$

is called a fuzzy real number.

We will denote by $\mathbb{R}_F$ - the set of all fuzzy real numbers.

Definition 2.2. (Mizumoto & Tanaka, 1979) The basic arithmetic operations $+, -, \cdot, /$ on $\mathbb{R}_F$ are defined by:

1. *Addition*:
$$\left(\widetilde{A} + \widetilde{B}\right)(x) = \bigvee_{y \in \mathbb{R}} min\left\{\widetilde{A}(y), \widetilde{B}(x-y)\right\},\ (\forall)x \in \mathbb{R} \tag{2.1}$$
2. *Subtraction*:
$$\left(\widetilde{A} - \widetilde{B}\right)(x) = \bigvee_{y \in \mathbb{R}} min\left\{\widetilde{A}(y), \widetilde{B}(y-x)\right\},\ (\forall)x \in \mathbb{R} \tag{2.2}$$
3. *Multiplication*:
$$\left(\widetilde{A} \cdot \widetilde{B}\right)(x) = \bigvee_{y \in \mathbb{R}^*} min\left\{\widetilde{A}(y), \widetilde{B}(x/y)\right\},\ (\forall)x \in \mathbb{R} \tag{2.3}$$
4. *Division*:
$$\left(\widetilde{A}/\widetilde{B}\right)(x) = \bigvee_{y \in \mathbb{R}} min\left\{\widetilde{A}(x \cdot y), \widetilde{B}(y)\right\},\ (\forall)x \in \mathbb{R}. \tag{2.4}$$

Remark. A triangular fuzzy number is defined by its membership function

$$x(t) = \begin{cases} 0, & \text{if } \ t < a_1 \\ \frac{t-a_1}{a_2-a_1}, & \text{if } \ a_1 \leq t < a_2 \\ \frac{a_3-t}{a_3-a_2}, & \text{if } \ a_2 \leq t < a_3 \\ 0, & \text{if } \ t > a_3 \end{cases} \text{, where } a_1 \leq a_2 \leq a_3, \tag{2.5}$$

and it is denoted $\widetilde{x} = (a_1, a_2, a_3)$.

Remark. (Chang & Wang, 2009; Elomda & Hefny, 2013; Hanss, 2005; Nădăban *et al.*, 2016) Let $\widetilde{x} = (a_1, a_2, a_3)$, $\widetilde{y} = (b_1, b_2, b_3)$ be two non negative triangular fuzzy numbers and $\alpha \in \mathbb{R}_+$. According to the extension principle, the arithmetic operations are defined as follows:

1. $\widetilde{x} + \widetilde{y} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $\widetilde{x} - \widetilde{y} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$
3. $\alpha\widetilde{x} = (\alpha a_1, \alpha a_2, \alpha a_3)$
4. $\widetilde{x}^{-1} = (1/a_3, 1/a_2, 1/a_1)$
5. $\widetilde{x} \times \widetilde{y} \cong (a_1 b_1, a_2 b_2, a_3 b_3)$
6. $\widetilde{x}/\widetilde{y} \cong (a_1/b_3, a_2/b_2, a_3/b_1)$

We denote that the results of (4) – (6) are not triangular fuzzy numbers, but they can be approximated by triangular fuzzy numbers.

Definition 2.3. (Fu & Shen, 2011) An fuzzy complex number, $\widetilde{z}$, is defined in the form of:

$$\widetilde{z} = \widetilde{A} + i\widetilde{B}, \tag{2.6}$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{R}_F$; $\widetilde{A}$ is the real part of $\widetilde{z}$ while $\widetilde{B}$ represents the imaginary part, i.e. $Re(\widetilde{z}) = \widetilde{A}$ and $Im(\widetilde{z}) = \widetilde{B}$.

We will denote by $\mathbb{C}_F$ - the set of all fuzzy complex numbers. The operations on $\mathbb{C}_F$ are a straightforward extension of those on real complex numbers.

Definition 2.4. (Fu & Shen, 2011) Let $\widetilde{z} = \widetilde{A} + i\widetilde{B}$, $\widetilde{z}_2 = \widetilde{C} + i\widetilde{D} \in \mathbb{C}_F$ where $\widetilde{A}, \widetilde{B}, \widetilde{C}$ and $\widetilde{D}$ are fuzzy real numbers. The basic arithmetic operations are defined as follows:

1. *Addition*:

$$\widetilde{z}_1 + \widetilde{z}_2 = (\widetilde{A} + \widetilde{C}) + i(\widetilde{B} + \widetilde{D}), \tag{2.7}$$

where

$$(\widetilde{A} + \widetilde{C})(y) = \bigvee_{y = x_1 + x_2} \left(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2)\right)$$
$$(\widetilde{B} + \widetilde{D})(y) = \bigvee_{y = x_1 + x_2} \left(\widetilde{B}(x_1) \wedge \widetilde{D}(x_2)\right)$$

2. *Subtraction*:

$$\widetilde{z}_1 - \widetilde{z}_2 = (\widetilde{A} - \widetilde{C}) + i(\widetilde{B} - \widetilde{D}), \tag{2.8}$$

where

$$(\widetilde{A} - \widetilde{C})(y) = \bigvee_{y = x_1 - x_2} \left(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2)\right)$$
$$(\widetilde{B} - \widetilde{D})(y) = \bigvee_{y = x_1 - x_2} \left(\widetilde{B}(x_1) \wedge \widetilde{D}(x_2)\right)$$

3. *Multiplication*:
$$\widetilde{z_1} \times \widetilde{z_2} = (\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}) + i(\widetilde{B}\widetilde{C} + \widetilde{A}\widetilde{D}), \tag{2.9}$$
where
$$(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D})(y) = \bigvee_{y=x_1x_2-x_3x_4} \left(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{B}(x_3) \wedge \widetilde{D}(x_4)\right)$$
$$(\widetilde{B}\widetilde{C} + \widetilde{A}\widetilde{D})(y) = \bigvee_{y=x_1x_2+x_3x_4} \left(\widetilde{B}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{A}(x_3) \wedge \widetilde{D}(x_4)\right)$$

4. *Division*:
$$\widetilde{z_1}/\widetilde{z_2} = \left(\frac{\widetilde{A}\widetilde{C} + \widetilde{B}\widetilde{D}}{\widetilde{C}^2 + \widetilde{D}^2}\right) + i\left(\frac{\widetilde{B}\widetilde{C} - \widetilde{A}\widetilde{D}}{\widetilde{C}^2 + \widetilde{D}^2}\right), \tag{2.10}$$
where $\frac{\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{D}}{\widetilde{C}^2+\widetilde{D}^2} = \widetilde{t_1}$ and $\frac{\widetilde{B}\widetilde{C}-\widetilde{A}\widetilde{D}}{\widetilde{C}^2+\widetilde{D}^2} = \widetilde{t_2}$ are fuzzy real numbers:
$$\widetilde{t_1}(y) = \bigvee_{y=\frac{x_1x_3+x_2x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} \left(\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)\right)$$
$$\widetilde{t_2}(y) = \bigvee_{y=\frac{x_2x_3-x_1x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} \left(\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)\right)$$

Remark. Fuzzy complex number $\widetilde{z} = \widetilde{A} + i\widetilde{B}$ admits a matrix representation, namely:
$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ -\widetilde{B} & \widetilde{A} \end{pmatrix},$$
where $\widetilde{A}, \widetilde{B} \in \mathbb{R}_F$.

3. On arithmetic operation with fuzzy quaternion numbers

In paper (Sida *et al.*, 2016) it was introduced fuzzy quaternion number as well as basic arithmetic operations with fuzzy quaternion numbers.

Definition 3.1. A fuzzy quaternion number is an element of the form
$$\widetilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k, \tag{3.1}$$
where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F$ and $i^2 = j^2 = k^2 = -1$; $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.

We will denote by $\mathbb{H}_F$ - the set of all fuzzy quaternion numbers.

Remark. $\widetilde{A}$ is called the real part of $\widetilde{q}$ and sometimes denoted $\widetilde{A} = Re(\widetilde{q})$, and $\widetilde{B}, \widetilde{C}, \widetilde{D}$, are called imaginary parts of $\widetilde{q}$ and denoted $\widetilde{B} = Im_1(\widetilde{q}), \widetilde{C} = Im_2(\widetilde{q}), \widetilde{D} = Im_3(\widetilde{q})$.

Definition 3.2. If $\widetilde{q_1} = \widetilde{A}_1 + \widetilde{B}_1 i + \widetilde{C}_1 j + \widetilde{D}_1 k$ and $\widetilde{q_2} = \widetilde{A}_2 + \widetilde{B}_2 i + \widetilde{C}_2 j + \widetilde{D}_2 k \in \mathbb{H}_F$ the basic arithmetic operations are defined as follows:

(i)
$$\widetilde{q_1} + \widetilde{q_2} = (\widetilde{A}_1 + \widetilde{A}_2) + (\widetilde{B}_1 + \widetilde{B}_2)i + (\widetilde{C}_1 + \widetilde{C}_2)j + (\widetilde{D}_1 + \widetilde{D}_2)k \tag{3.2}$$

(ii)
$$\widetilde{q_1} - \widetilde{q_2} = (\widetilde{A}_1 - \widetilde{A}_2) + (\widetilde{B}_1 - \widetilde{B}_2)i + (\widetilde{C}_1 - \widetilde{C}_2)j + (\widetilde{D}_1 - \widetilde{D}_2)k \tag{3.3}$$

(iii)
$$\widetilde{q_1} \cdot \widetilde{q_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k. \tag{3.4}$$

where

$$\begin{aligned}
\widetilde{A} &= (\widetilde{A}_1 \cdot \widetilde{A}_2 - \widetilde{B}_1 \cdot \widetilde{B}_2 - \widetilde{C}_1 \cdot \widetilde{C}_2 - \widetilde{D}_1 \cdot \widetilde{D}_2) \\
\widetilde{B} &= (\widetilde{A}_1 \cdot \widetilde{B}_2 + \widetilde{B}_1 \cdot \widetilde{A}_2 + \widetilde{C}_1 \cdot \widetilde{D}_2 - \widetilde{D}_1 \cdot \widetilde{C}_2) \\
\widetilde{C} &= (\widetilde{A}_1 \cdot \widetilde{C}_2 - \widetilde{B}_1 \cdot \widetilde{D}_2 + \widetilde{C}_1 \cdot \widetilde{A}_2 + \widetilde{D}_1 \cdot \widetilde{B}_2) \\
\widetilde{D} &= (\widetilde{A}_1 \cdot \widetilde{D}_2 + \widetilde{B}_1 \cdot \widetilde{C}_2 - \widetilde{C}_1 \cdot \widetilde{B}_2 + \widetilde{D}_1 \cdot \widetilde{A}_2).
\end{aligned}$$

Definition 3.3. If $\widetilde{q_1} = \widetilde{A}_1 + \widetilde{B}_1 i + \widetilde{C}_1 j + \widetilde{D}_1 k$ and $\widetilde{q_2} = \widetilde{A}_2 + \widetilde{B}_2 i + \widetilde{C}_2 j + \widetilde{D}_2 k \in \mathbb{H}_F$, then:

$$\frac{\widetilde{q_1}}{\widetilde{q_2}} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k, \tag{3.5}$$

where

$$\begin{aligned}
\widetilde{A} &= \frac{\widetilde{A}_1 \cdot \widetilde{A}_2 + \widetilde{B}_1 \cdot \widetilde{B}_2 + \widetilde{C}_1 \cdot \widetilde{C}_2 + \widetilde{D}_1 \cdot \widetilde{D}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{B} &= \frac{-\widetilde{A}_1 \cdot \widetilde{B}_2 + \widetilde{B}_1 \cdot \widetilde{A}_2 - \widetilde{C}_1 \cdot \widetilde{D}_2 + \widetilde{D}_1 \cdot \widetilde{C}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{C} &= \frac{-\widetilde{A}_1 \cdot \widetilde{C}_2 + \widetilde{B}_1 \cdot \widetilde{D}_2 + \widetilde{C}_1 \cdot \widetilde{A}_2 - \widetilde{D}_1 \cdot \widetilde{B}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{D} &= \frac{-\widetilde{A}_1 \cdot \widetilde{D}_2 - \widetilde{B}_1 \cdot \widetilde{C}_2 + \widetilde{C}_1 \cdot \widetilde{B}_2 + \widetilde{D}_1 \cdot \widetilde{A}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2}.
\end{aligned}$$

Proposition 3.1. *The expressing of the product $\widetilde{q_1}\widetilde{q_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ can be made as follows:*

(i)

$$\widetilde{A}(y) = \bigvee_{y = x_1x_2 - x_3x_4 - x_5x_6 - x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{A}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{B}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{C}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{D}_2(x_8)\Big)$$

(ii)

$$\widetilde{B}(y) = \bigvee_{y = x_1x_2 + x_3x_4 + x_5x_6 - x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{B}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{A}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{D}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{C}_2(x_8)\Big)$$

(iii)

$$\widetilde{C}(y) = \bigvee_{y=x_1x_2-x_3x_4+x_5x_6+x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{C}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{D}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{A}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{B}_2(x_8)\Big)$$

(iv)

$$\widetilde{D}(y) = \bigvee_{y=x_1x_2+x_3x_4-x_5x_6+x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{D}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{C}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{B}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{A}_2(x_8)\Big).$$

Proposition 3.2. *The expressing of the quotient* $\frac{\widetilde{q}_1}{\widetilde{q}_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ *can be made as follows:*

(i)

$$\widetilde{A}(y) = \bigvee_{y=\frac{x_1x_2+x_3x_4+x_5x_6+x_7x_8}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{A}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{B}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{C}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{D}_2(x_8)\Big)$$

(ii)

$$\widetilde{B}(y) = \bigvee_{y=\frac{-x_1x_4+x_2x_3-x_5x_8+x_6x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{B}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{A}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{D}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{C}_2(x_8)\Big)$$

(iii)

$$\widetilde{C}(y) = \bigvee_{y=\frac{-x_1x_6+x_3x_8+x_2x_5-x_4x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{C}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{D}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{A}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{B}_2(x_8)\Big)$$

(iv)

$$\widetilde{D}(y) = \bigvee_{y=\frac{-x_1x_8-x_3x_6+x_4x_5+x_2x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{D}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{C}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{B}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{A}_2(x_8)\Big).$$

4. Matrix representations of fuzzy quaternion numbers

Just as the quaternion numbers are represented as matrices (Moţ & Popa, 2014), so the fuzzy quaternion numbers have a matrix representations.

The first way in which we can represent the fuzzy quaternion numbers as a matrix is to use 2×2 fuzzy complex matrices and the representation given is one of a family of linearly related representations.

For this, we denote:

$$\begin{aligned} \mathbb{H}_{1F} &= \left\{ Q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \middle| z_1, z_2 \in \mathbb{C}_F \right\} = \\ &= \left\{ Q = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix} \middle| \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F \right\}. \end{aligned} \tag{4.1}$$

Operations of addition and multiplication in $\mathbb{H}_{1F}$ are made according to the rules of addition and multiplication of matrices.

For Q_1 and Q_2 in $\mathbb{H}_{1F}$ we have:

$$\begin{aligned} Q_1 + Q_2 &= \begin{pmatrix} \widetilde{A}_1 + \widetilde{D}_1 i & \widetilde{C}_1 + \widetilde{B}_1 i \\ -\widetilde{C}_1 + \widetilde{B}_1 i & \widetilde{A}_1 - \widetilde{D}_1 i \end{pmatrix} + \begin{pmatrix} \widetilde{A}_2 + \widetilde{D}_2 i & \widetilde{C}_2 + \widetilde{B}_2 i \\ -\widetilde{C}_2 + \widetilde{B}_2 i & \widetilde{A}_2 - \widetilde{D}_2 i \end{pmatrix} = \\ &= \begin{pmatrix} \widetilde{A}_1 + \widetilde{A}_2 + (\widetilde{D}_1 + \widetilde{D}_2)i & \widetilde{C}_1 + \widetilde{C}_2 + (\widetilde{B}_1 + \widetilde{B}_2)i \\ -(\widetilde{C}_1 + \widetilde{C}_2) + (\widetilde{B}_1 + \widetilde{B}_2)i & \widetilde{A}_1 + \widetilde{A}_2 - (\widetilde{D}_1 + \widetilde{D}_2)i \end{pmatrix}, \end{aligned}$$

where $\widetilde{A}_1 + \widetilde{A}_2$, $\widetilde{B}_1 + \widetilde{B}_2$, $\widetilde{C}_1 + \widetilde{C}_2$ and $\widetilde{D}_1 + \widetilde{D}_2$ were defined in previous section. The product of two matrices Q_1 and Q_2 also follows the usual definition for matrix multiplication.

It is easy to verify that the $\mathbb{H}_{1F}$ is closed under the operation of addition and multiplication. Moreover, any matrix of $\mathbb{H}_{1F}$ admits the opposed matrix, namely

$$-Q = \begin{pmatrix} -z_1 & -z_2 \\ \bar{z}_2 & -\bar{z}_1 \end{pmatrix} = \begin{pmatrix} -\widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i \\ \widetilde{C} - \widetilde{B}i & -\widetilde{A} + \widetilde{D}i \end{pmatrix},$$

which belongs to $\mathbb{H}_{1F}$ as well. For any non null matrix of $\mathbb{H}_{1F}$ we have that:

$$\begin{vmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{vmatrix} = \begin{vmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{vmatrix} = \widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2,$$

which is equal to zero, if and only if $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$. It result that any non null matrix of $\mathbb{H}_{1F}$ admits the inverse matrix, namely:

$$Q^{-1} = \frac{1}{\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2} \begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix} = \frac{1}{\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2} \begin{pmatrix} \widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i \\ \widetilde{C} - \widetilde{B}i & \widetilde{A} + \widetilde{D}i \end{pmatrix}$$

Therefore $\mathbb{H}_{1F}$ has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{1F}$ where $\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1$ and $\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{array}{llll} \widetilde{A}_1 = [1,2,4] & \widetilde{B}_1 = [0,1,3] & \widetilde{C}_1 = [1,2,3] & \widetilde{D}_1 = [0,2,4] \\ \widetilde{A}_2 = [2,4,6] & \widetilde{B}_2 = [2,4,5] & \widetilde{C}_2 = [0,1,4] & \widetilde{D}_2 = [1,3,4] \end{array}$$

We have

$$\widetilde{Q}_1\widetilde{Q}_2 = \begin{pmatrix} [-41,-4,24]+[-9,21,55]i & [-18,5,46]+[-10,8,53]i \\ [-46,-5,18]+[-10,8,53]i & [-41,-4,24]+[-55,-21,9]i \end{pmatrix}$$

and

$$\widetilde{Q}_2\widetilde{Q}_1 = \begin{pmatrix} [-41,-4,24]+[-14,7,50]i & [-10,15,54]+[-13,16,50]i \\ [-54,-15,10]+[-13,16,50]i & [-41,-4,24]+[-50,-7,14]i \end{pmatrix}$$

Based on the previous results we obtain the following theorem:

Theorem 4.1. *$\mathbb{H}_{1F}$ has a noncommutative field structure.*

Theorem 4.2. *$\mathbb{H}_F$ and $\mathbb{H}_{1F}$ are isomorphic fields.*

Proof. We consider the mapping $\varphi : \mathbb{H}_F \to \mathbb{H}_{1F}$,

$$\varphi(\widetilde{q}) = \varphi(\widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k) = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix},$$

which is a bijective application and it mantains the operations:

(i) $\varphi(\widetilde{q}_1 + \widetilde{q}_2) = \varphi(\widetilde{q}_1) + \varphi(\widetilde{q}_2),\ \forall \widetilde{q}_1, \widetilde{q}_2 \in \mathbb{H}_F$

(ii) $\varphi(\widetilde{q}_1 \cdot \widetilde{q}_2) = \varphi(\widetilde{q}_1) \cdot \varphi(\widetilde{q}_2),\ \widetilde{q}_1, \widetilde{q}_2 \in \mathbb{H}_F$

Hence it is a isomorphism of fields. Thus $\mathbb{H}_F \simeq \mathbb{H}_{1F}$. □

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{1F}$ allows us to state that each fuzzy quaternion number of $\mathbb{H}_F$ admits a matrix representation under the form of the elements of $\mathbb{H}_{1F}$, namely $\begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix}$.

Proposition 4.1. *Any element of $\mathbb{H}_{1F}$ admits:*

$$Q = \widetilde{A}\boldsymbol{1} + \widetilde{B}I + \widetilde{C}J + \widetilde{D}k, \tag{4.2}$$

where $\boldsymbol{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ *represent the matrix quaternion units.*

Proof. It is verified by direct calculus:

$$Q = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix} = \widetilde{A}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \widetilde{B}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \widetilde{C}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \widetilde{D}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \widetilde{A}\mathbf{1} + \widetilde{B}I + \widetilde{C}J + \widetilde{D}k.$$

□

Remark. The matrix quaternion units $I,\ J,\ K$ can be written with the help of Pauli matrix $\sigma_x,\ \sigma_y,\ \sigma_z$, namely:

$$I = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\sigma_x,\ J = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\sigma_y,\ K = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\sigma_z.$$

Remark. If constraining any two of $\widetilde{B}, \widetilde{C}$ and $\widetilde{D}$ to zero it results a representation of fuzzy complex numbers:

i) If $\widetilde{B} = \widetilde{C} = \widetilde{0}$, then it results a diagonal fuzzy complex matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix}\widetilde{A} + \widetilde{D}i & \widetilde{0}\\ \widetilde{0} & \widetilde{A} - \widetilde{D}i\end{pmatrix}.$$

ii) If $\widetilde{B} = \widetilde{D} = \widetilde{0}$, then it results a fuzzy real matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix}\widetilde{A} & \widetilde{C}\\ -\widetilde{C} & \widetilde{A}\end{pmatrix}.$$

Proposition 4.2. *The conjugate of a fuzzy quaternion corresponds to the Hermitian transpose (conjugate transpose) of the matrix*

$$Q^* = (\overline{Q})^T = \overline{Q^T} = \begin{pmatrix}\widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i\\ \widetilde{C} - \widetilde{B}i & \widetilde{A} + \widetilde{D}i\end{pmatrix},$$

where Q^T denotes the transpose of Q and $\overline{Q}$ denotes the matrix with complex conjugated entries.

The second way in which we can represent the fuzzy quaternion numbers as a matrix is to use 4×4 fuzzy real matrices and the representation given is one of a family of linearly related representations.

In order to obtain such a representation we consider:

$$\mathbb{H}_{2F} = \left\{ Q_1 = \begin{pmatrix}\widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D}\\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C}\\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B}\\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A}\end{pmatrix} \middle| \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F \right\}.$$

Operations of addition and multiplication in $\mathbb{H}_{2F}$ is made according to the rules of addition and multiplication of matrices.

It is easy to verify that the set of matrices $\mathbb{H}_{2F}$ is closed under the operation of addition and multiplication of matrices. Moreover, any matrix of $\mathbb{H}_{2F}$, admits the opposed matrix and any non null matrix of $\mathbb{H}_{2F}$ admits its inverse. Indeed, as

$$\begin{vmatrix}\widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D}\\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C}\\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B}\\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A}\end{vmatrix} = (\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2)^2,$$

is equal to zero, if and only if $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$, it results that for any non null matrix of $\mathbb{H}_{2F}$, there exists the inverse matrix:

$$Q_1^{-1} = \frac{1}{(\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2)^2} \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & -\widetilde{D} & -\widetilde{C} \\ -\widetilde{C} & -\widetilde{D} & \widetilde{A} & -\widetilde{B} \\ -\widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}.$$

Thus $\mathbb{H}_{2F}$ has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{2F}$ where $\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1$ and $\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{aligned} &\widetilde{A}_1 = [0,2,5], && \widetilde{B}_1 = [1,3,5], && \widetilde{C}_1 = [0,1,4], && \widetilde{D}_1 = [0,2,5] \\ &\widetilde{A}_2 = [1,2,3], && \widetilde{B}_2 = [0,3,4], && \widetilde{C}_2 = [2,4,5], && \widetilde{D}_2 = [0,2,4]. \end{aligned}$$

We have

$$\widetilde{Q}_1\widetilde{Q}_2 = \begin{pmatrix} [-60,-13,15] & [-51,-6,24] & [-57,-10,20] & [-60,-17,14] \\ [-24,6,51] & [-60,-13,15] & [-60,-17,14] & [-20,10,57] \\ [-20,10,57] & [-14,17,60] & [-60,-13,15] & [-51,-6,24] \\ [-14,17,60] & [-57,-10,20] & [-24,6,51] & [-60,-13,15] \end{pmatrix}$$

and

$$\widetilde{Q}_2\widetilde{Q}_1 = \begin{pmatrix} [-60,-13,15] & [-60,-18,15] & [-57,-10,20] & [-49,1,25] \\ [-15,18,60] & [-60,-13,15] & [-49,1,25] & [-20,10,57] \\ [-20,10,57] & [-25,-1,49] & [-60,-13,15] & [-60,-18,15] \\ [-25,-1,49] & [-57,-10,20] & [-15,18,60] & [-60,-13,15] \end{pmatrix}.$$

Based on the previous results we obtain the following theorem:

Theorem 4.3. *$\mathbb{H}_{2F}$ has a noncommutative field structure.*

Theorem 4.4. *$\mathbb{H}_F$ and $\mathbb{H}_{2F}$ are isomorphic fields.*

Proof. We consider the mapping $\psi : \mathbb{H}_F \to \mathbb{H}_{2F}$,

$$\psi(\widetilde{q}) = \varphi(\widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k) = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix}.$$

We note that ψ is bijective and preserves the operations - it follows immediately by direct calculation, thus it is a isomorphism of fields. Therefore $\mathbb{H}_F \simeq \mathbb{H}_{2F}$. □

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{2F}$ allows us to stat that each fuzzy quaternion number of $\mathbb{H}_F$ admits a matrix representation under the form of the elements of $\mathbb{H}_{2F}$.

Remark. Any element of $\mathbb{H}_{2F}$ can be written

$$Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A}\begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B}\begin{pmatrix} \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C}\begin{pmatrix} \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. Similarly, a fuzzy quaternion number $\widetilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ can be represented as

$$Q_2 = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ -\widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ -\widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A}\begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B}\begin{pmatrix} \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. The 48 possible matrix representation of fuzzy quaternion numbers are in fact the matrices whose transpose is its negation (skew-symmetric matrices, i.e. $-Q = Q^T$).

Proposition 4.3. *In this representations, the conjugate of a fuzzy quaternion number corresponds to the transpose of the matrix* Q_n, $n = \overline{1, 48}$

$$\overline{q} = \widetilde{A} - \widetilde{B}i - \widetilde{C}j - \widetilde{D}k = (Q_n)^T.$$

For example, for the representation Q_1 of a fuzzy quaternion number, we have

$$\overline{q} = (Q_1)^T = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & \widetilde{D} & -\widetilde{C} \\ -\widetilde{C} & -\widetilde{D} & \widetilde{A} & \widetilde{B} \\ -\widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}.$$

Or, for second representation Q_2,

$$\overline{q} = (Q_2)^T = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & \widetilde{D} & -\widetilde{C} \\ \widetilde{C} & -\widetilde{D} & \widetilde{A} & \widetilde{B} \\ \widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}$$

Remark. If $\widetilde{C} = \widetilde{D} = \widetilde{0}$ then $\widetilde{q} = \widetilde{A} + \widetilde{B}i$ and it results the representation of fuzzy complex numbers as diagonal matrices with two 2×2 blocks.

For example, for the representation Q_1 when $\widetilde{C} = \widetilde{D} = \widetilde{0}$ it results

$$Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & \widetilde{0} & \widetilde{0} \\ \widetilde{B} & \widetilde{A} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{A} & -\widetilde{B} \\ \widetilde{0} & \widetilde{0} & \widetilde{B} & \widetilde{A} \end{pmatrix}$$

and for Q_2 we obtain

$$Q_2 = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{0} & \widetilde{0} \\ -\widetilde{B} & \widetilde{A} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{A} & -\widetilde{B} \\ \widetilde{0} & \widetilde{0} & \widetilde{B} & \widetilde{A} \end{pmatrix}.$$

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