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New Subclasses of Analytic and Bi-Univalent Functions Involving a New Integral Operator Defined by Polylogarithm Function

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Abstract

In the present investigation, we introduce two new subclasses of the function class σ of bi-univalent functions in the open unit disc. Also we find coefficient estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class and several related classes are also considered and connections to earlier known results are made.

Keywords: Analytic functions, univalent functions, bi-univalent functions, coefficient bounds.
2010 MSC: 30C45.

1. Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Further, the class of all functions in A which are univalent in U is denoted by the symbol S.
The Koebe one-quarter theorem (Duren, 1983) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\,, \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w\,, \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, see (Srivastava *et al.*, 2010). The concept of bi-univalent function class was firstly studied by Lewin (Lewin, 1967) and obtained that the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie (Brannan & Clunie, 1980) conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu (Netanyahu, 1969) showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha (Brannan & Taha, 1986) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0 < \alpha \leq 1)$ respectively. The classes $\delta_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\delta^{\star}(\alpha)$ and $K(\alpha)$, were also introduced similarly. For each of the function classes $\delta_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$, non-sharp estimates on the initial coefficients were found by them. In recent years, bounds for various subclasses of bi-univalent functions were investigated by many authors ((Frasin & Aouf, 2011), (Srivastava *et al.*, 2010), (Xu *et al.*, 2012*b*)). For each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$, the problem of determining coefficient estimate is still an open problem. In the year 2010, the following subclasses of the bi-univalent function class Σ was introduced by Srivastava et al. (Srivastava *et al.*, 2010) and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ was obtained.

Definition 1.1. (Srivastava *et al.*, 2010) A function $f(z)$ given by the TaylorMaclaurin series expansion (1.1) is said to be in the class $\mathcal{H}_{\sigma}^{\alpha}$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|\arg\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(g'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

where the function g is given by

$$f^{-1}(w) = g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Theorem 1.1. *(Srivastava* et al.*, 2010) Let the function $f(z)$ given by (1.1) be in the class* $\mathrm{H}_{\Sigma}^{\alpha}(0 < \alpha \leq 1)$. *Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha + 2}} \qquad \text{and} \qquad |a_3| \leq \frac{\alpha(3\alpha + 2)}{3}.$$

Definition 1.2. (Srivastava *et al.*, 2010) A function $f(z)$ given by (1.1) is said to be in the class $H_{\Sigma}^{\beta}(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|Re\left(f'(z)\right)\right| > \beta \quad (0 \leq \beta < 1,\ z \in U)$$

and

$$\left|Re\left(g'(w)\right)\right| > \beta \quad (0 \le \beta < 1,\ w \in U)$$

where the function g is given by $f^{-1}(w) = g(w)$.

Theorem 1.2. *(Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class $\mathrm{H}_{\Sigma}^{\beta}\,(0 \le \beta < 1)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3}} \qquad and \qquad |a_3| \le \frac{(1-\beta)(5-3\beta)}{3}.$$

Here, in our present sequel to some of the aforecited works (especially [15]), the following subclass of the analytic function class A is introduced. Also, by using the method of (Srivastava *et al.*, 2010), (Frasin & Aouf, 2011), (Xu *et al.*, 2012*b*) and (Xu *et al.*, 2012*a*) different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Porwal and Darus and get more accurate estimates than that given in (Porwal & Darus, 2013). For the functions $f \in A$ given by (1.1) and $g \in A,\ g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their Hadamard product or convolution (Duren, 1983) is defined by the power series

$$(f * g)(z) = z + \sum_{k=2} a_k b_k z^k.$$

For $f(z) \in A$, Al-Shaqsi (AL-Shaqsi, 2014) defined the following integral operator:

$$\begin{aligned} L_c^{\delta} f(z) &= (1+c)^{\delta} \Phi_{\delta}(c; z) * f(z) \\ &= -\frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} \log\left(\frac{1}{t}\right)^{\delta-1} f(zt)\, dt \end{aligned} \tag{1.2}$$

$(c > 0, \delta > 1, z \in U)$

where Γ stands for the usual gamma function, $\Phi_{\delta}(c; z)$ is the well known generalization of the Riemann-zeta and polylogarithm functions, or the δth polylogarithm function, given by

$$\Phi_{\delta}(c; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^{\delta}}$$

where any term without $k + c = 0$ (see (Lerch, 1887) and (Bateman, 1953)(sections 1.10 and 1.12)). Also, $\Phi_{-1}(0; z) = \frac{z}{(1-z)^2}$ is Koebe function. One can find more details about polylogarithms in theory of univalent functions in the study of Ponnusamy and Sabapathy (Ponnusamy & Sabapathy, 1996).

We also state that the operator $\mathcal{L}_c^{\delta} f(z)$ given by the relation (1.2) can be expressed by the series expansions as follows:

$$\mathcal{L}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^{\delta} a_k z^k.$$

First of all, we present the following lemma to prove our main result

Lemma 1.1. *(Pommerenke & Jensen, 1975) If $h \in \mathbb{P}$ then $|\, c_k \,| \leq 2$ for each k, where $\mathbb{P}$ is the family of all functions h analytic in E for which $Re(h(z)) > 0$, then*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots .$$

2. Coefficient Estimates for the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$

Definition 2.1. The class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$ of the functions $f(z)$ determined by the equality (1.1) consists of those functions $f(z)$ that satisfy the following conditions:
$f \in \sigma$,

$$Re\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z}\right) > \beta \tag{2.1}$$

where $0 \leq \beta < 1, \lambda \geq 1, c > 0, Re\delta > 1, z \in U$ and

$$Re\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w}\right) > \beta. \tag{2.2}$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function introduced and studied by Al-Shaqsi and the function g is given by $g(w) = f^{-1}(w)$.

Remark. If we let $c = 0$ and $\delta = -n$, for $n \in \mathbb{N} \cup \{0\}$, then we obtain

$$\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c) = H_{\Sigma}(n, \beta, \lambda)$$

studied by Porwal and Darus (Porwal & Darus, 2013). This class contains the function $f \in \Sigma$ satisfying

$$Re\left(\frac{(1-\lambda)\mathcal{D}^{n} f(z) + \lambda \mathcal{D}^{n+1} f(z)}{z}\right) > \beta$$

and

$$Re\left(\frac{(1-\lambda)\mathcal{D}^{n} g(w) + \lambda \mathcal{D}^{n+1} g(w)}{w}\right) > \beta.$$

where $\mathcal{D}^n$ stands for Salagean derivative introduced by Sâlâgean (Salagean, 1983).

The class $\mathcal{B}_{\sigma}^{-n}(\beta, \lambda, 0)$ includes many earlier classes, which are mentioned below:

1. If we let $n = 0$, then we have

$$\mathcal{B}_{\sigma}^{-n}(\beta, \lambda, 0) = H_{\Sigma}^{\lambda}(\beta)$$

studied by Frasin and Aouf (Frasin & Aouf, 2011). This class contains the functions $f \in \Sigma$ satisfying

$$Re\left(\frac{(1-\lambda) f(z)}{z} + \lambda f'(z)\right) > \beta$$

and

$$Re\left(\frac{(1-\lambda) g(w)}{w} + \lambda g'(w)\right) > \beta.$$

2. If we let $n = 0$ and $\lambda = 1$, then we have

$$\mathcal{B}_{\sigma}^{-n}(\beta, 1, 0) = H_{\Sigma}(\beta)$$

studied by Srivastava et al.(Srivastava *et al.*, 2010). This class contains the functions $f \in \Sigma$ satisfying

$$Re\,(f'(z)) > \beta$$

and

$$Re\,(g'(w)) > \beta.$$

The next theorem gives the estimate on coefficient of the function in the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$ given in Definition 2.1.

Theorem 2.1. *Let the function $f(z)$ given by equation (1.1) be in the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}} \tag{2.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(1 + \frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}} \tag{2.4}$$

where $0 \leq \beta < 1$ and $\lambda \geq 1$.

Proof. Let $f \in \mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c), \lambda \geq 1$ and $0 \leq \beta < 1$. Using argument inequalities in (2.1) and (2.2), we can state their forms as follows:

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta} f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z} = \beta + (1-\beta)p(z) \quad (z \in U) \tag{2.5}$$

and

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta} g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w} = \beta + (1-\beta)q(w) \quad (w \in U) \tag{2.6}$$

where $p(z)$ and $q(w)$ given by the equalities

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \tag{2.8}$$

satisfy the inequalities $Re(p(z)) > 0$ and $Re(q(w)) > 0$ respectively. Equating coefficients (2.5) and (2.6) yields

$$\left(1 + \frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta} a_2 = (1-\beta)p_1, \tag{2.9}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} a_3 = (1-\beta)p_2, \tag{2.10}$$

and

$$-\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta} a_2 = (1-\beta)q_1 \tag{2.11}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} (2a_2^2 - a_3) = (1-\beta)q_2. \tag{2.12}$$

From (2.9) and (2.11) , we have

$$p_1 = -q_1 \tag{2.13}$$

and

$$2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta} a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{2.14}$$

Also, adding (2.10) to (2.12) , we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} a_2^2 = (1-\beta)(p_2+q_2). \tag{2.15}$$

Applying Lemma 1.1 for equality (2.15), we have

$$|a_2|^2 \leq \frac{(1-\beta)(|p_2|+|q_2|)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}} \leq \frac{2(1-\beta)}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}$$

This gives the bound on $|a_2|$ as asserted in (2.3).

Next, to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} (a_3 - a_2^2) = (1-\beta)(p_2-q_2) \tag{2.16}$$

which, upon substitution of value of a_2^2 from (2.14) yields

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{(1-\beta)(p_2-q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}.$$

Applying the lemma 1 for the coefficients p_1, q_1, p_2 and q_2, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}.$$

□

Remark. Choosing $c = 0$ in Theorem 2.1, we have the following corollaries:

1. If we let $\delta = -n, (n \in \mathbb{N} \cup \{0\})$, then we obtain the following:

Corollary 2.1. *(Porwal & Darus, 2013) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}(n,\beta,\lambda)$,*

$0 \le \beta < 1, \lambda \ge 1, n \in \mathbb{N}_0$. *Then,*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(1-\lambda)\,3^n + \lambda 3^{n+1}}}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{\left[(1-\lambda)\,2^n + \lambda 2^{n+1}\right]^2} + \frac{2(1-\beta)}{(1-\lambda)\,3^n + \lambda 3^{n+1}}.$$

2. Especially, choosing $n = 0$ in Corollary 2.1, we have the following result:

Corollary 2.2. *(Frasin & Aouf, 2011) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}^{\lambda}(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{1+2\lambda}}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(1+2\lambda)}.$$

Remark. The estimates for $|a_2|$ and $|a_3|$ of Corollary 2.2 and Corollary 2.3 show that Theorem 2.1 coincides with the the estimates obtained by Frasin and Aouf (Frasin & Aouf, 2011).

3. If we choose $n = 0$ and $\lambda = 1$, then we obtain the following corollary:

Corollary 2.3. *(Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3}}$$

and

$$|a_3| \le \frac{(1-\beta)(5-3\beta)}{3}.$$

3. Coefficient Estimates for the class $\mathcal{H}_{\sigma}^{\delta}(\alpha, \lambda, c)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_{\sigma}^{\delta}(\alpha, \lambda, c)$ if the following conditions are satisfied:

$$f \in \sigma, \quad \left|\arg\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ \lambda \ge 1,\ z \in U) \tag{3.1}$$

and

$$\left|\arg\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ \lambda \ge 1,\ w \in U) \tag{3.2}$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function and the function (by Al-Shaqsi) $g(w) = f^{-1}(w)$.

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_\sigma^\delta(\alpha,\lambda,c)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\alpha-(\alpha-1)\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}}} \tag{3.3}$$

and

$$|a_3| \le \frac{4\alpha^2}{\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}}+\frac{2\alpha}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}, where 0\le\beta<1 and \lambda\ge 1. \tag{3.4}$$

Proof. Let $f\in\mathcal{H}_\sigma^\delta(\alpha,\lambda,c)$, $\lambda\ge 1$ and $0<\alpha\le 1$. We can write the argument inequalities in (3.1) and (3.2) as follows:

$$\frac{(1-\lambda)\mathcal{L}_c^\delta f(z)+\lambda\mathcal{L}_c^{\delta-1}f(z)}{z}=[p(z)]^\alpha, \quad z\in U \tag{3.5}$$

$$\frac{(1-\lambda)\mathcal{L}_c^\delta g(w)+\lambda\mathcal{L}_c^{\delta-1}g(w)}{w}=[q(w)]^\alpha, \quad w\in U \tag{3.6}$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8) and satisfy the inequalities $Re(p(z))>0$ and $Re(q(w))>0$ respectively. Now, equating the coefficients of (3.5) and (3.6), we have

$$\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta}a_2=\alpha p_1, \tag{3.7}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{3.8}$$

$$-\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta}a_2=\alpha q_1, \tag{3.9}$$

and

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}(2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2, \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1=-q_1 \tag{3.11}$$

and

$$2\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}a_2^2=\alpha^2(p_1^2+q_1^2) \tag{3.12}$$

Also from (3.8) and (3.10), we obtain

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_2^2=\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2). \tag{3.13}$$

By using the relation (3.12) in (3.13), we find that

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_2^2=\alpha(p_2+q_2)+\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}a_2^2.$$

Thus we get

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\alpha-(\alpha-1)\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.14}$$

Then, applying Lemma1.1 for the aforementioned equality, we get desired estimate on $|a_2|$ as asserted in (3.3). Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\left(a_3-a_2^2\right)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{3.15}$$

Also from (3.11) , (3.12) and (3.15) we find that

$$a_3 = \frac{\alpha(p_2-q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}+\frac{\alpha^2(p_1^2+q_1^2)}{2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.16}$$

By applying the Lemma1 for the equality (3.16), we obtain desired estimate and this complats the proof of the theorem. □

Remark. If we let $c = 0$ in Theorem 3.1 and

1. $\delta = -n$, we obtain the following corollary:

Corollary 3.1. *(Porwal & Darus, 2013) Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(n,\alpha,\lambda), 0<\alpha\leq 1, \lambda\geq 1, n\in \mathbb{N}_0$. Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4^n(\lambda+1)^2+\alpha\left(2.3^n(1+2\lambda)-4^n(\lambda+1)^2\right)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}+\frac{2\alpha}{(1-\lambda)3^n+\lambda 3^{n+1}}.$$

2. Choosing $\delta = 0$, we obtain the following corollary:

Corollary 3.2. *(Frasin & Aouf, 2011). Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(\lambda,\beta), 0<\alpha\leq 1, \lambda\geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha\left(1+2\lambda-\lambda^2\right)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2}+\frac{2\alpha}{2\lambda+1}.$$

3. Also, if we choose $\lambda = 1$, we have the following corollary:

Corollary 3.3. *(Srivastava et al., 2010). Let the function $f(z)$ given by (1.1) be in the class $H^{\alpha}_{\Sigma}, 0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha \sqrt{\frac{2}{2+\alpha}}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

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References

AL-Shaqsi, K. (2014). Strong differential subordinations obtained with new integral operator defined by polylogarithm function. *International Journal of Mathematics and Mathematical Sciences* **Vol2014**(Art ID 260198), 6 pages.

Bateman, H. (1953). *Higher transendental functions, vol. 1*. Mc Graw-Hill, New York, NY, USA.

Brannan, D. A. and J. G. Clunie (1980). Aspects of comtemporary complex analysis. In: *Proceedings of the NATO Advanced Study Instute Held at University of Durham: July 1-20, 1979. New York: Academic Press.*

Brannan, D.A. and T.S. Taha (1986). On some classes of bi-univalent functions. *Studia Univ. Babeş-Bolyai Math.* **31**(2), 70–77.

Duren, P. L. (1983). *Univalent Functions*. Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 259.

Frasin, B.A. and M.K. Aouf (2011). New subclasses of bi-univalent functions. *Applied Mathematics Letters* **24**(9), 1569 – 1573.

Lerch, M. (1887). Note sur la fonction $k(w, x, s) = \sum_{k=0}^{\infty} e^{2k\pi ix}$. *Acta Mathematica* **11**(1-4), 19–24.

Lewin, M. (1967). On a coefficient problem for bi-univalent functions. *Proceedings of the American Mathematical Society* **18**(1), 63–68.

Netanyahu, E. (1969). The minimal distance of the image boundary from the orijin and the second coefficient of a univalent function in $|z| < 1$,. *Archive for Rational Mechanics and Analysis* **32**(2), 100–112.

Pommerenke, Christian. and Gerd. Jensen (1975). *Univalent functions / Christian Pommerenke, with a chapter on quadratic differentials by Gerd Jensen*. Vandenhoeck und Ruprecht Gottingen.

Ponnusamy, S. and S. Sabapathy (1996). Polylogarithms in the theory of univalent functions. *Results in Mathematics* **30**(1), 136–150.

Porwal, Saurabh and M. Darus (2013). On a new subclass of bi-univalent functions. *Journal of the Egyptian Mathematical Society* **21**(3), 190 – 193.

Salagean, Grigore Stefan (1983). *Subclasses of univalent functions*. pp. 362–372. Springer Berlin Heidelberg. Berlin, Heidelberg.

Srivastava, H.M., A.K. Mishra and P. Gochhayat (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters* **23**(10), 1188 – 1192.

Xu, Qing-Hua, Hai-Gen Xiao and H.M. Srivastava (2012*a*). A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. *Applied Mathematics and Computation* **218**(23), 11461 – 11465.

Xu, Qing-Hua, Ying-Chun Gui and H.M. Srivastava (2012*b*). Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Applied Mathematics Letters* **25**(6), 990 – 994.