Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (2) (2017) 51 – 62

On a Subclass of Harmonic Univalent Functions Based on Subordination

Serkan Çakmak[a], Sibel Yalçın[a], Şahsene Altınkaya[a,*]

[a]*Department of Mathematics, Faculty of Arts and Science, Uludag University, 16059, Bursa, Turkey*

Abstract

In this paper, we introduce a new class of harmonic univalent functions defined by subordination with a linear operator. Certain properties of this class are discussed.

Keywords: Harmonic functions, univalent functions, Hadamard product, modified generalized Sălăgean operator, subordination and modified multiplier transformation.
2010 MSC: 30C45, 30C80.

1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small (Clunie & Sheil-Small, 1984)). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k. \tag{1.1}$$

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\overline{z}}(0) = 0$.

*Corresponding author
Email addresses: serkan.cakmak64@gmail.com (Serkan Çakmak), syalcin@uludag.edu.tr (Sibel Yalçın), sahsene@uludag.edu.tr (Şahsene Altınkaya)

In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Also note that SH reduces to the class S of normalized analytic univalent functions in U, if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^n ($n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) of f was introduced by Sălăgean (Sălăgean, 1983). For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. (Jahangiri *et al.*, 2002) defined the modified Sălăgean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k.$$

Next, for functions $f \in A$, Al-Oboudi (Al-Oboudi, 2004) defined multiplier transformations. For $n \in \mathbb{N}_0$, $\lambda \geq 1$ and $f \in SH^0$ of the form (1.1), Yaşar and Yalçın (Yaşar & Yalcin, 2012) defined the modified Al-Oboudi operator $D_\lambda^n : SH^0 \to SH^0$ by

$$D_\lambda^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D_\lambda^1 f(z) = (1-\lambda) D^0 f(z) + \lambda D^1 f(z), \quad \lambda \geq 1, \tag{1.2}$$

$$D_\lambda^n f(z) = D_\lambda^1 \left(D_\lambda^{n-1} f(z) \right). \tag{1.3}$$

If f is given by (1.1), then from (1.2) and (1.3) we see that (see (Yaşar & Yalcin, 2012))

$$D_{1,\lambda}^n h(z) = z + \sum_{k=2}^{\infty} [\lambda (k-1) + 1]^n a_k z^k,$$

$$D_{2,\lambda}^n g(z) = \sum_{k=2}^{\infty} [\lambda (k+1) - 1]^n b_k z^k,$$

$$D_\lambda^n f(z) = D_{1,\lambda}^n h(z) + (-1)^n \overline{D_{2,\lambda}^n g(z)}$$

or

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [\lambda (k-1) + 1]^n a_k z^k + (-1)^n \sum_{k=2}^{\infty} [\lambda (k+1) - 1]^n \overline{b_k z^k}. \tag{1.4}$$

When $\lambda = 1$, we get modified Sălăgean differential operator (Jahangiri *et al.*, 2002). If we take the co-analytic part of $f = h + g$ of the form (1.1) is identically zero, $D_\lambda^n f$ reduces to the Al-Oboudi operator (Al-Oboudi, 2004).

The Hadamard product (or convolution) of functions $f_1$and f_2 of the form

$$f_t(z) = z + \sum_{k=2}^{\infty} a_{t,k} z^k + \sum_{k=2}^{\infty} \overline{b_{t,k} z^k} \quad (z \in U, t = \{1, 2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k}a_{2,k}z^k + \sum_{k=2}^{\infty} \overline{b_{1,k}b_{2,k}z^k} \quad (z \in U)$$

Also if f is given by (1.1), then we have

$$\begin{aligned} D_\lambda^n f(z) \quad &: \quad = f(z) * \underbrace{\left(\phi_1(z) + \overline{\phi_2(z)}\right) * \ldots * \left(\phi_1(z) + \overline{\phi_2(z)}\right)}_{n \text{ times}}, \\ &= \quad h(z) * \underbrace{\phi_1(z) * \ldots * \phi_1(z)}_{n \text{ times}} + \overline{g(z)} * \underbrace{\overline{\phi_2(z)} * \ldots * \overline{\phi_2(z)}}_{n \text{ times}}. \end{aligned}$$

where

$$\phi_1(z) = \frac{(\lambda - 1)z^2 + z}{(1 - z)^2}, \quad \phi_2(z) = \frac{(\lambda - 1)z^2 + (1 - 2\lambda)z}{(1 - z)^2}.$$

We say that a function $f : U \to \mathbb{C}$ is subordinate to a function $g : U \to \mathbb{C}$, and write $f(z) \prec g(z)$, if there exists a complex valued function w which maps U into itself with $w(0) = 0$, such that

$$f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Denote by $SH^0(\lambda, n, A, B)$ the subclass of SH^0 consisting of functions f of the form (1.1) that satisfy the condition

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -B \le A < B \le 1 \tag{1.5}$$

where $D_\lambda^n f(z)$ is defined by (1.4).

By suitably specializing the parameters, the classes $SH^0(\lambda, n, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH^0(1, \lambda, A, B) = H_\lambda(A, B)$, $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (Dziok *et al.* (2016)),

(ii) $SH^0(1, 1, A, B) = S_H^*(A, B) \cap SH^0$ ((Dziok, 2015*a*)),

(iii) $SH^0(\lambda, n, 2\alpha - 1, 1) = SH(\lambda, n, \alpha) \cap SH^0$ (Yaşar & Yalcin, 2012),

(iv) $SH^0(1, n, 2\alpha - 1, 1) = H^0(n, \alpha)$ (Jahangiri *et al.*, 2002),

(v) $SH^0(1, 0, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ((Jahangiri, 1999), (Silverman, 1998), (Silverman & Silvia, 1999)),

(vi) $SH^0(1, 1, 2\alpha - 1, 1) = S_{H^0}^c(\alpha)$ ((Jahangiri, 1999)),

(vii) $SH^0(\lambda, n, 2\alpha - 1, 1) = \overline{SH}(\lambda, 1 - \lambda, n, \alpha)$ ((Bayram & Yalcin, 2017)).

Making use of the techniques and methodology used by Dziok (see (Dziok, 2015*a*), (Dziok, 2015*b*)), Dziok et al. (Dziok *et al.*, 2016), in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $SH^0(\lambda, n, A, B)$.

2. Main Results

First, we provides a necessary and sufficient convolution condition for the harmonic functions in $SH^0(\lambda, n, A, B)$.

Theorem 2.1. *For $z \in U\backslash\{0\}$,let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if*

$$D_\lambda^n f(z) * \varphi(z;\zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1),$$

where

$$\begin{aligned}\varphi(z;\zeta) &= \frac{[(A-B)\zeta + \lambda(1+B\zeta)]z^2 + (B-A)\zeta z}{(1-z)^2} \\ &\quad -(-1)^n \frac{[-\lambda(1+B\zeta) + (B-A)\zeta]\overline{z}^2 + [2\lambda(1+B\zeta) + (A-B)\zeta]\overline{z}}{(1-\overline{z})^2}.\end{aligned}$$

Proof. Let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if the condition (1.5) holds or equivalently

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1). \tag{2.1}$$

Now for

$$D_\lambda^n f(z) = D_\lambda^n f(z) * \left(\frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}}\right),$$

and

$$D_\lambda^{n+1} f(z) = D_\lambda^n f(z) * \left(\phi_1(z) + \overline{\phi_2(z)}\right),$$

the inequality (2.1) yields

$$\begin{aligned}&(1+B\zeta) D_\lambda^{n+1} f(z) - (1+A\zeta) D_\lambda^n f(z) \\ &= D_\lambda^n f(z) * \left\{\frac{(1+B\zeta)\left[(\lambda-1)z^2 + z\right]}{(1-z)^2} + \frac{(1+B\zeta)\left[(\lambda-1)\overline{z}^2 + (1-2\lambda)\overline{z}\right]}{(1-\overline{z})^2}\right\} \\ &\quad -D_\lambda^n f(z) * \left\{\frac{(1+A\zeta)z}{1-z} + \frac{(1+A\zeta)\overline{z}}{1-\overline{z}}\right\} \\ &= D_\lambda^n f(z) * \left\{\frac{[(A-B)\zeta + \lambda(1+B\zeta)]z^2 + (B-A)\zeta z}{(1-z)^2}\right. \\ &\quad \left. -\frac{[2-\lambda(1+B\zeta) + (B+A)\zeta]\overline{z}^2 + [-2+2\lambda(1+B\zeta) + (B-A)\zeta]\overline{z}}{(1-\overline{z})^2}\right\} \\ &= D_\lambda^n f(z) * \varphi(z;\zeta) \neq 0.\end{aligned}$$

□

A sufficient coefficient for the functions in $f \in SH^0(\lambda, n, A, B)$ is provided in the following.

Theorem 2.2. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1).Then $f \in SH^0(\lambda, n, A, B)$, if*

$$\sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \leq B - A \tag{2.2}$$

where

$$M_k = [(k-1)\lambda + 1]^n [\lambda (k-1)(B+1) + B - A] \tag{2.3}$$

and

$$N_k = [(k+1)\lambda - 1]^n [\lambda (k+1)(B+1) + A - B]. \tag{2.4}$$

Proof. It is easy to see that the theorem is true for $f(z) = z$. So, we assume that $a_k \neq 0$ or $b_k \neq 0$ for $k \geq 2$. Since $M_k \geq k(B-A)$ and $N_k \geq k(B-A)$ by (2.2), we obtain

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} - \sum_{k=2}^{\infty} k |b_k| |z|^{k-1} \\ &\geq 1 - |z| \sum_{k=2}^{\infty} (k |a_k| + k |b_k|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Therefore f is sense preserving and locally univalent in U. For the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^k - z_2^k}{z_1 - z_2} \right| = \left| \sum_{m=1}^{k} z_1^{m-1} z_2^{k-m} \right| \leq \sum_{m=1}^{k} \left| z_1^{m-1} \right| \left| z_2^{k-m} \right| < k, \quad k \geq 2.$$

Hence

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=2}^{\infty} b_k \left(z_1^k - z_2^k \right)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k \right)} \right| \\ &> 1 - \frac{\sum_{k=2}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \geq 1 - \frac{\sum_{k=2}^{\infty} \frac{N_k}{B-A} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{M_k}{B-A} |a_k|} \geq 0 \end{aligned}$$

which proves univalence.

On the other hand, $f \in SH^0(\lambda, n, A, B)$ if and only if there exists a complex valued function w; $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$ such that

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

or equivalently

$$\left| \frac{D_\lambda^{n+1} f(z) - D_\lambda^n f(z)}{BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z)} \right| < 1, \quad (z \in U). \tag{2.5}$$

The above inequality (2.5) holds, since for $|z| = r$ $(0 < r < 1)$ we obtain

$$\begin{aligned}
&\left|D_\lambda^{n+1} f(z) - D_\lambda^n f(z)\right| - \left|BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z)\right| \\
= &\left| \sum_{k=2}^{\infty} (k-1)\lambda\left[(k-1)\lambda + 1\right]^n a_k z^k - (-1)^n \sum_{k=2}^{\infty} (k+1)\lambda\left[(k+1)\lambda - 1\right]^n \overline{b_k z^k} \right| \\
&- \left| (B-A)z + \sum_{k=2}^{\infty} \left[(k-1)\lambda B + B - A\right]\left[(k-1)\lambda + 1\right]^n a_k z^k \right. \\
&\left. -(-1)^n \sum_{k=2}^{\infty} \left[(k+1)\lambda B + A - B\right]\left[(k+1)\lambda - 1\right]^n \overline{b_k z^k} \right| \\
\leq &\sum_{k=2}^{\infty} (k-1)\lambda\left[(k-1)\lambda + 1\right]^n |a_k| r^k + \sum_{k=2}^{\infty} (k+1)\lambda\left[(k+1)\lambda - 1\right]^n |b_k| r^k - (B-A)r \\
&+ \sum_{k=2}^{\infty} \left[(k-1)\lambda B + B - A\right]\left[(k-1)\lambda + 1\right]^n |a_k| r^k + \sum_{k=2}^{\infty} \left[(k+1)\lambda B + A - B\right]\left[(k+1)\lambda - 1\right]^n |b_k| r^k \\
\leq &\, r\left\{ \sum_{k=2}^{\infty} \left(M_k |a_k| + N_k |b_k|\right) r^{k-1} - (B-A) \right\} < 0,
\end{aligned}$$

therefore $f \in SH^0(\lambda, n, A, B)$, and so the proof is complete. □

Next we show that the condition (2.2) is also necessary for the functions $f \in H$ to be in the class $SH_T^0(\lambda, n, A, B) = T^n \cap SH^0(\lambda, n, A, B)$ where T^n is the class of functions $f = h + \overline{g} \in SH^0$ so that

$$f = h + \overline{g} = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k \quad (z \in U). \tag{2.6}$$

Theorem 2.3. *Let $f = h + \overline{g}$ be defined by (2.6). Then $f \in SH_T^0(\lambda, n, A, B)$ if and only if the condition (2.2) holds.*

Proof. The "if" part follows from Theorem 2.2. For the "only-if" part, assume that $f \in SH_T^0(\lambda, n, A, B)$, then by (2.5) we have

$$\left| \frac{\sum_{k=2}^{\infty} (k-1)\lambda[(k-1)\lambda+1]^n |a_k| z^k + (k+1)\lambda[(k+1)\lambda-1]^n |b_k| \overline{z}^k}{(B-A)z - \sum_{k=2}^{\infty} [(k-1)\lambda B + B - A][(k-1)\lambda+1]^n |a_k| z^k + [(k+1)\lambda B + A - B][(k+1)\lambda-1]^n |b_k| \overline{z}^k} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum_{k=2}^{\infty} \{(k-1)\lambda[(k-1)\lambda+1]^n |a_k| + (k-+1)\lambda[(k+1)\lambda-1]^n |b_k|\} r^{k-1}}{B-A-\sum_{k=2}^{\infty} \{[(k-1)\lambda B + B - A][(k-1)\lambda+1]^n |a_k| + [(k+1)\lambda B + A - B][(k+1)\lambda-1]^n |b_k|\} r^{k-1}} < 1.$$

Thus, for M_k and N_k as defined by (2.3) and (2.4), we have

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|] r^{k-1} < B - A \quad (0 \le r < 1). \tag{2.7}$$

Let $\{\sigma_k\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|].$$

Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.7) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} [M_k |a_k| + N_k |b_k|] = \lim_{k\to\infty} \sigma_k \le B - A.$$

This gives the condition (2.2). □

In the following we show that the class of functions of the form (2.6) is convex and compact.

Theorem 2.4. *The class $SH_T^0(\lambda, n, A, B)$ is a convex and compact subset of SH.*

Proof. Let $f_t \in SH_T^0(\lambda, n, A, B)$, where

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \overline{z^k} \quad (z \in U,\ t \in \mathbb{N}). \tag{2.8}$$

Then $0 \le \eta \le 1$, let $f_1, f_2 \in SH_T^0(\lambda, n, A, B)$ be defined by (2.8). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1-\eta) f_2(z) \\ &= z - \sum_{k=2}^{\infty} \left(\eta |a_{1,k}| + (1-\eta) |a_{2,k}|\right) z^k \\ &\quad + (-1)^n \sum_{k=2}^{\infty} \left(\eta |b_{1,k}| + (1-\eta) |b_{2,k}|\right) \overline{z^k} \end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left\{M_k\left[\eta\left|a_{1,k}\right|+(1-\eta)\left|a_{2,k}\right|\right]+N_k\left[\eta\left|b_{1,k}\right|+(1-\eta)\left|b_{2,k}\right|\right]\right\}\\
=\ &\eta\sum_{k=2}^{\infty}\left\{M_k\left|a_{1,k}\right|+N_k\left|b_{1,k}\right|\right\}+(1-\eta)\sum_{k=2}^{\infty}\left\{M_k\left|a_{2,k}\right|+N_k\left|b_{2,k}\right|\right\}\\
\leq\ &\eta(B-A)+(1-\eta)(B-A)=B-A.
\end{aligned}$$

Thus, the function $\kappa=\eta f_1+(1-\eta)f_2$ belongs to the class $SH_T^0(\lambda,n,A,B)$. This means that the class $SH_T^0(\lambda,n,A,B)$ is convex.

On the other hand, for $f_t\in SH_T^0(\lambda,n,A,B)$, $t\in\mathbb{N}$ and $|z|\leq r(0<r<1)$, we get

$$\begin{aligned}
|f_t(z)| &\leq r+\sum_{k=2}^{\infty}\left\{\left|a_{t,k}\right|+\left|b_{t,k}\right|\right\}r^k\\
&\leq r+\sum_{k=2}^{\infty}\left\{M_k\left|a_{t,k}\right|+N_k\left|b_{t,k}\right|\right\}r^k\\
&\leq r+(B-A)r^2.
\end{aligned}$$

Therefore, $SH_T^0(\lambda,n,A,B)$ is locally uniformly bounded. Let

$$f_t(z)=z-\sum_{k=2}^{\infty}\left|a_{t,k}\right|z^k+(-1)^n\sum_{k=2}^{\infty}\left|b_{t,k}\right|\overline{z^k}\quad(z\in U,\ t\in\mathbb{N})$$

and let $f=h+\overline{g}$ be so that h and g are given by (1.1). Using Theorem 2.3 we obtain

$$\sum_{k=2}^{\infty}\left\{M_k\left|a_{t,k}\right|+N_k\left|b_{t,k}\right|\right\}\leq(B-A). \tag{2.9}$$

If we assume that $f_t\to f$, then we conclude that $\left|a_{t,k}\right|\to|a_k|$ and $\left|b_{t,k}\right|\to|b_k|$ as $k\to\infty$ ($t\in\mathbb{N}$). Let $\{\sigma_k\}$ be the sequence of partial sums of the series $\sum_{k=2}^{\infty}\{M_k|a_k|+N_k|b_k|\}$. Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.9) it is bounded above by $B-A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty}\{M_k|a_k|+N_k|b_k|\}=\lim_{k\to\infty}\sigma_k\leq B-A.$$

Therefore $f\in SH_T^0(\lambda,n,A,B)$ and therefore the class $SH_T^0(\lambda,n,A,B)$ is closed. In consequence, the class $SH_T^0(\lambda,n,A,B)$ is compact subset of SH, which completes the proof. □

We continue with the following lemma due to Jahangiri (Jahangiri, 1999).

Lemma 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty}\left\{\frac{k-\alpha}{1-\alpha}|a_k| + \frac{k+\alpha}{1-\alpha}|b_k|\right\} \leq 1 \quad (z \in U)$$

where $0 \leq \alpha < 1$. Then f is harmonic, orientation preserving, univalent in U and f is starlike of order α.

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $SH_T^0(\lambda, n, A, B)$.

Theorem 2.5. *Let $0 \leq \alpha < 1$, M_k and N_k be defined by (2.3) and (2.4). Then*

$$r_\alpha^*(SH_T^0(\lambda, n, A, B)) = \inf_{k\geq 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha}\right\}\right]^{\frac{1}{k-1}}. \tag{2.10}$$

Proof. Let $f \in SH_T^0(\lambda, n, A, B)$ be of the form (2.6). Then, for $|z| = r < 1$, we get

$$\begin{aligned}
&\left|\frac{D_1 f(z) - (1+\alpha)f(z)}{D_1 f(z) + (1-\alpha)f(z)}\right| \\
&= \left|\frac{-\alpha z - \sum\limits_{k=2}^{\infty}(k-1-\alpha)|a_k|z^k - (-1)^n\sum\limits_{k=2}^{\infty}(k+1+\alpha)|b_k|\overline{z}^k}{(2-\alpha)z - \sum\limits_{k=2}^{\infty}(k+1-\alpha)|a_k|z^k - (-1)^n\sum\limits_{k=2}^{\infty}(k-1+\alpha)|b_k|\overline{z}^k}\right| \\
&\leq \frac{\alpha + \sum\limits_{k=2}^{\infty}\{(k-1-\alpha)|a_k| + (k+1+\alpha)|b_k|\}r^{k-1}}{2-\alpha - \sum\limits_{k=2}^{\infty}\{(k+1-\alpha)|a_k| + (k-1+\alpha)|b_k|\}r^{k-1}}.
\end{aligned}$$

Note (see Lemma 2.1) that f is starlike of order α in U_r if and only if

$$\left|\frac{D_1 f(z) - (1+\alpha)f(z)}{D_1 f(z) + (1-\alpha)f(z)}\right| < 1, \ z \in U_r$$

or

$$\sum_{k=2}^{\infty}\left\{\frac{k-\alpha}{1-\alpha}|a_k| + \frac{k+\alpha}{1-\alpha}|b_k|\right\}r^{k-1} \leq 1. \tag{2.11}$$

Moreover, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty}\left\{\frac{M_k}{B-A}|a_k| + \frac{N_k}{B-A}|b_k|\right\}r^{k-1} \leq 1.$$

The condition (2.11) is true if

$$\frac{k-\alpha}{1-\alpha}r^{k-1} \leq \frac{M_k}{B-A}r^{k-1},$$

$$\frac{k+\alpha}{1-\alpha}r^{k-1} \leq \frac{N_k}{B-A}r^{k-1} \quad (k=2,3,\ldots)$$

or if

$$r \leq \frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha}\right\}^{\frac{1}{k-1}} \quad (k=2,3,\ldots).$$

It follows that the function f is starlike of order α in the disk $U_{r_\alpha^*}$ where

$$r_\alpha^* := \inf_{k\geq 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha}\right\}\right]^{\frac{1}{k-1}}.$$

The function

$$f_k(z) = h_k(z) + \overline{g_k(z)} = z - \frac{B-A}{M_k}z^k + (-1)^n\frac{B-A}{N_k}\overline{z}^k$$

proves that the radius r_α^* cannot be any larger. Thus we have (2.10). □

Using a similar argument as above we obtain the following.

Theorem 2.6. *Let $0 \leq \alpha < 1$ and M_k and N_k be defined by (2.3) and (2.4). Then*

$$r_\alpha^c(SH_T^0(\lambda,n,A,B)) = \inf_{k\geq 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k(k-\alpha)}, \frac{N_k}{k(k+\alpha)}\right\}\right]^{\frac{1}{k-1}}.$$

Our next theorem is on the extreme points of $SH_T^0(\lambda,n,A,B)$.

Theorem 2.7. *Extreme points of the class $SH_T^0(\lambda,n,A,B)$ are the functions f of the form (1.1) where $h = h_k$ and $g = g_k$ are of the form*

$$\begin{gathered} h_1(z) = z, \quad h_k(z) = z - \tfrac{B-A}{M_k}z^k, \\ g_k(z) = (-1)^n\tfrac{B-A}{N_k}\overline{z^k} \quad (z\in U,\ k\geq 2). \end{gathered} \tag{2.12}$$

Proof. Let $g_k = \eta f_1 + (1-\eta)f_2$ where $0<\eta<1$ and $f_1, f_2 \in SH_T^0(\lambda,n,A,B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty}\left|a_{t,k}\right|z^k + (-1)^n\sum_{k=2}^{\infty}\left|b_{t,k}\right|\overline{z^k} \quad (z\in U,\ t\in\{1,2\}).$$

Then, by (2.12), we have

$$\left|b_{1,k}\right| = \left|b_{2,k}\right| = \frac{B-A}{N_k},$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t\in\{2,3,\ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t\in\{2,3,\ldots\}\setminus\{k\}$. It follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme points of the function class $SH_T^0(\lambda,n,A,B)$. Similarly, we can verify that the functions $h_k(z)$ are the extreme points of the class $SH_T^0(\lambda,n,A,B)$. Now, suppose that a function f of the form (1.1) is in the family of extreme points of the class $SH_T^0(\lambda,n,A,B)$ and f is not of the form (2.12). Then there exists $m\in\{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{B-A}{[(m-1)\lambda+1]^n\,[\lambda(m-1)(B+1)+B-A]}$$

or

$$0 < |b_m| < \frac{B - A}{[(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]}.$$

If

$$0 < |a_m| < \frac{B - A}{[(m-1)\lambda + 1]^n [\lambda (m-1)(B+1) + B - A]},$$

then putting

$$\eta = \frac{|a_m| [(m-1)\lambda + 1]^n [\lambda (m-1)(B+1) + B - A]}{B - A}$$

and

$$\phi = \frac{f - \eta h_m}{1 - \eta},$$

we have $0 < \eta < 1,\ h_m \neq \phi$.

Therefore, f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$. Similarly, if

$$0 < |b_m| < \frac{B - A}{[(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]},$$

then putting

$$\eta = \frac{|b_m| [(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]}{B - A}$$

and

$$\phi = \frac{f - \eta g_m}{1 - \eta},$$

we have $0 < \eta < 1,\ g_m \neq \phi$.

It follows that f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$ and so the proof is completed. □

Therefore, by Theorem 2.7, we have the following corollary.

Corollary 2.1. *Let $f \in SH_T^0(\lambda, n, A, B)$ and $|z| = r < 1$. Then*

$$r - \frac{B - A}{(\lambda + 1)^n [\lambda (B+1) + A - B]} r^2 \leq |f(z)| \leq r + \frac{B - A}{(\lambda + 1)^n [\lambda (B+1) + A - B]} r^2.$$

The following covering result follows from Corollary 2.1.

Corollary 2.2. *If $f \in SH_T^0(\lambda, n, A, B)$ then $U_r \subset f(U)$ where*

$$r = 1 - \frac{B - A}{(\lambda + 1)^n [\lambda (B+1) + A - B]}.$$

References

Al-Oboudi, F. M. (2004). On univalent functions defined by a generalized Sălăgean operator. *International Journal of Mathematics and Mathematical Sciences* **27**, 1429–1436.

Bayram, H. and S. Yalcin (2017). A subclass of harmonic univalent functions defined by a linear operator. *Palestine Journal of Mathematics* **6**, 320–326.

Clunie, J. and T. Sheil-Small (1984). Harmonic univalent functions. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9**, 3–25.

Dziok, J. (2015*a*). Classes of harmonic functions defined by subordination. *Abstr. Appl. Anal.*

Dziok, J. (2015*b*). On Janowski harmonic functions. *J. Appl. Anal.* **21**, 99–107.

Dziok, J., J. M. Jahangiri and H. Silverman (2016). Harmonic functions with varying coefficients. *Journal of Inequalities and Applications*.

Jahangiri, J.M. (1999). Harmonic functions starlike in the unit disk. *J. Math. Anal. Appl.* **235**, 470–477.

Jahangiri, J.M., G. Murugusundaramoorthy and K. Vijaya (2002). Sălăgean-type harmonic univalent functions. *South J. Pure Appl. Math* **2**, 77–82.

Silverman, H. (1998). Harmonic univalent functions with negative coefficients. *J. Math. Anal. Appl.* **220**, 283–289.

Silverman, H. and E. M. Silvia (1999). Subclasses of harmonic univalent functions. *N. Z. J. Math.* **28**, 275–284.

Sălăgean, G.S. (1983). Subclasses of univalent functions. *Lecture Notes in Math. Springer- Verlag Heidelberg* **1013**, 362–372.

Yaşar, E. and S. Yalcin (2012). Generalized Sălăgean-type harmonic univalent functions. *Stud. Univ. Babeş-Bolyai Math.* **57**(3), 395–403.