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Non-Computable, Indiscernible and Uncountable Mathematical Constructions. Sub-Cardinals and Related Paradoxes

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Abstract

One of the most important achievements of the last century is the knowledge of the existence of non-countable sets. The proof by Cantor's diagonal method requires the assumption of actual infinity. By two paradoxes we show that this method sometimes proves nothing because of it can involve self-referential definitions. To avoid this inconvenient, we introduce another proving method based upon the information in the involved object definitions. We also introduce the concepts of indiscernible mathematical construction and sub-cardinal. In addition, we show that the existence of indiscernible mathematical constructions is an unavoidable consequence of uncountability.

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1. Introduction and preliminaries

In this article that two sets X and Y have different cardinalities only means that it is impossible to define one bijection between them. The existence of a bijection between two sets leads to size equality, but the converse implication need not be true, at least, with respect to infinite sets. The term "size" is ambiguous when applied to infinite sets. Recall that there is always a bijection between every infinite set and some proper subset of it. Bijections not only compare set-sizes but information and complexity in the definitions of their members. In particular, we consider those mathematical constructions that cannot be defined or determined by any finite expression. We say these objects to be *indiscernible*.

If $f : X \to Y$ is a bijection, the expression $y = f(x)$ specifies the object y uniquely. Thus, we can consider $y = f(x)$ as a definition for y, whenever both f and x are definable. As a consequence, if the information lying in any definition for y is always greater than the one in both definitions of f and x, we cannot assume the local equality $y = f(x)$ at x. We can find this situation when y is

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an indiscernible object. This method leads to non-existence of a bijection $f : X \to Y$ when there is, at least, one indiscernible object in Y. Even singletons can satisfy this condition. However, it is trivial to define a bijection between two singletons, what suggests that this proving method is wrong. Fortunately, in Theorem 2.2 we show that, if a set E contains one indiscernible object, then E must be infinite, and satisfies the following relation.

$$\text{“}E \text{ contains an indiscernible object”} \Rightarrow (\text{“}E \text{ is infinite”} \wedge \text{“}E \text{ is uncountable.”})$$

Analogously, by virtue of Theorem 2.1,

$$\text{“}E \text{ is uncountable”} \Rightarrow \text{“}E \text{ contains an infinite subset each member of which is indiscernible.”}$$

We can find similar facts using complexity instead of information. For instance, let Ω denote the Chaitin's constant (Chaitin, 2012). If $f : \mathbb{N} \to E \subseteq \mathbb{R}$ is a bijection, being computable by a Turing machine T, then E cannot contain Ω; otherwise, T could compute Ω, simply, as the value of f at some positive integer $n \in \mathbb{N}$. Accordingly, if $\Omega \in E$, then there is no bijection between $\mathbb{N}$ and E, being computable by any Turing machine. Nevertheless, depending on their nature, we could define a non-computable bijection F between $\mathbb{N}$ and E. This fact allows us to define the concept of sub-cardinal. Thus, comparing sets by computable bijections, we define sub-cardinals as sub-equivalences of those that we can state by definable one-to-one maps.

1.1. Preliminaries

To avoid any ambiguity, we use the concepts of “extensible language,” “identifier” and “discernible mathematical construction” with particular meanings that we explain below.

Let A be a finite alphabet. We term “vocabulary generated by A” every nonempty subset of the class Voc(A) of all finite and infinite sequences of symbols in A. Likewise, we term “language” every partial (syntactic) free-monoid $\mathcal{L}(\mathrm{Voc}(A))$, generated by Voc($A$), provided that we assign a meaning to each of its members. Thus, $\mathcal{L}(\mathrm{Voc}(A))$ consists of sentences. Two sentences s_1 and s_2 are equivalent when both denote the same concept.

Definition 1.1. We say that a language $\mathcal{L}(\mathrm{Voc}(A))$ is “extensible” when it satisfies the following conditions.

1. If there is a set E of sentences in some language $\mathfrak{L}$ that have no equivalent in $\mathcal{L}(\mathrm{Voc}(A))$, we can add sentences and symbols to $\mathcal{L}(\mathrm{Voc}(A))$ denoting all members of E.
2. When we extend $\mathcal{L}(\mathrm{Voc}(A))$ with a set W of new symbols and sentences, the way that we assign meanings and interpret each member of W can be described by “finite” sentences in $\mathcal{L}(\mathrm{Voc}(A))$.

The second condition is important for our purposes. For instance, consider the sentence: S = *“We can compute the area of a polygon of three angles.”* Using the word “triangle” we can transform S into the shorter sentence: *“We can compute the area of a triangle.”* Since we can describe the meaning of the term “triangle” by a finite expression, this substitution satisfies the former definition.

Definition 1.2. We say that a sentence S in an extensible language is an "identifier" for a mathematical construction X, provided that S specifies X uniquely. Thus, S can denote a definition, a predicate, or any procedure that can be associated with X in an ambiguity-free way.

Identifiers can be infinite, for instance, the binary expansion $0.d_1d_2d_3\ldots$ of an irrational number consists of an infinite symbol sequence.

Definition 1.3. We say that a mathematical construction X is discernible when there is, at least, one finite identifier for it; otherwise, we say that X is indiscernible.

To illustrate the concept in the former definition, consider the real number π. There are endless expressions that identify π. For instance, the infinite digit sequence $3.14159\ldots$ of its decimal expansion. In spite of being an endless symbol sequence, the number π is discernible because it can be specified by finite sentences. For instance,

$$S = \text{“}\textit{the ratio of a circle's circumference to its diameter.}\text{”}$$

Although the symbol π denotes the infinite expression $3.141592\ldots$, the language extension obtained adding π satisfies the second condition in Definition 1.1 because its meaning can be described by the finite sentence S. Likewise, every algebraic number is discernible by any equation that it satisfies. Every computable mathematical construction X is discernible because any algorithm (finite) that calculates X also identifies it.

A noticeable property related with indiscernible mathematical constructions is the impossibility of handling them individually. We show this topic in Theorem 2.2. There are sets of indiscernible objects that can be denoted by finite expressions. For instance, *"the subset K of all indiscernible members of* $[0,1]$*"* is a finite definition; hence K is discernible, but each of its members is not. Likewise, we show in Theorem 3.1 that, if there is one non-computable number in $[0,1]$, there is also an infinite subset no member of which is computable.

From now on, we frequently use the term "infinite," that can be regarded either as potential or actual. By potential infinity, we understand the concept stated by Poincaré (1854-1912) in the following quotation.

> *Actual infinity does not exist. What we call infinite is only the endless possibility of creating new objects no matter how many exist already.*

When we do not state the concept of infinity as "potential," we implicitly mean that it is actual.

2. Discernible mathematical constructions

In this section, we show that every uncountable set contains an infinite subset no member of which is discernible. In addition, indiscernible mathematical constructions cannot be identified and compared to each other by finite procedures. This property gives rise to some undecidable statements together with the impossibility of proving that a map satisfies the one-to-one property. By undecidable, we mean that either its truthfulness or falsehood cannot be proved by some finite procedure. Thus, if we assume any undecidable statement, there is no finite procedure to reject or confirm our assumption.

Lemma 2.1. *Every computable mathematical construction is discernible.*

Proof. Let O be a mathematical construction that is computable by a Turing machine T. The transition map of T is determined by a finite matrix T_M. We can denote T_M by a finite symbol sequence in any extensible language $\mathcal{L}(\mathrm{Voc}(A))$, simply, adding some end-row symbol to A, whenever it does not contain one.

Now, suppose that there is no Turing machine computing O, but we can compute it by some more complex algorithm Alg, for instance, a super-recursive one. In this case, we need a finite definition Def for Alg. Accordingly, in some extensible language there is a finite symbol sequence S that denotes Def. Thus, S determines Def; this denotes Alg; the later determines O. As a consequence, O is discernible through S. □

The concept of discernibility stated in Definition 1.3 can be either intrinsic or circumstantial. For instance, the expression E = *"the positive solution of the equation* $x^2 - 2 = 0$*"* determines $\sqrt{2}$. Since E is finite, $\sqrt{2}$ is discernible. Likewise, every algebraic number is discernible. The discernibility of $\sqrt{2}$ is intrinsic because arises from an intrinsic property of $\sqrt{2}$. By contrast, the expression

$$E = \text{"the second object in the sequence } S = \square, \sqrt{2}, 6, \gamma\text{"}$$

determines $\sqrt{2}$ by a circumstance (position) that can be satisfied by any object O, simply, substituting $\sqrt{2}$ by O in S. This is a positional determination in the scenario denoted by S. If S can be defined by a finite symbol sequence, then this description together with the expression S is also an identifier for $\sqrt{2}$.

Theorem 2.1. *Every uncountable set X contains an uncountable subset each member of which is indiscernible. In addition, the subset of all discernible members of X is countable.*

Proof. Let X be a nonempty set each member of which is discernible, hence, for every $x \in X$, there is a finite expression $S_x = \alpha_1\alpha_2 \ldots \alpha_{k_x}$ in some extensible language $\mathcal{L}(\mathrm{Voc}(A))$ that determines x. Let $\beta : A \to \mathbb{N}$ be an injective map and $\{p_1, p_2, p_3 \ldots\}$ the set of all positive prime integers. We can define a numbering function γ that sends each symbol sequence $\alpha_1\alpha_2 \ldots \alpha_{k_x}$ in $\mathcal{L}(\mathrm{Voc}(A))$ into the positive integer

$$\gamma(\alpha_1\alpha_2 \ldots \alpha_{k_x}) = \prod_{i=1}^{k_x} p_i^{\beta(\alpha_i)} \in \mathbb{N} \tag{2.1}$$

By construction, γ is injective. Since, by definition, there is a finite expression determining each discernible object x, the set X is countable because its image under γ is a subset of $\mathbb{N}$. As a consequence, to be uncountable, X must contain a nonempty set U of indiscernible objects. Since each member of $X \setminus U$ is discernible, as we have just seen, it is a countable subset of X. If U were also countable, $X = (X \setminus U) \cup U$ would be the union of two countable sets. Consequently, U must be uncountable. □

A straightforward consequence of the former theorem is the existence of indiscernible real numbers in the unit interval $[0, 1]$ because it is uncountable. This is an example of discernible set that contains indiscernible members. The set $[0, 1]$ is discernible because it can be defined by a finite sequence of finite expressions in any extensible language. For instance, *"the set of limits of all convergent sequences of rational numbers that are greater than or equal to* 0 *and less than or equal to* 1."

Axiom 2.1. *Let O_1 and O_2 be two mathematical constructions. If $O_1 \neq O_2$, there is, at least, one finitely representable predicate $p(x)$, in some extensible language, that O_1 satisfies and O_2 does not.*

The former axiom is widely satisfied. As the lemma below shows, the real number set satisfies it.

Lemma 2.2. *The real number set $\mathbb{R}$ satisfies Axiom 2.1.*

Proof. Let r_1 and r_2 be two real numbers. If

$$r_1 = c_1 c_2 \cdots c_k \,.\, c_{k+1} c_{k+2} c_{k+3} \cdots$$

and

$$r_2 = d_1 d_2 \cdots d_j \,.\, d_{j+1} d_{j+2} d_{j+3} \cdots$$

are their decimal expansions and $r_1 \neq r_2$, for some $n \in \mathbb{N}$, the inequality $c_n \neq d_n$ holds. Thus, r_1 is the only of them that satisfies the following predicate:

$p(x)$=*"The n-th digit of the decimal expansion of x is c_n."*

□

Lemma 2.3. *Let $E = \{O_n \mid n \in \{0, 1, \ldots k\}\}$ be a finite subset of $\mathbb{N}$. If the members of E satisfy Axiom 2.1 pairwisely, for every positive integer $n \leq k$, there is a predicate $q_n(x)$ that O_n satisfies and any other member of E does not.*

Proof. By Axiom 2.1, for every positive integer $m \leq k$, if $m \neq n$, there is a predicate $p_{n,m}(x)$ that O_n satisfies and O_m does not. Thus, O_n is the only member of E that satisfies the conjunction

$$q_n(x) = \bigwedge_{\substack{m \neq n \\ m \in \{0,1\ldots k\}}} p_{n,m}(x) \tag{2.2}$$

□

Now, we show that every finitely representable predicate that is satisfied by one indiscernible member of a finitely definable set K, it is also satisfied by every member of an infinite subset U of K; therefore K must be infinite.

Theorem 2.2. *Let X be a set the members of which satisfy Axiom 2.1 pairwisely. If X is finitely definable in an extensible language $\mathcal{L}(\mathrm{Voc}(A))$, and the subset U of all indiscernible members of X is nonempty, the following statements are true.*

1. *Let $p(x)$ be a finitely representable predicate in $\mathcal{L}(\mathrm{Voc}(A))$. If a member O_0 of U satisfies $p(x)$, then there is an infinite subset U_0 of U each member of which satisfies $p(x)$ too; therefore U must be infinite.*

2. *With the same assumptions as in the preceding statement, there is no finitely definable bijective map from any nonempty subset* $\mathbf{K}$ *of* $\mathbb{N}$ *onto* U_0.

Proof.

1. Suppose that O_0 is the only member of U that satisfies $p(x)$. In this case, $p(x)$ determines O_0 uniquely. Since, by assumption, $p(x)$ can be represented by a finite expression E_1 in $\mathcal{L}(\mathrm{Voc}(A))$, then O_0 is discernible and belongs to $X \setminus U$; which contradicts our assumption. Thus, O_0 cannot be the only object that satisfies $p(x)$. Since we assume that X is finitely definable in $\mathcal{L}(\mathrm{Voc}(A))$ by a finite expression E_0, the subset U of X is also finitely representable in $\mathcal{L}(\mathrm{Voc}(A))$ by the expression

   E_2 = *"The subset of all indiscernible members of the set denoted by* E_0*."*

   If U does not contain any other member of the set $\{O \in X \mid p(O)\}$, both finite expressions *"x belongs to the set denoted by* E_2*"* and *"x satisfies the predicate denoted by* E_1*"* form a finite expression that determines O_0, and both expressions E_1 and E_2 consist of finite symbol sequences in A. As in the previous case, O_0 would be discernible. Accordingly, there is at least one $O_1 \in U$ that satisfies $p(x)$ too. By Axiom 2.1 and taking into account Lemma 2.3, there is a finitely representable predicate $p_1(x)$ that O_0 satisfies and O_1 does not. The conjunction of both predicates $p(x) \wedge p_1(x)$ determines O_0, unless there is $O_2 \in U$ that also satisfies this conjunction. Iterating the procedure, we obtain an infinite subset U_0 of U each member of which satisfies $p(x)$.

2. If there is a bijection $f : \mathbb{N} \to U_0$ that can be defined by a finite expression E_f in $\mathcal{L}(\mathrm{Voc}(A))$, for every $n \in \mathbf{K}$, the expression

   "$f(n)$ is the image of the integer n under the map f defined by E_f*"*

   determines $f(n) \in U_0$ uniquely. Since, by hypothesis, E_f can be denoted by a finite symbol sequence, the former expression is finite, and $f(n)$ is discernible, for every $n \in \mathbb{N}$. Thus, f cannot be surjective. Accordingly, the existence of any finitely definable bijection f is not compatible with the indiscernibility of the members of U_0.

□

Corollary 2.1. *With the same assumptions as in the preceding theorem, if a discernible set E contains one indiscernible member O, then there is an infinite subset U of E each member of which is also indiscernible; hence E must be infinite. In addition, if E is uncountable, so is U.*

Proof. If E is discernible, there is at least one identifier Q for E that can be described by a finite symbol sequence in some extensible language. Now, let $P(x)$ denote the predicate:

$$P(x) = \text{“}x \text{ is a member of the set that } Q \text{ specifies.”}$$

It is straightforward that an object O satisfies $P(x)$ if and only if it is a member of E. Since $P(x)$ can be denoted by a finite expression, by virtue of the preceding theorem, there is an infinite set U each of its members satisfies also $P(x)$; hence $U \subseteq E$. Thus, E must be also infinite.

Finally, if E is uncountable, by virtue of Theorem 2.1, the subset $V \subseteq E$ of all discernible members of E is countable. As a consequence, U must be uncountable; otherwise $E = V \cup U$ is the union of two countable sets, which contradicts our assumption of the uncountability of E. □

Remark. As a consequence of Statement 1 in the former theorem, we can only handle infinite sets of indiscernible objects. Procedures involving indiscernible singletons require endless expressions. For instance, the short sentence

$$S = \text{“\textit{the subset of all undiscernible real numbers,}”}$$

denotes a subset $\mathbf{K}$ of $\mathbb{R}$. In spite of being discernible, because we can denote $\mathbf{K}$ by the short sentence S, each of its members requires an endless expression to be handled or identified. As a consequence, when $\mathbf{K}$ belongs to the image of a map f, by no finite method or procedure is possible to discern whether f is a one-to-one map because of the indiscernibility of the members of $\mathbf{K}$.

Each non-finitely definable map $f : X \to Y$ between infinite sets can be stated through a two column table, each of its rows consists of a member of X in the first column followed by its image under f in the second one. Its indiscernibility can be a consequence of its infinite size whenever by no predictable pattern we can determine its values. If for every finite subset K of X the restriction of f to K is finitely representable, we say f to be “first-kind indiscernible”. In this case, each restriction of f to any finite subset of its domain is discernible. By contrast, if for some $x_0 \in X$ the image $f(x_0)$ can only be denoted by an infinite symbol sequence in any language, then we say that f is “second-kind indiscernible”.

Theorem 2.3. *Let X be a set that contains a nonempty subset of indiscernible objects. If the members of X satisfy Axiom 2.1 pairwisely, for every one-to-one map $f : \mathbb{N} \to X$, the following statements are true.*

1. *If f is a finitely definable map in some language $\mathcal{L}(\mathrm{Voc}(A))$, $\forall K \in \mathbb{N}$, there is no positive integer $n \leq K$ such that its image $y = f(n)$ is indiscernible.*
2. *If f is a first-kind indiscernible map, the image of f does not contain any indiscernible member of X consequently f cannot be surjective.*

Proof.

1. By hypothesis, the map f is finitely definable. Thus, the predicate $p(n, y)$ denoted by the expression
$$p(n, y) = \text{“\textit{y is the image of n under f,}”}$$
satisfies Definition 1.1; therefore it is also finitely definable, for every $n \in \mathbb{N}$. As a consequence of Statement (1) in Theorem 2.2, there is an infinite subset of indiscernible members of X each member of which satisfies $p(n, y)$ too, which contradicts our hypothesis because f would be a relation, but not a map.

2. Let T be a table that describes f, and M any positive integer. The restriction of f to the finite set $\{1, 2, 3 \dots M\}$ consists of a finite sub-table T_M of T. Since, by hypothesis, f is first-kind indiscernible, the finite sub-table T_M can be described by a finite sequence S_M of symbols in an extensible language $\mathcal{L}(\mathrm{Voc}(A))$. The result of substituting S_M by the corresponding finite symbol sequence in the expression

   E =*"x is the member of X denoted by the expression lying in the second column and the n-th row of the table denoted by* S_M*"*

   is also finite and determines the image of f at n, for each $n \leq M$. Thus, the image of every positive integer n under f is discernible. As a consequence, f cannot be surjective, whenever its codomain contains some nonempty subset of indiscernible members. By hypothesis, X satisfies this condition.

□

It is worth pointing out that Theorem 2.3 is very similar to the well-known Cantor's Theorem. However, the non-existence of a first-kind indiscernible bijection is a consequence of indiscernible members of X, which is a local property. By contrast, the diagonal proof is built under the assumption of actual infinity and depends on the complete bijection domain. In the proof of Cantor's Theorem, there is no mention of the nature of the table denoting the map $f : \mathbb{N} \to [0, 1]$. We suppose that, at most, f must be a first-kind indiscernible map; otherwise, assuming that f cannot be injective, to reject this claim we need an endless procedure.

To proceed more accurately, we say that a set X is ω-countable to denote the cardinality equivalence between X and $\mathbb{N}$ is based on the existence of a *first-kind indiscernible* bijection $f : \mathbb{N} \to X$.

Theorem 2.3 is built by a pointwise method, which does not depend on the set sizes. We are aware that this method can seem dark because, in finite sets, cardinality strongly depends on size. This is why, in the section below, we show similar results from another viewpoint. In any case, it is worth mentioning that, by virtue of Corollary 2.1, every finitely definable set that contains some indiscernible member must be infinite. The following result shows the existence of indiscernible numbers in the unit interval $[0, 1]$.

Theorem 2.4. *If there is one non-computable number* ϖ *in the unit interval* $[0, 1]$*, there is also an infinite subset* $U \subseteq [0, 1]$*, each member of which is indiscernible.*

Proof. For each $n \in \mathbb{N}$, let $p_n(x, d_n)$ denote the predicate

$$p_n(x, d_n) = \text{"The n-th digit of the decimal mantissa of } x \text{ is } d_n\text{."}$$

It is straightforward that, for each map $\lambda : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the conjunction

$$\bigwedge_{n \in \mathbb{N}} p_n(x, \lambda(n))$$

together with the predicate $Q(x) =$ *"x belongs to* $[0, 1]$*"* define a unique member of $[0, 1]$.

Now, for each $r \in [0, 1]$, let $\Upsilon(n, r, \bigwedge_{m \in \mathbb{N}} p_m(x, c_m))$ be the conjunction obtained by substituting $p_n(x, c_n)$ by $p_n(x, d_n)$ in $\bigwedge_{m \in \mathbb{N}} p_m(x, c_m)$, where the members of $\{d_n \mid n \in \mathbb{N}\}$ are the digits of the decimal expansion of r.

Let $r_1 = 0.c_1c_2c_3\cdots \in [0,1]$ be a number the figures of which are chosen at random. Taking into account Theorem 2.2, if r is indiscernible the theorem is true. If it is not, consider the number r_2 defined by the conjunction $P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m\in\mathbb{N}} p_m(x, c_m))$. If there is no finitely representable predicate being equivalent to $P_1(x)$, then the number r_2 that it defines is indiscernible; otherwise, let $P_2(x)$ be the conjunction $\Upsilon(4, \varpi, P_1(x))$. Once again, if there is no finitely representable predicate being equivalent to $P_2(x)$, the number r_3 that it defines is indiscernible, and the theorem is true. Iterating the process, we obtain a sequence $\mathbf{S} = P_1(x), P_2(x), P_3(x)\ldots$ defined recursively as follows.

$$\begin{cases} P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m\in\mathbb{N}} p_m(x, c_m)) \\ P_{n+1} = \Upsilon\left(2^{(n+1)}, \varpi, P_n(x)\right) \end{cases} \tag{2.3}$$

The process stops when, for some $n \in \mathbb{N}$, the predicate $P_n(x)$ defines an indiscernible number, and the theorem is true. Otherwise, the process becomes infinite. In this case, if the infinite sequence $\mathbf{S}$ defines a member r of $[0,1]$, since this is an endless definition, r is indiscernible. Notice, that by no finite method is possible to build the sequence $\mathbf{S}$ because it involves the digits of ϖ and, by hypothesis, this is not computable. Now, we show that $\mathbf{S}$ defines a real number in $[0,1]$.

Let $S = r_1, r_2, r_3\ldots$ be the sequence that the members of $\mathbf{S}$ define. As a consequence of (2.3), for every positive integer $m > 0$, and each pair $(i, j) \in \mathbb{N}\times\mathbb{N}$, if $i, j > 2^m$, the 2^m-th first digits of the mantissas of both r_i and r_j are the same. Accordingly, for each couple r_i and r_j in $[0,1]$, the relation $i, j > 2^m$ leads to

$$|r_i - r_j| \leq \frac{1}{10^{2^m}}. \tag{2.4}$$

Thus, S is a Cauchy sequence and converges to some $r \in \mathbb{R}$. Since $[0,1]$ is closed, $r \in [0,1]$. Taking into account Theorem 2.2, the subset U of indiscernible members of $[0,1]$ is infinite. □

It is worth pointing out, that the information in the sequence $\mathbf{S}$ is infinite. No finite expression can contain the same information as $\mathbf{S}$. Since the definition of $\mathbf{S}$ involves figures obtained at random, in each instance determines different numbers. As a consequence, $\mathbf{S}$ defines a class of indiscernible objects. Recall that, by finite methods, we only can handle infinite sets of indiscernible objects (Theorem 2.2). Finally, by virtue of Theorem 2.3, the former result leads to the ω-uncountability of $[0,1]$.

2.1. Remote-cardinals and identifier supports

Let $\mathfrak{R}$ be a class of discernible maps and relations, and X the union of all domains and parameter sets of all members of $\mathfrak{R}$. We assume that X can contain hidden parameters too (Palomar Tarancón, 2016). From now on, we denote by $\mathfrak{C}[X, \mathfrak{R}]$ the class of all mathematical constructions such that, for each of which, we can build an identifier consisting of "finite compositions" of maps and relations in $\mathfrak{R}$. Likewise, for each mathematical construction K, we say each class $\mathfrak{C}[X, \mathfrak{R}]$ that contains K, to be an identifier support for it, provided that $\mathfrak{R}$ is finite and X is nonempty.

NOTATION. For every mathematical construction K, we denote by $\mathrm{Supp}(K)$ the class of all identifier supports for K.

Definition 2.1. We say that $\mathfrak{E}[X, \mathcal{R}] \in \mathrm{Supp}(K)$ is a *basic identifier support for* K provided that X satisfies the relation

$$\forall \mathfrak{E}(Y, \mathfrak{S}) \in \mathrm{Supp}(K) : \quad \#(X) \leq \#(Y) \tag{2.5}$$

In this case, we also say that $\#(X)$ is the remote-cardinal of K, that we denote by the symbol $\flat$. In this case, $\flat(K) = \#(X)$.

Lemma 2.4. *The remote-cardinal of the set* $\mathbb{Z}$ *of all integers and the one of every of its members is* 1.

Proof. We can build every positive integer n in $\mathbb{Z}$ iterating n-times the map $\mathrm{suc} : m \to (m+1)$ with the only argument 0. Likewise, we can build each negative integer iterating the inverse map suc^{-1}. To obtain 0 we can apply the identity map. Thus, if $X = \{0\}$ and $\mathfrak{R} = \{\mathrm{suc}, \mathrm{suc}^{-1}, \mathrm{id}_{\mathbb{Z}}\}$, then $\mathfrak{E}[X, \mathfrak{R}]$ is an identifier support for $\mathbb{Z}$, and so is also for every of its members. Since $\#(\{0\}) = 1$ is the smallest possible cardinal of any nonempty set, $\flat(\mathbb{Z}) = \#(\{0\}) = 1$. Analogously, $\forall n \in \mathbb{Z} : \flat(n) = \#(\{0\}) = 1$. □

Theorem 2.5. *If* $f : X \to Y$ *is a discernible bijection, and for every* $x \in X$ *there are both remote-cardinals* $\flat(x)$ *and* $\flat(f(x))$*, then*

$$\forall x \in X : \quad \flat(x) = \flat(f(x)). \tag{2.6}$$

Proof. First, we show that $\flat(f(x)) \leq \flat(x)$. Let $\mathfrak{E}(A_0, \mathfrak{R}_0)$ be a basic identifier support for x, and $\mathfrak{E}(A_1, \mathfrak{R}_1)$ a basic one for $f(x)$; hence $\flat(x) = \#(A_0)$, and $\flat(f(x)) = \#(A_1)$. Since we can obtain $f(x)$, simply, applying the map f to x, and by assumption f is discernible, the expression $\mathfrak{E}(A_0, \mathfrak{R}_0 \cup \{f\})$ is also an identifier support for $f(x)$. Now, taking into account (2.5), the following relation holds.

$$\flat(f(x)) = \#(A_1) \leq \#(A_0) = \flat(x). \tag{2.7}$$

Likewise, because of f is a bijection there is the inverse f^{-1}; therefore we can also show that

$$\#(A_0) = \flat(x) = \flat\left(f^{-1}(f(x))\right) \leq \flat(f(x)) = \#(A_1) \tag{2.8}$$

Both equations (2.7) and (2.8) lead to (2.6). □

Definition 2.2. We say that a prime factorization $p_{k_1}^{n_1} p_{k_2}^{n_2} \cdots p_{k_j}^{n_j}$ of an integer N is exhaustive if it contains every prime smaller than p_{k_j}, perhaps with exponent 0.

For instance, $20 = 2^2 \cdot 5^1$, but the exhaustive factorization is $20 = 2^2 \cdot 3^0 \cdot 5^1$, which contains every prime smaller than 5.

Corollary 2.2. *The remote-cardinal of every discernible mathematical construction is* 1.

Proof. If X is a discernible mathematical construction, the map γ defined in (2.1) specifies X by an integer $M = \prod_{i=1}^{m} p_i^{\beta(\alpha_i)}$ because we can obtain the parameters $\alpha_1\alpha_2 \ldots \alpha_m$, simply, through the prime factorization of M as follows.

Let Φ denote the map that sends each integer M into the m-tuple of its factors

$$\left(p_1^{k_1}, p_2^{k_2}, \cdots p_m^{k_m}\right) = \left(p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \cdots p_m^{\beta(\alpha_m)}\right)$$

in its exhaustive prime factorization, and ω the map

$$\left(p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \cdots p_m^{\beta(\alpha_m)}\right) \mapsto \alpha_1\alpha_2 \ldots \alpha_m.$$

Accordingly, $\mathfrak{E}[\{M\}, \{\Phi, \omega\}]$ is an identifier support for X. Since $\{M\}$ is a singleton, $\flat(X) = \#(\{M\}) = 1$.

We can also show this result as a corollary of Theorem 2.5. By Theorem 2.1 we know that, for every nonempty set E, if every of its members is discernible, the discernible injective map γ sends E into a subset K of $\mathbb{N}$. Since the restriction of γ to its image is bijective, by virtue of Theorem 2.5, for each X in E, $\flat(X) = 1$. □

Lemma 2.5. *The remote-cardinal of every indiscernible real number is* $\aleph_0$.

Proof. On the one hand, a real number r is defined as the limit of a sequence $(a_n)_{n\in\mathbb{N}}$ of rationals. On the other hand, since $\mathbb{R}$ with the standard topology T is a Hausdorff space, the binary relation Lim_T between each converging sequence and its limit is a map. If $f : \mathbb{N} \to \mathbb{Q}$ is the map $n \mapsto a_n$, then $\mathfrak{E}\left[\mathrm{img}(f), \{\mathrm{Lim}_T, f\}\right]$ is an identifier support for r. Now, we show that, for every indiscernible real number, this support is basic.

If r is indiscernible, every of its identifiers must be infinite. By definition, for every $\mathfrak{E}[A, \mathfrak{R}] \in \mathrm{Supp}(r)$, each identifier E for r, associated with $\mathfrak{E}[A, \mathfrak{R}]$, must be built by finite compositions of members of $\mathfrak{R}$, and each of which is discernible. Under this condition, E can only be infinite if so is A. Accordingly, $\#(\mathbb{N}) = \aleph_0 \leq \#(A)$. Since $\mathrm{img}(f)$ is countable, then $\#(\mathrm{img}(f)) = \aleph_0$, and $\mathfrak{E}\left[\mathrm{img}(f), \{\mathrm{Lim}_T, f\}\right]$ is basic. Thus, $\flat(r) = \#\left(\mathrm{img}(f)\right) = \#(\mathbb{N}) = \aleph_0$. □

Corollary 2.3. *If the set* $[0, 1]$ *contains any indiscernible number, there is no discernible bijection between* $\mathbb{N}$ *and* $[0, 1]$.

Proof. Let $f : \mathbb{N} \to E \subseteq [0, 1]$ be a bijection. As a consequence of Theorem 2.5 and Lemma 2.5, for every n in $\mathbb{N}$, $\flat(n) = 1$ and $\flat\left(f(n)\right) = 1$; hence $\mathrm{img}(f)$ does not contain any indiscernible real number. Consequently, $E = \mathrm{img}(f)$ is a proper subset of $[0, 1]$ and f cannot be surjective. □

3. Non-computable real numbers

In this section, we say that a real number r is non-computable, whenever there is no Turing machine computing r. We show that the existence of any non-computable number in $[0, 1]$ leads to an infinite set of them. To this end, we assign an automaton to each positive integer, which accepts every binary representation of integers. We simplify their structures using one-way automaton classes. Consider the equivalence between two-way and one-way finite automata (Hulden, 2015).

Definition 3.1. Let $\mathbf{Aut}(\mathbb{N})$ be the class of all automata each of its members is a 7-tuple $\mathbf{A}_n = (Q_n, \Gamma, \varnothing, \Sigma, \delta, q_0, F)$; where

- $Q_n = \{p_1, p_2, \dots p_n\} \cup \{HALT\}$ is the set of states, for some positive integer n; where each member is associated with a prime integer p_k, and $q_0 = p_1$ is the initial state.

- $\Gamma = \{0, 1, \varnothing\}$ is the tape alphabet, and $\varnothing$ is the blank symbol.

- $\Sigma \subset \Gamma$ is the set $\{0, 1\}$ of input symbols.

- $\delta : (Q_n \setminus F) \times \Gamma \to Q_n \times \Gamma \times \{R\}$ is the transition map, the codomain of which only contains the move R (right).

- $F = \{HALT\} \subseteq Q_n$ is the subset of final states.

From now one, we assume that each member of $\mathbf{Aut}(\mathbb{N})$ satisfies the following conditions.

1. For every $k \in \mathbb{N}$, p_k denotes the k-th prime integer. Thus, the initial state is $q_0 = p_1 = 2$. Likewise, $p_2 = 3$, $p_3 = 5$ and so on.
2. For every state $s \in (Q_n \setminus F)$: $\quad \delta(s, \varnothing) = (HALT, 1, R)$.
3. The image $(p_j, y, R) = \delta(p_k, x)$ under δ of every pair $(p_k, x) \in (Q_n \setminus F) \times \Gamma$ satisfies the following conditions.

$$p_j = \begin{cases} p_{k+1} \text{ if } k < n \text{ and } x \neq \varnothing, \\ p_n \text{ if } k = n \text{ and } x \neq \varnothing, \\ HALT \text{ if } x = \varnothing. \end{cases} \tag{3.1}$$

Now, we assign a Turing machine to each non-negative integer through the map $\Psi : \mathbb{N} \to \mathbf{Aut}(\mathbb{N})$ defined as follows.

- If $n \leq 2$, then $\Phi(n)$ is the Turing-machine in $\mathbf{Aut}(\mathbb{N})$ with the only states p_1 and $HALT$, and with the transition map

$$\delta(p_1, x) = \begin{cases} (p_1, x, R) \text{ if } x \in \{0, 1\} \\ (HALT, 1, R) \text{ otherwise.} \end{cases} \tag{3.2}$$

- If $n > 2$ and $p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m} = n$ is its exhaustive factorization into prime integers, then $\Phi(n)$ is the member of $\mathbf{Aut}(\mathbb{N})$ with the transition map δ defined in the following table.

$$\left| \begin{array}{cllcl} x & \text{state } p_1 & \text{state } p_2 & \dots & \text{state } p_m \\ \hline 0 & (p_2, f_{k_1}(0), R) & (p_3, f_{k_2}(0), R) & \dots & (p_m, f_{k_m}(0)) \\ 1 & (p_2, f_{k_1}(1), R) & (p_3, f_{k_2}(1), R) & \dots & (p_m, f_{k_m}(1)) \\ \varnothing & (HALT, 1, R) & (HALT, 1, R) & \dots & (HALT, 1, R) \end{array} \right| \tag{3.3}$$

where for each $k \in \mathbb{N}$, $f_k : \{0,1\} \to \{0,1\}$ is the function defined as follows.

$$f_m(x) = \begin{cases} 0 \text{ if } m \equiv 0 \mod (4) \\ x \text{ if } m \equiv 1 \mod (4) \\ 1 \text{ if } x = 0 \text{ and } m \equiv 2 \mod (4) \\ 0 \text{ if } x = 1 \text{ and } m \equiv 2 \mod (4) \\ 1 \text{ if } m \equiv 3 \mod (4). \end{cases} \tag{3.4}$$

Notation. For every positive integer n and each finite tape-symbol sequence $c_1c_2\dots c_j$, we denote by $\Phi(n)[c_1c_2\dots c_j]$ the output that $\Phi(n)$ carries out from the initial tape configuration $c_1c_2\dots c_j$.

Lemma 3.1. *For every $n \in \mathbb{N}$, the Turing machine $\Phi(n)$ accepts every finite tape-symbol sequence.*

Proof. Let $s_1, s_2, s_3 \dots s_m$ be any finite symbol sequence in the tape of $\Phi(n)$. Since the only move is R, after a finite step sequence the focused cell is the one containing the symbol s_m. In the following step, the head reads the $(m+1)$-th cell that must contain the blank symbol $\varnothing$. According to (3.3), $\varnothing$ is substituted by 1, and the process gets the final state HALT. □

Theorem 3.1. *The set $[0,1] \subset \mathbb{R}$ contains an infinite subset of non-computable discernible numbers.*

Proof. We show that if there is one non-computable number in $[0,1]$, it also contains an infinite subset $\mathbf{K} \subset [0,1]$ of them. Let $r = 0.d_1d_2d_3\dots d_j\dots$ a non-computable real number in $[0,1]$ written in the binary numeration system. For each integer $k > 0$, let n_k be the positive one the binary figures of which are

$$d_{10k+1}d_{10k+2}\dots d_{(10k+10)} \tag{3.5}$$

Likewise, let $c_{k,1}c_{k,2}\dots c_{k,j_k}$ be the result of the computation

$$\Phi(n_k)[d_{10k+1}d_{10k+2}\dots d_{(10k+10)}] = c_{k,1}c_{k,2}\dots c_{k,j_k} \tag{3.6}$$

The real number

$$r^* = 0.c_{1,1}c_{1,2}\dots c_{1,j_1}c_{2,1}c_{2,2}\dots c_{2,j_2}\cdots_{k,1} c_{k,2}\dots c_{k,j_k}\dots \tag{3.7}$$

is non-computable because its binary figures depend on the ones of r, and by hypothesis so is r. In addition, the set of Turing machines

$$\{\Phi(n_1), \Phi(n_2)\dots\Phi(n_k)\dots\}$$

by which we compute r^*, is infinite and requires to know previously the binary expansion of r.

Iterating the process over r^*, we obtain an infinite sequence of non-computable real numbers in $[0,1]$. Although in (3.5) we take sequences of ten binary figures, with figure sequences of other length we can also build infinite sets of non-computable real numbers.

Finally, the non-computable numbers involved in this proof can be determined by the finite expressions (3.5), (3.6) and (3.7); hence each of them is discernible. □

4. Sub-cardinals

From now on, we denote by **Dsc** the subset of all discernible members of $[0, 1]$. Likewise, by **Cmp** we denote the subset of all computable members of **Dsc**. As we have just seen in Theorem 3.1, there exist non-computable numbers that are discernible; hence, **Cmp** is a proper subset of **Dsc**.

Lemma 4.1. *There is no computable bijective map between any nonempty subset* **K** *of* $\mathbb{N}$ *and* $\complement_{[0,1]}$**Cmp**.

Proof. Suppose that there is a computable bijection $f : \mathbf{K} \to \complement_{[0,1]}\mathbf{Cmp}$. For every positive integer n_0 in **K**, $f(n_0)$ would be computable, which contradicts $f(n_0) \in \complement_{[0,1]}\mathbf{Cmp}$. □

Both results, Statement (2) in Theorem 2.2 and the former lemma, allow us to compare infinite sets through several criteria based upon the nature of the comparing bijections. According to Theorem 2.1 and Lemma 2.1, both sets $\mathbf{Dsc} \subset [0, 1]$ and $\mathbf{Cmp} \subset [0, 1]$ are countable. However, there is no "computable" bijection between these subsets of $[0, 1]$. This peculiarity allows us to state the following definitions.

Definition 4.1. Two sets X and Y are of the same λ-cardinality if there is, at least, one "computable" bijection between them (Radó, 1962). Likewise, we say X and Y to have the same δ-cardinal whenever there is a "discernible" bijection $f : X \to Y$.

Remark. The set **Cmp** of all computable real numbers in $[0, 1]$ is a proper subset of the set **Dsc** of all finitely definable members of $[0, 1]$. As we have seen in Lemma 2.1 and Theorem 2.1, both are countable subsets of $[0, 1]$, hence there is, at least, one finitely definable bijection $f : \mathbf{Cmp} \to \mathbf{Dsc}$. However, as a consequence of Lemma 4.1, there is no computable bijection $k : \mathbf{Cmp} \to \mathbf{Dsc}$. In other words, although both sets have the same cardinal $\aleph_0$, they are not of the same λ-cardinality. That is a consequence of the information that each member of **Cmp** and **Dsc** contains. Both, the information and complexity of non-computable numbers are greater than that lying in any computable one (Li & Vitányi, 2008).

Taking into account these properties, we can consider the λ-cardinal of **Cmp** as a sub-cardinal of $\aleph_0 = \#(\mathbb{N})$, which we denote by $\aleph_{-2}$. Since $\aleph_{-2} \neq \aleph_0$, then $\aleph_{-2}$ is a "proper" sub-cardinal of $\aleph_0$. By sub-cardinal we do not mean "smaller than." According to both Theorem 2.2 and Lemma 4.1, cardinalities are, simply, equivalence relations based on the comparing bijection nature. This viewpoint needs some explanation. The existence of an isomorphism between structured sets leads to size equality (Hume's principle). The converse statement need not be true. The size equality between structured sets need not lead to the existence of isomorphisms. Nevertheless, we can define bijections between the underlying sets, which are structure-free constructions. However, we cannot always forget the structure of any mathematical construction. For instance, real numbers are defined as limits of rational sequences. If we forget the topology of $\mathbb{R}$, the concept of limit vanishes, and we lose the real number definition. Indiscernibility is also an unforgettable structure property because it is an intrinsic object-definition attribute.

We denote by the expression $\aleph_{-1}$ the cardinal of **Dsc**. Since **Cmp** is a proper subset of **Dsc**, $\aleph_{-1}$ is a sub-cardinal of $\aleph_0$; and $\aleph_{-2}$ is a sub-cardinal of $\aleph_{-1}$. Finally, we denote by $\aleph_\omega$ the cardinal of each set E such that there is a first-kind indiscernible bijection between $\mathbb{N}$ and E.

Conjecture 4.1. *Each of the following incompatible statements is undecidable.*

1) $\aleph_\omega$ *is a proper sub-cardinal of* $\aleph_0$.
2) $\aleph_\omega = \aleph_0$.

In the former conjecture, by undecidable we mean the impossibility of rejecting or proving any of these statements by finite procedures. Take into account that, in general, by no finite procedure we can discern the equality or inequality of two indiscernible objects. Discerning the equality is necessary to know whether any map is a one-to-one correspondence. This is a consequence of Theorem 2.2.

5. Some paradoxes arising from Cantor's method

In this section, we introduce three paradoxes to show that Cantor's methods can involve inadequate self-referential definitions. The first paradox is built through an instance of the diagonal method, which leads to a contradiction. This situation occurs if the method involves any self-referential definition that fits into the following pattern.

$$X = \text{"Expression that contains } X\text{"} \tag{5.1}$$

The former expression is an abstract equation. Since there are unsolvable equations, it is possible that some self-referential definitions define nothing. Self-referential definitions can be implicitly stated as in the following pattern.

$$X = \text{"Expression that contains a class } E \text{ containing } X \text{ implicitly."} \tag{5.2}$$

This is the case of Cantor's diagonal method under the assumption of actual infinity. For instance, let the following table denote a bijection $f : \mathbb{N} \to X \subseteq [0, 1]$.

$$\begin{cases} f(1) = 0.d_{11}d_{12}d_{13}\dots \\ f(2) = 0.d_{21}d_{22}d_{23}\dots \\ \dots\dots\dots\dots\dots\dots\dots \\ f(n) = 0.d_{n1}d_{n2}d_{n3}\dots \\ \dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{5.3}$$

By the diagonal method we define a member $r = 0.c_1c_2c_3\dots$ of $[0, 1]$ that satisfies the condition

$$\forall n \in \mathbb{N} : \quad c_n \neq d_{nn} \tag{5.4}$$

If $r \notin X$ the statement above is an adequate definition, but if $r \in X$, is implicitly self-referential.

Paradox 5.1. According to Theorem 2.1, the subset **Dsc** of all discernible members or $[0, 1]$ is countable. Let $\gamma : \mathbf{Dsc} \to \mathbb{N}$ be the injective map finitely defined in (2.1), and $\gamma_0 : \mathbf{Dsc} \to \text{img}(\gamma)$ the restriction to its image, which is bijective; hence there exists its inverse

$$\gamma_0^{-1} : \text{img}(\gamma) \subseteq \mathbb{N} \to \mathbf{Dsc}.$$

Since γ is finitely defined in (2.1), so are both γ_0 and γ_0^{-1}; consequently, the three maps are discernible.

Now, for every positive integer n, let $0.c_{n1}c_{n2}c_{n3}\dots$ be the decimal mantissa of $\gamma_0^{-1}(n)$. By Cantor's diagonal method we can build a real number $r = 0.d_1d_2d_3\dots$ in $[0,1]$ as follows.

$$\forall n \in \mathbb{N}: \quad d_n = \begin{cases} c_{nn}+1 \text{ if } c_{nn} < 9 \\ 0 \text{ otherwise.} \end{cases} \tag{5.5}$$

The former equation is a finite expression that determines r uniquely; hence it is discernible, and the relation

$$r \in \mathbf{Dsc}. \tag{5.6}$$

is true. Nevertheless, equation (5.5) leads to $\forall n \in \mathbb{N} : r \neq \gamma_0^{-1}(n)$, because $\forall n \in \mathbb{N} : c_{nn} \neq d_n$; consequently

$$r \notin \mathbf{Dsc}, \tag{5.7}$$

which contradicts (5.6).

We can solve the former paradox easily. Recall that, in the scope of actual infinity, every infinite set is a construction that can be completed. In this case, when the image of an indiscernible bijection $f : \mathbb{N} \to E$ contains every number r_0 that can be obtained by Cantor's diagonal method, this method involves a self-referential definition for r_0. This is the case in the former paradox. To better understand this topic, we state the paradox below by the inverse method. We choose the number $r_0 \in [0,1]$ and build a bijection that does not contain it.

Paradox 5.2. Let $f : \mathbb{N} \to \mathbf{M} \subseteq [0,1]$ be a bijection that satisfies the following conditions.

1. Let $\mathbb{E}$ denote the set $\{2 \cdot n \mid n \in \mathbb{N}\}$ of all positive even integers. The restriction $f|_{\mathbb{E}} : \mathbb{E} \to \mathbf{M}$ of f to $\mathbb{E}$ is the subset $\mathbb{Q} \cap [0,1]$ of all rationals in $[0,1]$. Recall that both $\mathbb{E}$ and $\mathbb{Q}$ are countable; so then we can build the map f under the condition $\text{img}(f|_{\mathbb{E}}) = \mathbb{Q} \cap [0,1]$.
2. The image $f(n)$ of each odd integer n is an irrational member of $\mathbf{M}$. Thus, $\text{img}(f)$ contains every rational in $[0,1]$ and an infinite and countable subset of irrationals.

Let $r \in \mathbb{Q} \cap [0,1] \subset \text{img}(f)$ be a rational number and $0.d_1d_2d_3\dots$ the figures of its decimal expansion. Suppose that, for every $n \in \mathbb{N}$, the expansion of $f(n)$ is $0.c_{n1}c_{n2}c_{n3}\dots$. Now, we define a new bijection f_1 as follows. If $d_1 \neq c_{11}$, then $f_1 = f$; otherwise, if n_1 is the smallest integer such that $c_{n1} \neq d_1$, then

$$f_1(n) = \begin{cases} f(n_1) \text{ if } n = 1 \\ f(1) \text{ if } n = n_1 \\ f(n) \text{ otherwise;} \end{cases} \tag{5.8}$$

We obtain the bijection f_1, simply, by the transposition of $f(1)$ and $f(n_1)$ in the image of f, hence $\text{img}(f) = \text{img}(f_1)$, besides, the first figure in the mantissa of $f_1(1)$ differs from d_1.

Now, we define another bijection f_2 by a similar procedure. If the second figure in the mantissa of $f_1(2)$ is not equal to d_2, then $f_2 = f_1$; otherwise, if n_2 is the smallest integer such that $n_2 > 2$ and $c_{n2} \neq d_2$, then

$$f_2(n) = \begin{cases} f(n_2) \text{ if } n = 2 \\ f(2) \text{ if } n = n_2 \\ f(n) \text{ otherwise;} \end{cases} \tag{5.9}$$

As in the preceding case, the bijection f_2 is obtained, simply, by the transposition of $f(2)$ and $f(n_2)$; hence $\mathrm{img}(f) = \mathrm{img}(f_1) = \mathrm{img}(f_2)$, besides, the second figure in the mantissa of $f_1(2)$ differs from d_2.

Iterating the procedure m-times, we obtain a bijection $f_m : \mathbb{N} \to \mathbf{M}$ such that $\mathrm{img}(f_m) = \mathrm{img}(f)$, and $\forall n \leq m$: $r \neq f_m(n)$. Under the scope of potential infinity, the iteration never ends. By contrast, assuming infinity as an actual entity, the iteration can be completed obtaining a bijection $f_\infty : \mathbb{N} \to \mathbf{M}$ with $\mathrm{img}(f_\infty) = \mathrm{img}(f) = \mathbf{M}$, and $\forall n \in \mathbb{N} : r \neq f_\infty(n)$, hence $r \notin \mathbf{M}$; which contradicts that $r \in \mathbb{Q} \cap [0, 1] \subseteq \mathbf{M}$, that we assume implicitly in statement (1), and f satisfies this condition.

Paradox 5.3. Let $\mathcal{F}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N})$ be the subset of all finitely definable subsets of $\mathbb{N}$. Since we can denote every member of $\mathcal{F}(\mathbb{N})$ by a finite symbol sequence in some extensible language, we can also define a map $\zeta : \mathcal{F}(\mathbb{N}) \to \mathbb{N}$ similar to the one γ defined in (2.1). Let $\zeta_0 : \mathcal{F}(\mathbb{N}) \to \mathrm{img}(\zeta)$ be the restriction of ζ to its image, which is a bijection.

Now, we define the subset $\mathbf{K}$ of $\mathbb{N}$ by the finite expression

$$\mathbf{K} = \{n \in \mathrm{img}(\zeta) \subseteq \mathbb{N} \mid n \notin \zeta_0^{-1}(n)\} \tag{5.10}$$

If $\zeta(\mathbf{K}) \in \mathbf{K}$, by virtue of (5.10), this relation leads to $\zeta(\mathbf{K}) \notin \mathbf{K}$, and vice versa. We can solve this contradiction supposing that ζ is not defined at $\mathbf{K}$. In other words: $\mathbf{K} \notin \mathcal{F}(\mathbb{N})$. However, $\mathbf{K}$ is a subset of $\mathbb{N}$ finitely defined in (5.10); hence $\mathbf{K} \in \mathcal{F}(\mathbb{N})$.

It is worth pointing out that, as in both paradoxes 5.1 and 5.2, the former one requires the assumption of actual infinity. Thus, if we assume that every infinity is potential, these paradoxes vanish.

6. Conclusion

On the one hand, we should not ignore the existence of indiscernible mathematical constructions, unless we reject the existence of uncountable sets. Unfortunately, as a consequence of Theorem 2.2, we can only handle infinite sets of indiscernible objects. Thus, when working with real world problems, we never can be involved with any "finite" set of indiscernible numbers.

On the other hand, in the scope of actual infinity, Cantor's diagonal method sometimes proves nothing. Fortunately, taking into account Corollary 2.1 and Theorem 2.3, we can deduce the existence of uncountable sets from indiscernible mathematical constructions. In addition, if Conjecture 4.1 is true, a set theory including it becomes incomplete, and we can apply Gödel's incompleteness theorems to it.

References

Chaitin, Gregory (2012). How much information can there be in a real number? [reprint of mr2346819]. In: *Computation, physics and beyond*. Vol. 7160 of *Lecture Notes in Comput. Sci.*. pp. 247–251. Springer, Heidelberg.

Hulden, Mans (2015). From two-way to one-way finite automata—three regular expression-based methods. In: *Implementation and application of automata*. Vol. 9223 of *Lecture Notes in Comput. Sci.*. pp. 176–187. Springer, Cham.

Inoue, Katsushi, Itsuo Takanami, Akira Nakamura and Tadashi Ae (1979). One-way simple multihead finite automata. *Theoret. Comput. Sci.* **9**(3), 311–328.

Kanamori, Akihiro (2012). Set theory from Cantor to Cohen. In: *Sets and extensions in the twentieth century*. Vol. 6 of *Handb. Hist. Log.*. pp. 1–71. Elsevier/North-Holland, Amsterdam.

Li, Ming and Paul Vitányi (2008). *An introduction to Kolmogorov complexity and its applications*. Texts in Computer Science. third ed.. Springer, New York.

Palomar Tarancón, Juan-Esteban (2016). Co-universal algebraic extension with hidden parameters. *Theory Appl. Math. Comput. Sci.* **6**(2), 150–169.

Rabin, M. O. and D. Scott (1959). Finite automata and their decision problems. *IBM J. Res. Develop.* **3**, 114–125.

Radó, T. (1962). On non-computable functions. *Bell System Tech. J.* **41**, 877–884.

Ye, Feng (2011). *Strict finitism and the logic of mathematical applications*. Vol. 355 of *Synthese Library. Studies in Epistemology, Logic, Methodology, and Philosophy of Science*. Springer, Dordrecht.