Local Fixed Point Theorems for Graphic Contractions in Generalized Metric Spaces

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Abstract

In this paper, we will present some local fixed point theorems for graphic contractions on a generalized metric space in the sense of Perov.

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1. Introduction

The classical Banach contraction principle was extended for single-valued contraction on spaces endowed with vector-valued metrics by Perov (Perov, 1964). Other fixed point results, given in the framework of a set endowed with a complete vector-valued metric, are given in (Agarwal, 1983), (Filip & Petruşel, 2009), (O'Regan *et al.*, 2007), (Petruşel *et al.*, 2015), (Precup, 2009), ...

On the other hand, the concept of graphic contraction is more general that that of contraction mapping, since the contraction condition is assumed to be satisfied only for pairs $(x, y) \in Graph(f) := \{(x, f(x)) : x \in X\}$. In this case, existence of the fixed point can be established under some additional continuity assumption on f. In this sense, several fixed point results for graphic contractions were proved in (Rus, 1972) (see also (Rus *et al.*, 2008), page 29), (Subrahmanyam, 1974) and (Hicks & Rhoades, 1979).

An existence and uniqueness result for graphic contractions in complete metric spaces was recently proved in (Chaoha & Sudprakhon, 2017).

For a synthesis and new results concerning fixed point theory for graphic contractions in complete metric spaces see (Petruşel & Rus, 2018).

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The purpose of this paper is to give some local fixed point theorems for graphic contractions in the context of complete vector-valued metric spaces. Our results extend, to the case of vector-valued metric spaces, a local variant of Banach's contraction principle, which was proved for the first time (to our best knowledge) by M.A. Krasnoselskii. Our results also extend some local fixed point theorems for graphic contractions in complete metric spaces given in (Petruşel, 2019).

2. Main results

We first some preliminary notions and results.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, By O_m the null $m \times m$ matrix and by I_m the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that, through this paper, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq O$ for all $x, y \in X$; if $d(x, y) = O$, then $x = y$; (where $O := \underbrace{(0, 0, \cdots, 0)}_{m-times}$)

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space in the sense of Perov (or a vector-valued metric space) and it will be denoted by (X, d). In this context, if $x_0 \in X$ and $r \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$, then we denote

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}, \ \tilde{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\}.$$

The notions of convergent sequence, Cauchy sequence, completeness, open, closed, bounded and compact subset are similar to those for usual metric spaces. Notice also that in Precup (Precup, 2009) are pointed out the advantages of working with vector-valued metrics with respect to the usual scalar ones.

Definition 2.1. ((Varga, 2000)) A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(A)$ is strictly less than 1. In other words, this means that all the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{m,m}(\mathbb{R})$.

A classical result in matrix analysis is the following theorem (see (Varga, 2000)).

Theorem 2.1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) A is convergent to zero;

(ii) $A^n \to O_m$ as $n \to \infty$;

(iii) The matrix $(I_m - A)$ is nonsingular and

$$(I_m - A)^{-1} = I_m + A + ... + A^n + ... \tag{2.1}$$

(iv) The matrix $(I_m - A)$ is nonsingular and $(I_m - A)^{-1}$ has nonnegative elements.

We recall now some contraction conditions in vector-valued metric spaces.

Definition 2.2. Let (X, d) be a generalized metric space in the sense of Perov and $f : X \to X$ be an operator. Then, f is called:

(i) an *A-contraction* if $A \in M_{mm}(\mathbb{R}_+)$ converges to zero and

$$d(f(x), f(y)) \le A d(x, y), \text{ for every } x, y \in X.$$

(ii) a graphic *A-contraction* if $A \in M_{mm}(\mathbb{R}_+)$ converges to zero and

$$d(f(x), f^2(x)) \le A d(x, f(x)), \text{ for every } x \in X.$$

Notice that any *A*-contraction is a graphic *A*-contraction, but not reversely.

The following local fixed point theorem in generalized metric space in the sense of Perov is an extension of a result proved by R. Agarwal in (Agarwal, 1983).

Theorem 2.2. *Let (X, d) be a complete generalized metric in the sense of Perov. Let $x_0 \in X$, $r = (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$ and let $f : \tilde{B}(x_0, r) \to X$ be an operator which has closed graph with respect to d. We suppose:*

(i) f is a graphic A-contraction on $\tilde{B}(x_0, r)$;

(ii) $(I_m - A)^{-1} d(x_0, f(x_0)) \le r$.

Then:

(a) $Fix(f) \ne \emptyset$;

(b) $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$ (where $R := (I_m - A)^{-1} d(x_0, f(x_0))$) and the sequence of successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ converges to a fixed point of f;

(c) if $x^* := \lim\limits_{n \to \infty} f^n(x_0)$, then the following apriori estimation holds

$$d(f^n(x_0), x^*) \le A^n (I_m - A)^{-1} d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}.$$

Proof. We can prove, by mathematical induction, that

$$d(x_0, f^n(x_0)) \le (I_m + A + \cdots + A^{n-1}) d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}, n \ge 2. \tag{2.2}$$

Indeed, we have

$$d(x_0, f^2(x_0)) \le d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) \le$$
$$d(x_0, f(x_0)) + A d(x_0, f(x_0)) = (I + A) d(x_0, f(x_0)).$$

Next, for the general case of (2.2), we have

$$d(x_0, f^n(x_0)) \le d(x_0, f^{n-1}(x_0)) + d(f^{n-1}(x_0), f^n(x_0)) \le$$
$$(I_m + A + \cdots + A^{n-2}) d(x_0, f(x_0)) + A^{n-1} d(x_0, f(x_0)) =$$
$$(I_m + A + \cdots + A^{n-1}) d(x_0, f(x_0)).$$

Thus, by (2.2), we obtain that

$$d(x_0, f^n(x_0)) \le (I_m - A)^{-1} d(x_0, f(x_0)) := R, \text{ for each } n \in \mathbb{N}, n \ge 2. \tag{2.3}$$

Hence, $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$. Then, by the graphic contraction condition, we obtain that $d(f^n(x_0), f^{n+1}(x_0)) \leq A^n d(x_0, f(x_0))$, for each $n \in \mathbb{N}$. Using this relation, we immediately obtain, for every $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq A^n(I_m + A + \cdots + A^{p-1})d(x_0, f(x_0)) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)). \tag{2.4}$$

The relation (2.4) shows that the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ is Cauchy and, by the completeness of the space, there exists $x^* \in \tilde{B}(x_0, R)$ such that $x^* := \lim\limits_{n \to \infty} f^n(x_0)$. The conclusions follow now by the closed graph condition of the operator f. The apriori evaluation follows by (2.4) letting $p \to \infty$. $\square$

Remark. In particular, if f is an A-contraction, we get Theorem 2.1 in (Agarwal, 1983).

A more general result can be proved using the framework of a complete metric space endowed with a partial order relation. Our next theorem result extends the main result given in (Ran & Reurings, 2004).

Theorem 2.3. *Let X be a nonempty set endowed with a partial order relation "$\preceq$" and let $d : X \times X \to \mathbb{R}_+^m$ be a complete generalized metric in the sense of Perov on X. Let $x_0 \in X$, $r = (r_1, \cdots, r_m) \in \mathbb{R}^m$ with $r_i > 0$ for every $i \in \{1, 2, \cdots, m\}$ and $f : \tilde{B}(x_0, r) \to X$ be an operator which has closed graph with respect to d and is increasing with respect to "$\preceq$". We suppose:*
(i) there exists $A \in M_{mm}(\mathbb{R}_+)$ convergent to zero such that

$$d(f(x), f^2(x)) \leq Ad(x, f(x)), \text{ for every } x \in X \text{ with } x \leq x_0;$$

(ii) $f(x_0) \preceq x_0$;
(iii) $(I_m - A)^{-1}d(x_0, f(x_0)) \leq r$.
Then $Fix(f) \neq \emptyset$ and the sequence of successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ converges to a fixed point of f. Moreover, if $x^* := \lim\limits_{n \to \infty} f^n(x_0)$, then the following apriori estimation holds

$$d(f^n(x_0), x^*) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}.$$

Proof. By (ii) and the monotonicity assumption on f we get that

$$x_0 \succeq f(x_0) \succeq f^2(x_0) \succeq \cdots \succeq f^n(x_0) \succeq \cdots$$

Next, as before, we can prove that

$$d(x_0, f^n(x_0)) \leq (I_m + A + \cdots + A^{n-1})d(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}, n \geq 2. \tag{2.5}$$

Thus, by (2.5), we obtain that

$$d(x_0, f^n(x_0)) \leq (I_m - A)^{-1}d(x_0, f(x_0)) := R, \text{ for each } n \in \mathbb{N}, n \geq 2. \tag{2.6}$$

Hence, $f^n(x_0) \in \tilde{B}(x_0, R)$ for each $n \in \mathbb{N}$. Then, by the graphic contraction condition, we obtain that $d(f^n(x_0), f^{n+1}(x_0)) \leq A^n d(x_0, f(x_0))$, for each $n \in \mathbb{N}$. Using this relation, we immediately obtain, for every $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq A^n(I_m + A + \cdots + A^{p-1})d(x_0, f(x_0)) \leq A^n(I_m - A)^{-1}d(x_0, f(x_0)). \tag{2.7}$$

The relation (2.7) shows that the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ is Cauchy and, thus, it converges to an element $x^* \in \tilde{B}(x_0, R)$. We notice that $x^* \in Fix(f)$, by the closed graph condition of the operator f. The apriori evaluation follows again letting $p \to \infty$ in (2.7). $\qquad\square$

Remark. It is an open question to obtain the convergence (to a fixed point) of the sequence of successive approximations $(f^n(x))_{n \in \mathbb{N}}$ for each $x \in \tilde{B}(x_0; R)$. Another open question to extend the above results to the multi-valued case.

References

Agarwal, R.P. (1983). Contraction and approximate contraction with an application to multi-point boundary value problems. *J. Comput. Applied Math.* **9**, 315–325.

Chaoha, P. and W. Sudprakhon (2017). Fixed point sets of Subrahmanyam maps. *Linear and Nonlinear Anal.* **3**, 149–154.

Filip, A.D. and A. Petruşel (2009). Fixed point theorems on spaces endowed with vector-valued metrics. *Fixed Point Theory and Applications* **2010**(Article ID 281381), 15 pages.

Hicks, T.L. and B.E. Rhoades (1979). A Banach type fixed point theorem. *Math. Japon.* **24**, 327–330.

O'Regan, D., N. Shahzad and R.P. Agarwal (2007). Fixed point theory for generalized contractive maps on spaces with vector-valued metrics. *Fixed Point Theory and Applications, Nova Sci. Publ., New York* pp. 143–149.

Perov, A.I. (1964). On the Cauchy problem for a system of ordinary differential equations. *Pviblizhen. Met. Reshen. Differ. Uravn.* **2**, 115–134.

Petruşel, A. (2019). Local fixed point results for graphic contractions. *J. Nonlinear Var. Anal., (to appear).*

Petruşel, A. and I.A. Rus (2018). Graphic contractions and applications. *Mathematical Analysis and Applications, Springer, (to appear).*

Petruşel, A., O. Mleşniţe and C. Urs (2015). Vector-valued metrics in fixed point theory. *Amer. Math. Soc., Contemporary Mathematics* **636**, 149–165.

Precup, R. (2009). The role of matrices that are convergent to zero in the study of semilinear operator systems. *Math. Comput. Model.* **49**, 703–708.

Ran, A.C.M. and M.C.B. Reurings (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Amer. Math. Soc.* **132**, 1435–1443.

Rus, I.A. (1972). On the method of successive approximations. *Revue Roum. Math. Pures et Appl.* **17**, 1433–1437.

Rus, I.A., A. Petruşel and G. Petruşel (2008). *Fixed Point Theory*. Cluj Univ. Press, Cluj-Napoca.

Subrahmanyam, P.V. (1974). Remarks on some fixed point theorems related to Banach's contraction principle. *J. Math. Phys. Sri.* **8**, 445–457.

Varga, R.S. (2000). *Matrix Iterative Analysis*. Springer Series in Computational Mathematics, Vol. 27, Springer, Berlin.