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On Normal Fuzzy Submultigroups of a Fuzzy Multigroup

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Abstract

In this paper, we propose the notion of normal fuzzy submultigroups of a fuzzy multigroup. Some properties of normal fuzzy submultigroups of a fuzzy multigroup are explored and some related results are obtained. It is shown that a fuzzy submultigroup of a fuzzy multigroup is normal if and only if its alpha-cut is a normal subgroup of a given group. The concepts of commutator and normalizer in fuzzy multigroup setting are introduced and some results are deduced.

Keywords: Fuzzy comultiset, Fuzzy multiset, Fuzzy multigroup, Fuzzy submultigroup, Normal fuzzy submultigroup.
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1. Introduction

The concept of fuzzy sets proposed by (Zadeh, 1965) is a mathematical tool for representing vague concepts. The theory of fuzzy sets has grown stupendously over the years giving birth to fuzzy groups proposed in (Rosenfeld, 1971). Several works have been done on fuzzy groups and fuzzy normal subgroups (see Ajmal & Jahan, 2012; Malik *et al.*, 1992; Mashour *et al.*, 1990; Mordeson *et al.*, 1996; Mukherjee & Bhattacharya, 1984; Seselja & Tepavcevic, 1997; Wu, 1981).

Motivated by the work in (Zadeh, 1965), the idea of fuzzy multisets was conceived in (Yager, 1986) as the generalization of fuzzy sets in multisets framework. For some details on fuzzy multisets (see Ejegwa, 2014; Miyamoto, 1996; Syropoulos, 2012). Recently, in (Shinoj *et al.*, 2015), the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory, and some properties of fuzzy multigroups were presented. In fact, fuzzy multigroup is a generalization of fuzzy groups. (Baby *et al.*, 2015) continued the algebraic study of fuzzy multisets by proposing the idea of abelian fuzzy multigroups.

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The work in (Ejegwa, 2018*c*), which was built on (Shinoj *et al.*, 2015), introduced the concept of fuzzy multigroupoids and presented the idea of fuzzy submultigroups with a number of results. More properties of abelian fuzzy multigroups were explicated in (Ejegwa, 2018*b*), in the same vein, the notions of centre and centralizer in fuzzy multigroup setting where established with some relevant results. In (Ejegwa, 2018*a*), the notion of homomorphism in the context of fuzzy multigroups was defined and some homomorphic properties of fuzzy multigroups in terms of homomorphic images and homomorphic preimages, respectively, were presented. Since the notions of fuzzy multigroups, fuzzy submultigroups and abelian fuzzy multigroups have been established in literature, then it is germane to consider when a fuzzy submultigroup is said to be normal. Hence the motivation for this present research. In fact, this study is an application of fuzzy multisets to group theoretical notions like normal subgroups.

In this paper, we propose the notion of normal fuzzy submultigroups of a fuzzy multigroup and discuss some of its properties. The concepts of commutator and normalizer in fuzzy multigroup setting are also introduced, and some related results are deduced. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy multisets, fuzzy multigroups and fuzzy submultigroups. In Section 3, we propose the idea of normal fuzzy submultigroups of a fuzzy multigroup and discuss some of its properties. Also, the concepts of commutator and normalizer in fuzzy multigroup setting are also introduced, and some related results are obtained. Finally, Section 4 concludes the paper and provides direction for future studies.

2. Preliminaries

In this section, we review some existing definitions and results which shall be used in the sequel.

Definition 2.1. (Yager, 1986) Assume X is a set of elements. Then a fuzzy bag/multiset A drwan from X can be characterized by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From (Syropoulos, 2012), a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By (Miyamoto & Mizutani, 2004), it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0, 1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0, 1]$ (see Biswas, 1999; Mizutani *et al.*, 2008).

We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.1. Let $X = \{a, b, c\}$ be a set. Then a fuzzy multiset of X is given as

$$A = \{\langle \frac{0.5, 0.4, 0.3}{a} \rangle, \langle \frac{0.6, 0.4, 0.4}{b} \rangle, \langle \frac{0.7, 0.4, 0.2}{c} \rangle\}.$$

Definition 2.2. (see Miyamoto, 1996) Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.3. (see Syropoulos, 2012) Let $A, B \in FMS(X)$. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $CM_{A\cap B}(x) = CM_A(x) \wedge CM_B(x)$,
(ii) $CM_{A\cup B}(x) = CM_A(x) \vee CM_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum, respectively.

Definition 2.4. (see Miyamoto, 1996) Let $A, B \in FMS(X)$. Then A and B are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5. A fuzzy multiset B of a set X is said to have sup-property if for any subset $W \subset X$ $\exists\, w_0 \in W$ such that

$$CM_B(w_0) = \bigvee_{w \in W} \{CM_B(w)\}.$$

Definition 2.6. (Shinoj *et al.*, 2015) Let X be a group. A fuzzy multiset A of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$,
(ii) $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$.

It follows immediately that,

$$CM_A(x^{-1}) = CM_A(x), \forall x \in X$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(x^n) \geq CM_A(x) \forall x \in X,\ n \in \mathbb{N}$$

since

$$\begin{aligned} CM_A(x^n) = CM_A(x^{n-1}x) &\geq CM_A(x^{n-1}) \wedge CM_A(x) \\ &\geq CM_A(x) \wedge ... \wedge CM_A(x) \\ &= CM_A(x). \end{aligned}$$

It can be easily verified that if A is a fuzzy multigroup of X, then

$$CM_A(e) = \bigvee_{x \in X} CM_A(x)\ \forall x \in X,$$

that is, $CM_A(e)$ is the tip of A. The set of all fuzzy multigroups of X is denoted by $FMG(X)$.

Example 2.2. Let $X = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e.$$

Again, let

$$A = \{\langle \frac{1, 0.9}{e} \rangle, \langle \frac{0.7, 0.5}{a} \rangle, \langle \frac{0.8, 0.6}{b} \rangle, \langle \frac{0.7, 0.5}{c} \rangle\}$$

be a fuzzy multiset of X. We investigate whether $A \in MG(X)$ using Definition 2.6.

$$CM_A(ea) = CM_A(a) = 0.7, 0.5 \geq CM_A(e) \wedge CM_A(a) = 0.7, 0.5$$

$$CM_A(eb) = CM_A(b) = 0.8, 0.6 \geq CM_A(e) \wedge CM_A(b) = 0.8, 0.6$$

$$CM_A(ec) = CM_A(c) = 0.7, 0.5 \geq CM_A(e) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(ab) = CM_A(c) = 0.7, 0.5 \geq CM_A(a) \wedge CM_A(b) = 0.7, 0.5$$

$$CM_A(ac) = CM_A(b) = 0.8, 0.6 \geq CM_A(a) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(bc) = CM_A(a) = 0.7, 0.5 \geq CM_A(b) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(aa) = CM_A(e) = 1, 0.9 \geq CM_A(a) \wedge CM_A(a) = 0.7, 0.5$$

$$CM_A(bb) = CM_A(e) = 1, 0.9 \geq CM_A(b) \wedge CM_A(b) = 0.8, 0.6$$

$$CM_A(cc) = CM_A(e) = 1, 0.9 \geq CM_A(c) \wedge CM_A(c) = 0.7, 0.5$$

$$CM_A(ee) = CM_A(e) = 1, 0.9 \geq CM_A(e) \wedge CM_A(e) = 1, 0.9$$

$$CM_A(a^{-1}) = CM_A(a) = 0.7, 0.5, CM_A(b^{-1}) = CM_A(b) = 0.8, 0.6$$

$$CM_A(c^{-1}) = CM_A(c) = 0.7, 0.5, CM_A(e^{-1}) = CM_A(e) = 1, 0.9$$

Because all the axioms in Definition 2.6 are satisfied $\forall x, y \in X$, it follows that A is a fuzzy multigroup of X.

Clearly, a fuzzy multigroup is a fuzzy group that admits repetition of membership values. That is, a fuzzy multigroup collapses into a fuzzy group whenever repetition of membership values is ignored.

Remark. We notice the following from Definition 2.6:

(i) every fuzzy multigroup is a fuzzy multiset but the converse is not always true.

(ii) a fuzzy multiset A of a group X is a fuzzy multigroup if $\forall x, y \in X$,

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM(y)$$

holds.

Definition 2.7. (Shinoj *et al.*, 2015) Let A be a fuzzy multigroup of a group X. Then A^{-1} is defined by $CM_{A^{-1}}(x) = CM_A(x^{-1})\ \forall x \in X$.

Thus, we notice that $A \in FMG(X) \Leftrightarrow A^{-1} \in FMG(X)$.

Definition 2.8. (Ejegwa, 2018c) Let $A, B \in FMG(X)$. Then the product $A \circ B$ of A and B is defined to be a fuzzy multiset of X as follows:

$$CM_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz}(CM_A(y) \wedge CM_B(z)), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

This definition is adapted from (Shinoj *et al.*, 2015).

Definition 2.9. (Ejegwa, 2018c) Let $A \in FMG(X)$. A fuzzy submultiset B of A is called a fuzzy submultigroup of A denoted by $B \sqsubseteq A$ if B is a fuzzy multigroup. A fuzzy submultigroup B of A is a proper fuzzy submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.10. (Baby *et al.*, 2015) Let $A \in FMG(X)$. Then A is said to be abelian (commutative) if for all $x, y \in X$, $CM_A(xy) = CM_A(yx)$.

Whenever A is a fuzzy multigroup of an abelian group X, it implies that A is abelian.

Definition 2.11. (see Ejegwa, 2018c; Shinoj *et al.*, 2015) Let $A \in FMG(X)$. Then the sets A_* and A^* are defined as

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ and

(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, where e is the identity element of X.

Proposition 2.1. (see Ejegwa, 2018c; Shinoj *et al.*, 2015) *Let $A \in FMG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.12. Let $A \in FMG(X)$. Then the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined as

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and

(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of A, where $\alpha \in [0, 1]$.

Definition 2.13. Let $A \in FMG(X)$. Then the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined as

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and

(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of A, where $\alpha \in [0, 1]$.

Theorem 2.1. *Let $A \in FMG(X)$. Then $A_{[\alpha]}$ is a subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0, 1]$.*

Proof. Let $x, y \in A_{[\alpha]}$, then $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Because $A \in FMG(X)$, we get

$$\begin{aligned} CM_A(xy^{-1}) &\geq (CM_A(x) \wedge CM_A(y)) \geq \alpha \\ &= CM_A(x) \geq \alpha \wedge CM_A(y) \geq \alpha. \end{aligned}$$

Thus, $xy^{-1} \in A_{[\alpha]}$. Hence, $A_{[\alpha]}, \alpha \in [0, 1]$ is a subgroup of X for all $\alpha \leq CM_A(e)$. The proof of the second part, that is, $A^{[\alpha]}$ is a subgroup of X $\forall$ $\alpha \geq CM_A(e)$ is similar. □

3. Main Results

In this section, some properties of normal subgroups in fuzzy multigroup setting are investigated by redefining some concepts in the light of fuzzy multigroups.

Definition 3.1. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is called a normal fuzzy submultigroup of B if for all $x, y \in X$, it satisfies

$$CM_A(xyx^{-1}) \geq CM_A(y).$$

Example 3.1. Let $X = \{0, 1, 2, 3\}$ be a group of modulo 4 with respect to addition. Then a fuzzy multigroup of X is given as

$$B = \{\langle\frac{1, 0.9, 0.8}{0}\rangle, \langle\frac{0.9, 0.7, 0.5}{1}\rangle, \langle\frac{0.8, 0.7, 0.4}{2}\rangle, \langle\frac{0.9, 0.7, 0.5}{3}\rangle\},$$

and

$$A = \{\langle\frac{1, 0.8, 0.7}{0}\rangle, \langle\frac{0.8, 0.6, 0.4}{1}\rangle, \langle\frac{0.7, 0.6, 0.4}{2}\rangle, \langle\frac{0.8, 0.6, 0.4}{3}\rangle\}$$

is a fuzzy submultigroup of B. It follows that A is a normal fuzzy submultigroup of B since

$$CM_A(1 + 2 + 1^{-1}) = CM_A(1 + 2 + 3) = 0.7, 0.6, 0.4 \geq CM_A(2)$$

$$CM_A(2 + 1 + 2^{-1}) = CM_A(2 + 1 + 2) = 0.8, 0.6, 0.4 \geq CM_A(1)$$

$$CM_A(3 + 2 + 3^{-1}) = CM_A(3 + 2 + 1) = 0.7, 0.6, 0.4 \geq CM_A(2)$$

$$CM_A(2 + 3 + 2^{-1}) = CM_A(2 + 3 + 2) = 0.8, 0.6, 0.4 \geq CM_A(3)$$

$$CM_A(1 + 3 + 1^{-1}) = CM_A(1 + 3 + 3) = 0.8, 0.6, 0.4 \geq CM_A(3)$$

$$CM_A(3 + 1 + 3^{-1}) = CM_A(3 + 1 + 1) = 0.8, 0.6, 0.4 \geq CM_A(1).$$

Definition 3.2. Let $A \in FMG(X)$ and $x, y \in X$. Then x and y are called conjugate elements in A if for some $z \in X$,

$$CM_A(x) = CM_A(zyz^{-1}).$$

Two fuzzy multigroups A and B of X are conjugate to each other if for all $x, y \in X$,

$$CM_A(y) = CM_B(xyx^{-1}) \text{ or } CM_A(y) = CM_{B^x}(y)$$

and

$$CM_B(x) = CM_A(yxy^{-1}) \text{ or } CM_B(x) = CM_{A^y}(x).$$

Remark. Let A be a fuzzy submultigroup of $B \in FMG(X)$. From Definitions 2.6 and 2.7, A is normal if and only if A^{-1} is normal.

Proposition 3.1. *If $B \in FMG(X)$ and A is a normal fuzzy submultigroup of B. Then A_* and A^* are normal subgroups of X. Also, A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*.*

Proof. We know that A_* and A^* are subgroups of X by Proposition 2.1. Now, we proof that A_* and A^* are normal subgroups of X.

Let $x, y \in A_*$. By the definition of A_*, it follows that $CM_A(x) > 0$ and $CM_A(y) > 0$. That is,

$$CM_A(xyx^{-1}) \geq CM_A(y) > 0.$$

So, $xyx^{-1} \in A_* \Rightarrow A_*$ is a normal subgroup of X.

Similarly, assume $x, y \in A^*$. By the definition of A^*, it follows that

$$CM_A(x) = CM_A(e) = CM_A(y).$$

That is,

$$CM_A(xyx^{-1}) \geq CM_A(y) = CM_A(e) \geq CM_A(xyx^{-1}).$$

Thus, $CM_A(xyx^{-1}) = CM_A(e)\ \forall x, y \in X$. Hence, $xyx^{-1} \in A^*$ and the result follows.

Recall that, A is a normal fuzzy submultigroup of B, and A_* and A^* are normal subgroups of X. Synthesizing these, it implies that A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*. □

Proposition 3.2. *Let A be a normal fuzzy submultigroup of $B \in FMG(X)$. Then $A_{[\alpha]}$ is a normal subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a normal subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X and $\alpha \in [0, 1]$. Consequently, $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$ and $A^{[\alpha]}$ is a normal subgroup of $B^{[\alpha]}$.*

Proof. It implies from Theorem 2.1 that, $A_{[\alpha]}$ is a subgroup of X for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where $\alpha \in [0, 1]$. Now, we proof that $A_{[\alpha]}$ and $A^{[\alpha]}$ are normal subgroups of X.

Let $x, y \in A_{[\alpha]}$. By the definition of $A_{[\alpha]}$, we get

$$CM_A(x) \geq \alpha \text{ and } CM_A(y) \geq \alpha.$$

That is,

$$CM_A(xyx^{-1}) = CM_A(y) \geq \alpha.$$

Thus, $xyx^{-1} \in A_{[\alpha]}$, so $A_{[\alpha]}$ is a normal subgroup of X. Similarly, it follows that $A^{[\alpha]}$ is a normal subgroup of X.

But we know that, A is a normal fuzzy submultigroup of B, and $A_{[\alpha]}$ and $A^{[\alpha]}$ are normal subgroups of X. Synthesizing these, it happens that $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$ and $A^{[\alpha]}$ is a normal subgroup of $B^{[\alpha]}$. □

Theorem 3.1. *For a fuzzy submultigroup A of $B \in FMG(X)$, the following statements are equivalent:*

(i) A is a normal submultigroup of B.

(ii) $A_{[\alpha]}$ (for $\alpha \in [0,1]$ and $\alpha \leq CM_A(e)$, where e is he identity element of X) is a normal subgroup of X. It also holds for$A^{[\alpha]}$, where $\alpha \in [0,1]$ and $\alpha \geq CM_A(e)$.

Proof. (i)⇒(ii). Let $x \in X$ and $y \in A_{[\alpha]}$. By the hypothesis, we have

$$CM_A(xyx^{-1}) = CM_A(y) \geq \alpha.$$

It follows that $y = xyx^{-1} \in A_{[\alpha]}$ and hence $A_{[\alpha]}$ is a normal subgroup of X.

(ii)⇒(i). Let $x, g \in X$. Take $\alpha = CM_A(x)$ and $\beta = CM_B(g)$, so that $x \in A_{[\alpha]}$ and $g \in B_{[\beta]}$.

Case 1: $\alpha \geq \beta$. This implies that $\alpha_0 \geq CM_A(x) \geq \beta = CM_B(g)$ for $\alpha \in [0, \alpha_0]$. Thus $\beta \in Im(B)$ and $\beta \leq \alpha_0$. By the hypothesis, $A_{[\beta]}$ is a normal subgroup of $B_{[\beta]}$. Also, $x \in A_{[\beta]}$ and $g \in B_{[\beta]}$. Hence $gxg^{-1} \in A_{[\beta]}$. So,

$$CM_A(gxg^{-1}) \geq \beta = CM_B(g) = CM_A(x) \wedge CM_B(g).$$

Case 2: $\beta \geq \alpha$. This implies that

$$CM_B(g) \geq \alpha = CM_A(x).$$

Thus $\alpha \in Im(A)$ and $x \in A_{[\alpha]}$, $g \in B_{[\alpha]}$. By the hypothesis, $A_{[\alpha]}$ is a normal subgroup of $B_{[\alpha]}$. Consequently, $gxg^{-1} \in A_{[\alpha]}$. So,

$$CM_A(gxg^{-1}) \geq \alpha = CM_A(x) = CM_A(x) \wedge CM_B(g).$$

Hence (i) holds. □

Proposition 3.3. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then the following statements are equivalent.*

(i) A is a normal fuzzy submultigroup of B.

(ii) $CM_A(xyx^{-1}) = CM_A(y)\ \forall x, y \in X$.

(iii) $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$.

Proof. (i)⇒(ii). Suppose A is a normal fuzzy submultigroup of B. From Definition 3.1, it implies that $CM_A(xyx^{-1}) = CM_A(y)\ \forall x, y \in X$.

(ii)⇒(iii). Suppose that $CM_A(xyx^{-1}) = CM_A(y)$. Then, it implies that

$$CM_A(xy) = CM_A(yx)\ \forall x, y \in X.$$

(iii)⇒(i). Assume that $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$. It follows that A is a normal fuzzy submultigroup of B since $A \subseteq B$. □

Remark. Every normal fuzzy submultigroup of a fuzzy multigroup is abelian.

Proposition 3.4. *If A is a fuzzy submultigroup of $B \in FMG(X)$ such that $CM_A(x) = CM_A(y)\ \forall x, y \in X$. Then the following assertions are equivalent.*

(i) A is a normal fuzzy submultigroup of B.
(ii) $CM_A(yx) = CM_A(xy) \wedge CM_B(y)\ \forall x, y \in X$.

Proof. (i)⇒(ii). Since A is a normal fuzzy submultigroup of B and $CM_A(x) = CM_A(y)$, it follows from Definition 3.1 and Proposition 3.3 that,

$$CM_A(yx) = CM_A(y(xy)y^{-1}) = CM_A(xy) \wedge CM_B(y)\ \forall x, y \in X.$$

(ii)⇒(i). Suppose $CM_A(yx) = CM_A(xy) \wedge CM_B(y)$. We infer that

$$CM_A(xy) = CM_A(yx) \wedge CM_B(y).$$

Then it implies that, $CM_A(xy) = CM_A(yx)$. Hence, the proof is completed by Proposition 3.3. □

Proposition 3.5. *Let A be a fuzzy submultigroup of $G \in FMG(X)$ and B be a fuzzy submultiset of G. If A and B are conjugate, then B is a fuzzy submultigroup of G.*

Proof. Since A and B are conjugate, then by Definition 3.2 it implies that $A = B$. And this completes the proof for the fact that, A is a fuzzy submultigroup of $G \in FMG(X)$. □

Proposition 3.6. *Let $A, B, C \in FMG(X)$ such that A and B are normal fuzzy submultigroups of C. If $A \subseteq B$, then $A \cap B$ and $A \cup B$ are normal fuzzy submultigroups of C.*

Proof. Since A and B are normal fuzzy submultigroups of C such that $A \subseteq B$, it follows that $A \cap B = A$ and $A \cup B = B$. Thus, $A \cap B$ and $A \cup B$ are normal fuzzy submultigroups of C. □

Theorem 3.2. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is a normal fuzzy submultigroup of B if and only if $x \in X$ is constant on the conjugacy classes of A.*

Proof. Suppose that A is a normal fuzzy submultigroup of B. Then

$$CM_A(y^{-1}xy) = CM_A(xyy^{-1}) = CM_A(x)\ \forall x, y \in X.$$

This implies that, $x \in X$ is constant on the conjugacy classes of A.

Conversely, let $x \in X$ be constant on each conjugacy classes of A. Then

$$CM_A(xy) = CM_A(xyxx^{-1}) = CM_A(x(yx)x^{-1}) = CM_A(yx)\ \forall x, y \in X.$$

Hence, A is a normal fuzzy submultigroup of B. □

We now give an alternative formulation of the notion of normal fuzzy submultigroup in terms of commutator of a group. First, we recall that if X is a group and $x, y \in X$, then the element $x^{-1}y^{-1}xy$ is usually depicted by $[x, y]$ and is called the commutator of x and y.

Theorem 3.3. *Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then A is a normal fuzzy submultigroup of B if and only if*

(i) $CM_A([x, y]) \geq CM_A(x)\ \forall x, y \in X$.

(ii) $CM_A([x, y]) = CM_A(e)\ \forall x, y \in X$*, where e is the identity of X.*

Proof. (i) Suppose A is a normal fuzzy submultigroup of B. Let $x, y \in X$, then

$$\begin{aligned} CM_A(x^{-1}y^{-1}xy) &\geq CM_A(x^{-1}) \wedge CM_A(y^{-1}xy) \\ &= CM_A(x) \wedge CM_A(x) = CM_A(x). \end{aligned}$$

Conversely, assume that A satisfies the inequality. Then for all $x, y \in X$, we have

$$\begin{aligned} CM_A(x^{-1}yx) &= CM_A(yy^{-1}x^{-1}yx) \\ &\geq CM_A(y) \wedge CM_A([y, x]) = CM_A(y). \end{aligned}$$

Thus, $CM_A(x^{-1}yx) \geq CM_A(y) \forall x, y \in X$. Hence, A is a normal fuzzy submultigroup of B.

(ii) Let $x, y \in X$. Suppose A is a normal fuzzy submultigroup of B. We know that A is a normal fuzzy submultigroup of B
$\Leftrightarrow CM_A(xy) = CM_A(yx)\ \forall x, y \in X$
$\Leftrightarrow CM_A(x^{-1}y^{-1}x) = CM_A(y^{-1})\ \forall x, y \in X$
$\Leftrightarrow CM_A(x^{-1}y^{-1}xyy^{-1}) = CM_A(y^{-1})\ \forall x, y \in X$
$\Leftrightarrow CM_A([x, y]y^{-1}) = CM_A(y^{-1})\ \forall x, y \in X$.
Consequently, $CM_A([x, y]) = CM_A(y^{-1}y) = CM_A(e)\ \forall x, y \in X$.

Conversely, assume $CM_A([x, y]) = CM_A(e)\ \forall x, y \in X$. Then

$$CM_A(x^{-1}y^{-1}xy) = CM_A(e) \Rightarrow CM_A((yx)^{-1}xy) = CM_A(e).$$

That is, $CM_A(xy) = CM_A(yx)\ \forall x, y \in X$. Thus, A is a normal fuzzy submultigroup of B. □

Theorem 3.4. *Let A be a normal fuzzy submultigroup of $G \in FMG(X)$. Then $\bigcap_{x\in X} A^x$ is normal and is the largest normal fuzzy submultigroup of G that is contained in A.*

Proof. Suppose $A^x \in FMG(X)\ \forall x \in X$. Then for all $y \in X$, we observe that $A^x = A^{xy}\ \forall x, y \in X$ since

$$CM_{A^x}(z) = CM_A(xzx^{-1}) = CM_A(z)$$

and

$$CM_{A^{xy}}(z) = CM_A((xy)z(xy)^{-1}) = CM_A(z).$$

That is, $A^x = A$ whenever A is normal. Thus,

$$\begin{aligned}\bigwedge_{x\in X} CM_{A^x}(yzy^{-1}) &= \bigwedge_{x\in X} CM_A(xyzy^{-1}x^{-1})\\ &= \bigwedge_{x\in X} CM_A((xy)z(xy)^{-1})\\ &= \bigwedge_{x\in X} CM_{A^{xy}}(z)\\ &= \bigwedge_{x\in X} CM_{A^x}(z)\ \forall y, z \in X.\end{aligned}$$

Hence, $\bigcap_{x\in X} A^x$ is a normal fuzzy submultigroup of G. Now, let B be a normal fuzzy submultigroup of G such that $B \subseteq A$. Then $B = B^x \subseteq A^x\ \forall x \in X$. Thus, $B \subseteq \bigcap_{x\in X} A^x$. Therefore, $\bigcap_{x\in X} A^x$ is the largest normal fuzzy submultigroup of G that is contained in A. □

Definition 3.3. Let A be a submultiset of $B \in FMG(X)$. Then the normalizer of A in B is the set given by

$$N(A) = \{g \in X \mid CM_A(gy) = CM_A(yg)\ \forall y \in X\}.$$

Theorem 3.5. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $N(A)$ is a subgroup of X.*

Proof. Let $g, h \in N(A)$. Then

$$CM_{A^{gh}}(x) = CM_{(A^h)^g}(x) \quad = \quad CM_{A^h}(x) = CM_A(x) \forall x \in X$$

since $CM_{A^g}(x) = CM_A(g^{-1}xg) = CM_A(x)$. Hence, $gh \in N(A)$. Again, let $g \in N(A)$. We show that $g^{-1} \in N(A)$. For all $y \in X$, $CM_A(gy) = CM_A(yg)$ and so $CM_A((gy)^{-1}) = CM_A((yg)^{-1})$. Thus, for all $y \in X$,

$$CM_A(y^{-1}g^{-1}) = CM_A(g^{-1}y^{-1})$$

and so $CM_A(yg^{-1}) = CM_A(g^{-1}y)$, since $CM_A(y) = CM_A(y^{-1})$. Thus,
$g^{-1} \in N(A)$. Hence, $N(A)$ is a subgroup of X. □

Theorem 3.6. *Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then A is a normal fuzzy submultigroup of B if and only if $N(A) = X$.*

Proof. Let A be a normal fuzzy submultigroup of B and $g \in X$. Then $\forall x \in X$, we have

$$\begin{aligned} CM_{A^g}(x) &= CM_A(g^{-1}xg) = CM_A((g^{-1}x)g) \\ &= CM_A(g(g^{-1}x)) = CM_A(x). \end{aligned}$$

Thus, $CM_{A^g}(x) = CM_A(x)$ and so $g \in N(A)$. Therefore, $N(A) = X$.

Conversely, suppose $N(A) = X$. Let $x, y \in X$. To prove that A is a normal fuzzy submultigroup of B, it is sufficient we show that $CM_A(xy) = CM_A(yx)$. Now

$$\begin{aligned} CM_A(xy) &= CM_A(xyxx^{-1}) = CM_A(x(yx)x^{-1}) \\ &= CM_{A^{x^{-1}}}(yx) = CM_A(yx), \end{aligned}$$

where the last equality follows since $N(A) = X$ and so, $x^{-1} \in N(A)$. Hence, $CM_{A^{x^{-1}}}(y) = CM_A(y)$ (that is, $A^{x^{-1}} = A = A^x$). Therefore, A is a normal fuzzy submultigroup of B. □

Remark. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then $S = N(A) = T$, if

$$S = \{x \in X \mid CM_A(xy(yx)^{-1}) = CM_A(e)\ \forall y \in X\}$$

and

$$T = \{x \in X \mid CM_A(xyx^{-1}) = CM_A(y)\ \forall y \in X\}.$$

Theorem 3.7. *Let A, B and C be fuzzy multigroups of an abelian group X such that $A \subseteq B \subseteq C$. Then*

(i) $N(A) \cap N(B) \subseteq N(A \cap B)$.
(ii) $N(A) \cap N(B) \subseteq N(A \circ B)$.

Proof. (i) Let $y \in N(A)$ and $y \in N(B) \Rightarrow y \in N(A) \cap N(B)$. For any $x, y \in X$, we get $CM_{A\cap B}(xy) = CM_{A\cap B}(yx) \Rightarrow CM_{A\cap B}(xyx^{-1}) = CM_{A\cap B}(y)$. Now,

$$\begin{aligned} CM_{A\cap B}(xyx^{-1}) &= CM_A(xyx^{-1}) \wedge CM_B(xyx^{-1}) \\ &= CM_A(yxx^{-1}) \wedge CM_B(yxx^{-1}) \\ &= CM_A(y) \wedge CM_B(y) \\ &= CM_{A\cap B}(y). \end{aligned}$$

Thus, $y \in N(A \cap B)$. Hence, $N(A) \cap N(B) \subseteq N(A \cap B)$.

(ii) Let $y \in N(A) \cap N(B)$, that is $y \in N(A)$ and $y \in N(B)$. Then for all $x \in X$,

$$\begin{aligned} CM_{A\circ B}(y) &= \bigvee_{y=ab} (CM_A(a) \wedge CM_B(b)), \forall a, b \in X \\ &= \bigvee_{y=ab} (CM_A(x^{-1}ax) \wedge CM_B(x^{-1}bx)), \forall a, b \in X \\ &\leq \bigvee_{x^{-1}yx=cd} (CM_A(c) \wedge CM_B(d)), \forall c, d \in X \\ &= CM_{A\circ B}(x^{-1}yx) \end{aligned}$$

$\Rightarrow CM_{A\circ B}(y) \leq CM_{A\circ B}(x^{-1}yx)$. The inequality holds since

$$y = ab \Rightarrow x^{-1}abx = cd \Rightarrow ab = xcdx^{-1} = (xcx^{-1})(xdx^{-1})$$

and since $a = xcx^{-1}$ and $b = xdx^{-1}$ imply $x^{-1}ax = c$ and $x^{-1}bx = d$. Again,

$$CM_{A\circ B}(x^{-1}yx) \leq CM_{A\circ B}(x(x^{-1}yx)x^{-1}) = CM_{A\circ B}(y).$$

So, $CM_{A\circ B}(y) \geq CM_{A\circ B}(x^{-1}yx)$. Thus,

$$CM_{A\circ B}(y) = CM_{A\circ B}(x^{-1}yx).$$

Hence, $y \in N(A \circ B)$. Therefore, $N(A) \cap N(B) \subseteq N(A \circ B)$. □

Corollary 3.1. *Let $A, B, C \in FMG(X)$ such that $A \subseteq B \subseteq C$ and $CM_A(e) = CM_B(e)$. Then $N(A) \cap N(B) = N(A \cap B)$.*

Proof. Recall that

$$\begin{aligned} N(A) &= \{x \in X \mid CM_A(xy) = CM_A(yx)\ \forall y \in X\} \\ &= \{x \in X \mid CM_A(xyx^{-1}y^{-1}) = CM_A(e)\ \forall y \in X\}. \end{aligned}$$

Let $y \in N(A \cap B)$. Then from the definition of $N(A)$, for all $x \in X$ we get

$$\begin{aligned} CM_{A\cap B}(xyx^{-1}y^{-1}) &= CM_A(xyx^{-1}y^{-1}) \wedge CM_B(xyx^{-1}y^{-1}) \\ &= CM_A(e) \wedge CM_B(e), \end{aligned}$$

implies $y \in N(A)$ and $y \in N(B)$. Thus, $y \in N(A) \cap N(B)$ since

$$CM_A(xyx^{-1}y^{-1}) = CM_A(e) \Rightarrow CM_A(xy) = CM_A(yx)$$

and similarly in the case of B because $CM_A(e) = CM_B(e)$. Hence, it follows that $N(A) \cap N(B) = N(A \cap B)$. □

Remark. If A and B are fuzzy submultigroups of $C \in FMG(X)$ such that $A \subseteq B$. Then $N(A) \subseteq N(B)$.

Definition 3.4. Let A be a fuzzy submultigroup of $G \in FMG(X)$. Then the fuzzy submultiset yA of G for $y \in X$ defined by

$$CM_{yA}(x) = CM_A(y^{-1}x)\ \forall x \in X$$

is called the left fuzzy comultiset of A. Similarly, the fuzzy submultiset Ay of G for $y \in X$ defined by

$$CM_{Ay}(x) = CM_A(xy^{-1})\ \forall x \in X$$

is called the right fuzzy comultiset of A.

Proposition 3.7. *Let A be a normal fuzzy submultigroup of $B \in FMG(X)$. Then $CM_{xA}(xz) = CM_{xA}(zx) = CM_A(z)\ \forall x, z \in X$.*

Proof. Let $x, z \in X$. Suppose A is a normal fuzzy submultigroup of B, then by Proposition 3.3 and Definition 3.4, we get

$$CM_{xA}(xz) = CM_{xA}(zx) = CM_A(x^{-1}zx) = CM_A(z).$$

Hence,

$$CM_{xA}(xz) = CM_{xA}(zx) = CM_A(z) \; \forall z \in X.$$

□

Theorem 3.8. *Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then A is a normal fuzzy submultigroup of B if and only if for all $x \in X$, $Ax = xA$.*

Proof. Suppose A is a normal fuzzy submultigroup of B. Then for all $x \in X$, we have

$$\begin{aligned} CM_{Ax}(y) &= CM_A(yx^{-1}) = CM_A(x^{-1}y) \\ &= CM_{xA}(y) \; \forall y \in X. \end{aligned}$$

Thus, $Ax = xA$.

Conversely, let $Ax = xA$ for all $x \in X$. We get,

$$\begin{aligned} CM_A(xy) &= CM_{x^{-1}A}(y) = CM_{Ax^{-1}}(y) \\ &= CM_A(yx) \; \forall y \in X. \end{aligned}$$

Hence, A is a normal fuzzy submultigroup of B by Proposition 3.3. □

Theorem 3.9. *Let X be a finite group and A be a fuzzy submultigroup of $B \in FMG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid CM_A(g) = CM_A(e)\}, \\ K &= \{x \in X \mid CM_{Ax}(y) = CM_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again, $H = K$.

Proof. Let $g, h \in H$. Then

$$\begin{aligned} CM_A(gh) &\geq CM_A(g) \wedge CM_A(h) \\ &= CM_A(e) \wedge CM_A(e) \\ &= CM_A(e) \end{aligned}$$

$\Rightarrow CM_A(gh) \geq CM_A(e)$.

But, $CM_A(gh) \leq CM_A(e)$ from Definition 2.6. Thus, $CM_A(gh) = CM_A(e)$, implying that $gh \in H$. Since X is finite, it follows that H is a subgroup of X.

Now, we show that $H = K$. Let $k \in K$. Then for $y \in X$ we get

$$CM_{Ak}(y) = CM_{Ae}(y) \Rightarrow CM_A(yk^{-1}) = CM_A(y).$$

Choosing $y = e$, we obtain

$$CM_A(k^{-1}) = CM_A(e) \Rightarrow k^{-1} \in H,$$

and so, $k \in H$ since H is a subgroup of X. Thus, $K \subseteq H$.

Again, let $h \in H$. Then $CM_A(h) = CM_A(e)$. Also,

$$CM_{Ah}(y) = CM_A(yh^{-1})\ \forall y \in X$$

and

$$CM_{Ae}(y) = CM_A(y)\ \forall y \in X.$$

Thus, to show that $h \in K$, it suffices to prove that

$$CM_A(yh^{-1}) = CM_A(y)\ \forall y \in X.$$

Now,

$$\begin{aligned} CM_A(yh^{-1}) &\geq CM_A(y) \wedge CM_A(h^{-1}) \\ &= CM_A(y) \wedge CM_A(h) \\ &= CM_A(y) \wedge CM_A(e) \\ &= CM_A(y). \end{aligned}$$

Again,

$$\begin{aligned} CM_A(y) &= CM_A(yh^{-1}h) \\ &\geq CM_A(yh^{-1}) \wedge CM_A(h) \\ &= CM_A(yh^{-1}) \wedge CM_A(e) \\ &= CM_A(yh^{-1}) \end{aligned}$$

$\Rightarrow CM_A(yh^{-1}) = CM_A(y)$, thus $H \subseteq K$. Hence, $H = K$. Therefore, K is a subgroup of X. □

Corollary 3.2. *With the same notation as in Theorem 3.9, H is a normal subgroup of X if A is a normal fuzzy submultigroup of B.*

Proof. Let $y \in X$ and $x \in H$. Then

$$\begin{aligned} CM_A(yxy^{-1}) &= CM_A(yy^{-1}x) \text{ since } A \text{ is normal in } B \\ &= CM_A(x) = CM_A(e). \end{aligned}$$

Thus, $yxy^{-1} \in H$. Hence, H is normal in X. □

Definition 3.5. Let A and B be fuzzy submultigroups of $C \in FMG(X)$. Then the commutator of A and B is the fuzzy multiset (A, B) of X defined as follows:

$$CM_{(A,B)}(x) = \begin{cases} \bigvee_{x=[a,b]}\{CM_A(a) \wedge CM_B(b)\}, & \text{if } x \text{ is a commutator in } X \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$CM_{(A,B)}(x) = \bigvee_{x=aba^{-1}b^{-1}} \{CM_A(a) \wedge CM_B(b)\}.$$

Since the supremum of an empty set is zero, $CM_{(A,B)}(x) = 0$ if x is not a commutator.

Definition 3.6. Let A and B be fuzzy submultigroups of $C \in FMG(X)$. Then the commutator fuzzy multigroup of A and B is the fuzzy multigroup generated by the commutator (A, B). It is denoted by $[A, B]$.

Definition 3.7. Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then the fuzzy submultigroup of B generated by A is the least fuzzy submultigroup of B containing A. It is denoted by $< A >$. That is

$$< A >= \bigcap \{A_i \in FMG(X) | CM_A(x) \leq CM_{A_i}(x)\}.$$

With the aid of Definitions 3.5 and 3.6, we obtain the result that follows.

Theorem 3.10. *Let A and B be normal fuzzy submultigroups of* $C \in FMG(X)$. *Then* $[A, B] \subseteq A \cap B$.

Proof. Let $x \in X$. Now if x is not a commutator, then $CM_{(A,B)}(x) = 0$ and therefore there is nothing to prove. Suppose that $x = aba^{-1}b^{-1}$ for some $a, b \in X$. Then

$$\begin{aligned}
CM_{A\cap B}(x) &= CM_A(x) \wedge CM_B(x) \\
&= CM_A(aba^{-1}b^{-1}) \wedge CM_B(aba^{-1}b^{-1}) \\
&\geq (CM_A(a) \wedge CM_A(ba^{-1}b^{-1})) \wedge (CM_B(aba^{-1}) \wedge CM_B(b^{-1})) \\
&\geq (CM_A(a) \wedge CM_C(b)) \wedge (CM_B(b) \wedge CM_C(a)) \\
&= CM_A(a) \wedge CM_B(b).
\end{aligned}$$

This implies that

$$\begin{aligned}
CM_{A\cap B}(x) &\geq \bigvee_{x=aba^{-1}b^{-1}} CM_A(a) \wedge CM_B(b) \\
&= CM_{(A,B)}(x).
\end{aligned}$$

Consequently, $CM_{A\cap B}(x) \geq CM_{(A,B)}(x)$. Thus $[A, B] \subseteq A \cap B$. □

4. Conclusion

We have introduced and also studied the concept of normal fuzzy submultigroups of a fuzzy multigroup and explored some of its properties. Also, the ideas of commutator and normalizer in fuzzy multigroup setting were proposed and some related results were established. However, more properties of normal fuzzy submultigroups could still be exploited.

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