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On Orthogonal Polynomials

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Abstract

The aim of this article is to present some general properties of orthogonal sequence of functions, respectively orthogonal polynomials. We obtain some connections between the classical orthogonal polynomials: Legendre, Chebyshev, Laguerre and Hermite.

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1. Introduction

The (classical) orthogonal polynomials represent one of the most important problems, extensively investigated in the literature from different points of view (see (Carlitz, 1968), (Chihara, 2011), (Dominici & Maier, 2008), (Grinshpun, 2004), (Mason & Handscomb, 2002) and references therein).

In (McCarthy *et al.*, 1993) the authors study the Dirichlet polynomials as a generalization of the Legendre polynomials and prove a formula of Rodrigues type and some results for the zeros of the generalized Legendre polynomials and an interesting approach is made in (Sun, 2014).

Relations for products of Chebyshev polynomials and the related generating functions are shown in (Cesarano, 2012) and in (Siyi, 2015) some connections between the Chebyshev polynomials, Fibonacci numbers and Lucas numbers are emphasized.

The Laguerre polynomials have been studied in mathematical physics, combinatorics and special functions (see (Koeph, 1997), (Micu & Papp, 2005)) and T. Kim, D. S. Kim, K. W. Hwang and J. J. Seo ((Kim *et al.*, 2016)) derive a family of ordinary differential equations from the generating function of the Laguerre polynomials and prove new identities for those polynomials.

Y. He and F. Yang ((He & Yang, 2018)) obtain recurrence formulas for the Hermite polynomials using generating function methods and Padé approximation techniques and in (Kim & Kim, 2013) a formula for a product of two Hermite polynomials is given. Also, recently, the differential equations associated with squared Hermite polynomials are treated in (Kim *et al.*, 2017).

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In this paper we treat the orthogonal sequence of functions and the (classical) orthogonal polynomials. Some relations for the classical orthogonal polynomials (Legendre, Chebyshev, Laguerre, Hermite) are proved.

2. The main results

Definition 2.1. We consider $\rho : (a, b) \to \mathbb{R}$ a nonnegative function on (a, b) and $\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a, b)$. We assume that there exist $\int\limits_a^b f_n^2(x)\rho(x)\,dx$. The mapping $\langle\cdot,\cdot\rangle : L^2_{\mathbb{R}}(a, b) \times L^2_{\mathbb{R}}(a, b) \to \mathbb{R}$,

$$\langle f_m, f_n\rangle = \int\limits_a^b f_m(x) f_n(x)\rho(x)\,dx$$

is *inner product* on $L^2_{\mathbb{R}}(a, b)$ and the function ρ is called *the weight function* of the inner product.

Proposition 1. *The inner product from Definition 2.1 satisfies the following properties:*

(*i*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}}, \{f_p\}_{p\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a, b) : \langle f_m + f_n, f_p\rangle = \langle f_m, f_p\rangle + \langle f_n, f_p\rangle;$

(*ii*) $\forall\alpha \in \mathbb{R},\ \forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a, b) : \langle\alpha f_m, f_n\rangle = \alpha\langle f_m, f_n\rangle;$

(*iii*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a, b) : \langle f_m, f_n\rangle = \langle f_n, f_m\rangle;$

(*iv*) $\forall\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a, b) : \langle f_n, f_n\rangle \geq 0.$

Definition 2.2. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is called *orthogonal* if

$$\langle f_m, f_n\rangle = 0,\ \forall m \neq n.$$

Proposition 2. *Any orthogonal sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *on* (a, b) *is a linearly independent system.*

Theorem 2.1. *Any linearly independent sequence of functions* $\{f_n\}_{n\in\mathbb{N}}$ *can be orthogonalized.*

Definition 2.3. The orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}}$ on (a, b) are called *classical orthogonal polynomials* relative to the weight function ρ if the following differential equation is checked:

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x),$$

where τ is a polynomial of degree 1 and σ is given by:

$$\sigma(x) = \begin{cases} (x-a)(b-x), & \text{if } a, b \in \mathbb{R} \\ x - a, & \text{if } a \in \mathbb{R}, b = \infty \\ b - x, & \text{if } a = -\infty, b \in \mathbb{R} \\ 1, & \text{if } a = -\infty, b = \infty \end{cases}$$

and

$$\lim_{x\to a} x^n\sigma(x)\rho(x) = \lim_{x\to b} x^n\sigma(x)\rho(x) = 0,\ \forall n \in \mathbb{N}.$$

Remark 1. From Definition 2.3, we obtain the expression of the function ρ:

$$\rho(x) = \begin{cases} (x-a)^{\alpha}(b-x)^{\beta}, & \alpha = \frac{\tau(a)}{b-a} - 1, \beta = -\frac{\tau(b)}{b-a} - 1 \ (a, b \in \mathbb{R}) \\ (x-a)^{\alpha} e^{x\tau'(x)}, & \alpha = \tau(a) - 1 \ (a \in \mathbb{R}, b = \infty) \\ (b-x)^{\beta} e^{-x\tau'(x)}, & \beta = -\tau(b) - 1 \ (a = -\infty, b \in \mathbb{R}) \\ e^{\int \tau(x)dx}, & a = -\infty, b = \infty \end{cases}$$

Remark 2. In particular, considering in Definition 2.3

- $(a, b) = (-1, 1)$ and $\rho(x) = 1$, $\sigma(x) = 1 - x^2$, $\tau(x) = -2x$, $\forall x \in (-1, 1)$, we obtain *the Legendre polynomials* denoted P_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \dfrac{1}{\sqrt{1-x^2}}$, $\sigma(x) = 1-x^2$, $\tau(x) = -x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the first kind* denoted T_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \sqrt{1-x^2}$, $\sigma(x) = 1-x^2$, $\tau(x) = -3x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the second kind* denoted U_n;
- $(a, b) = (0, \infty)$ and $\rho(x) = x^{\alpha} e^{-x}$, $\sigma(x) = x$, $\tau(x) = -x + \alpha + 1$, $\forall x \in (0, \infty)$ (where $\alpha > -1$), we obtain *the Laguerre polynomials* denoted L_n^{α};
- $(a, b) = (-\infty, \infty)$ and $\rho(x) = e^{-x^2}$, $\sigma(x) = 1$, $\tau(x) = -2x$, $\forall x \in (-\infty, \infty)$, we obtain *the Hermite polynomials* denoted H_n.

Remark 3. Also, the expression of the Chebyshev polynomials of the first, respectively second kind, can be given by:

$$T_n(x) = \cos(n \arccos x), \quad \text{respectively} \quad U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N}.$$

Now we emphasize some properties of the Legendre, Chebyshev, Laguerre and Hermite polynomials (see (Bochner, 1929), (Szego, 1939)).

Proposition 3. *The Legendre polynomials satisfy the following:*

(i) $P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n-k}^k \dfrac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-2k-1)}{(n-k)! 2^k} x^{n-2k}$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(ii) $P_n(x) = \dfrac{1}{2^n n!} \cdot \dfrac{d^n}{dx^n}(x^2-1)^n$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(iii) $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}$;

(iv) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$, $\forall x \in (-1, 1)$, $\forall n \in \mathbb{N}^*$;

(v) $\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$

Proposition 4. *The Chebyshev polynomials of the first kind verify the following:*

$$(i)\ \ T_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} x^{n-2k}(1-x^2)^k,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(ii)\ \ T_n(x) = \frac{(-1)^n\sqrt{1-x^2}}{(2n-1)!!}\cdot\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}},\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(iii)\ \ (1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(iv)\ \ T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

$$(v)\ \ \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \neq 0 \\ \pi, & \text{if } m = n = 0. \end{cases}$$

Proposition 5. *The Chebyshev polynomials of the second kind verify the following:*

$$(i)\ \ U_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n+1}^{2k+1} x^{n-2k}(1-x^2)^k,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(ii)\ \ U_n(x) = \frac{(-1)^n(n+1)}{(2n+1)!!\sqrt{1-x^2}}\cdot\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}},\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(iii)\ \ (1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N};$$

$$(iv)\ \ U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

$$(v)\ \ \int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases}$$

Proposition 6. *The Laguerre polynomials verify the following:*

$$(i)\ \ L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} C_{n+\alpha}^{n-k} x^k,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$$

$$(ii)\ \ L_n^\alpha(x) = \frac{1}{n!}x^{-\alpha}e^x\cdot\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}),\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$$

$$(iii)\ \ x(L_n^\alpha(x))'' + (1+\alpha-x)(L_n^\alpha(x))' + nL_n^\alpha(x) = 0,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N};$$

$$(iv)\ \ (n+1)L_{n+1}^\alpha(x) + (x-2n-1-\alpha)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N}^*.$$

$$(v)\ \ \int_{0}^{\infty} L_m^\alpha(x)L_n^\alpha(x)x^\alpha e^{-x}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{1}{n!}\Gamma(n+1+\alpha), & \text{if } m = n. \end{cases}$$

Proposition 7. *The Hermite polynomials satisfy the following:*

$$(i)\ \ H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k}\cdot(k+1)\cdot(k+2)\cdot\ldots\cdot(2k)\cdot(2x)^{n-2k},\ \forall x \in (-\infty,\infty),\ \forall n \in \mathbb{N};$$

$$(ii)\ \ H_n(x) = (-1)^n e^{x^2}\cdot\frac{d^n}{dx^n}(e^{-x^2}),\ \forall x \in (-\infty,\infty),\ \forall n \in \mathbb{N};$$

(*iii*) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$, $\forall x \in (-\infty, \infty)$, $\forall n \in \mathbb{N}$;

(*iv*) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$, $\forall x \in (-\infty, \infty)$, $\forall n \in \mathbb{N}^*$;

(*v*) $\int\limits_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 2^n n!\,\sqrt{\pi}, & \text{if } m = n. \end{cases}$

Further, we show other connections between the classical orthogonal polynomials indicated in Remark 2.

Proposition 8. *For the Legendre polynomials, the following relations hold:*

(*i*) $xP'_{n+1}(x) - P'_n(x) = (n+1)P_{n+1}(x)$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$;

(*ii*) $P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$.

Proof. Using Proposition 3, we deduce

$$P'_n(x) = \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2}, \ \forall x \in (-1,1), \ \forall n \in \mathbb{N}^*.$$

Thus, making the computations, we obtain:
(*i*)

$$\begin{aligned} &xP'_{n+1}(x) - P'_n(x) = \\ &= x \cdot \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1-x^2} - \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2} = \\ &= \frac{(2n+1)xP_n(x) - nP_{n-1}(x) - nx^2P_{n+1}(x) - x^2P_{n+1}(x)}{1-x^2} = \\ &= \frac{(n+1)P_{n+1}(x) - x^2(n+1)P_{n+1}(x)}{1-x^2} = (n+1)P_{n+1}(x), \ \forall x \in (-1,1), \ \forall n \in \mathbb{N}^*; \end{aligned}$$

(*ii*)

$$\begin{aligned} P'_{n+1}(x) - xP'_n(x) &= \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1-x^2} - x \cdot \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2} = \\ &= \frac{(n+1)P_n(x) - nxP_{n+1}(x) + nx^2P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x)}{1-x^2} = \\ &= \frac{(n+1)P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x) + xP_{n+1}(x) - nx^2P_n(x) - x^2P_n(x) + nxP_{n-1}(x)}{1-x^2} = \\ &= \frac{(n+1)P_n(x) - x^2(n+1)P_n(x)}{1-x^2} = (n+1)P_n(x), \ \forall x \in (-1,1), \ \forall n \in \mathbb{N}^*. \end{aligned}$$

□

Proposition 9. *The Chebyshev polynomials of the first, respectively second kind, verify:*

(*i*) $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$;

(*ii*) $U_n(x) - 2T_n(x) - U_{n-2}(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$.

Proof. From Remark 3, after some computations we obtain:

$$U_n(x) = \frac{T'_{n+1}(x)}{n+1}, \forall x \in (-1,1),\ \forall n \in \mathbb{N}.$$

(i) We have:

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - 2x\frac{T'_n(x)}{n} + \frac{T'_{n-1}(x)}{n-1} =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) + \sin((n-1)\arccos x) - 2x\sin(n\arccos x)] =$$

$$= \frac{1}{\sqrt{1-x^2}}[2x\sin(n\arccos x) - 2x\sin(n\arccos x)] = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(ii) Similarly with (i), we deduce:

$$U_n(x) - 2T_n(x) - U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} - 2T_n(x) =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) - \sin((n-1)\arccos x)] - 2\cos(n\arccos x) =$$

$$= \frac{1}{\sqrt{1-x^2}}2\sin(\arccos x)\cos(n\arccos x) - 2\cos(n\arccos x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

□

Proposition 10. *For the Hermite polynomials, the following relations are checked:*

(i) $H'_{n+1}(x) + H'_{n-1}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x)$*,* $\forall x \in (-\infty,\infty)$*,* $\forall n \geq 2$*;*

(ii) $H'_{n+1}(x) - 2xH'_n(x) = 2H_n(x) - 4n(n-1)H_{n-2}(x)$*,* $\forall x \in (-\infty,\infty)$*,* $\forall n \geq 2$*.*

Proof. From Proposition 7, $\forall x \in (-\infty,\infty)$, $\forall n \in \mathbb{N}^*$ it results

$$H'_n(x) = 2nH_{n-1}(x)$$

and then
(i)

$$H'_{n+1}(x) + H'_{n-1}(x) = 2(n+1)H_n(x) + 2(n-1)H_{n-2}(x) =$$

$$= (2n+1)H_n(x) + H_n(x) + 2(n-1)H_{n-2}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x);$$

(ii)

$$H'_{n+1}(x) - 2xH'_n(x) = 2(n+1)H_n(x) - 4nxH_{n-1}(x) =$$

$$= 2H_n(x) + 2n(H_n(x) - 2xH_{n-1}(x)) = 2H_n(x) - 4n(n-1)H_{n-2}(x).$$

□

Proposition 11. *The classical orthogonal polynomials from Remark 2 satisfy:*

(i) $(n+1)P_{n+1}(x) + (1-x^2)P'_n(x) - x(n+1)P_n(x) = 0$*,* $\forall x \in (-1,1)$*,* $\forall n \in \mathbb{N}^*$*;*

(ii) $nT_{n+1}(x) + (1-x^2)T'_n(x) - nxT_n(x) = 0$*,* $\forall x \in (-1,1)$*,* $\forall n \in \mathbb{N}^*$*;*

(*iii*) $nU_{n+1}(x) + (1 - x^2)U_n'(x) - nxU_n(x) - U_{n-1}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*$;

(*iv*) $(n+1)L_{n+1}^{\alpha+1}(x) - (L_n^{\alpha}(x))' - (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n+\alpha)L_{n-1}^{\alpha+1}(x) = 0,\ \forall x \in (0, \infty), \alpha > -1,\ \forall n \in \mathbb{N}^*$;

(*v*) $H_{n+1}(x) + H_n'(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*$.

Proof. (*i*) From Proposition 3, (*ii*), respectively (*iv*), we obtain

$$(1 - x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x),$$

respectively

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

and then

$$(n + 1)P_{n+1}(x) + (1 - x^2)P_n'(x) + (nx - 2n - 1)P_n(x) =$$
$$= (2n + 1)xP_n(x) - nP_{n-1}(x) + nP_{n-1}(x) - nxP_n(x) - x(n + 1)P_n(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*ii*) From Proposition 4, (*ii*), respectively (*iv*), we deduce

$$(1 - x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x),$$

respectively

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

and then

$$nT_{n+1}(x) + (1 - x^2)T_n'(x) - nxT_n(x) =$$
$$= 2nxT_n(x) - nT_{n-1}(x) - nxT_n(x) + nT_{n-1}(x) - nxT_n(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*iii*) From Proposition 5, (*ii*), respectively (*iv*), we observe

$$(1 - x^2)U_n'(x) = -nxU_n(x) + (n + 1)U_{n-1}(x),$$

respectively

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

and then

$$nU_{n+1}(x) + (1 - x^2)U_n'(x) - nxU_n(x) - U_{n-1}(x) =$$
$$= 2nxU_n(x) - nU_{n-1}(x) - nxU_n(x) + (n + 1)U_{n-1}(x) - nxU_n(x) - U_{n-1}(x) = 0, \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

(*iv*) From Proposition 6, (*ii*), respectively (*iv*), it results

$$(L_n^{\alpha})'(x) = -L_{n-1}^{\alpha+1}(x),$$

respectively

$$(n + 1)L_{n+1}^{\alpha} = (\alpha - x + 2n + 1)L_n^{\alpha}(x) - (n + \alpha)L_{n-1}^{\alpha}(x)$$

and then

$$(n + 1)L_{n+1}^{\alpha+1}(x) - (L_n^{\alpha}(x))' - (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n + \alpha)L_{n-1}^{\alpha+1}(x) =$$
$$= (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) - (n + \alpha + 1)L_{n-1}^{\alpha+1}(x) + L_{n-1}^{\alpha+1}(x) -$$
$$-(\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n + \alpha)L_{n-1}^{\alpha+1}(x) = 0,\ \forall x \in (0, \infty),\ \alpha > -1,\ \forall n \in \mathbb{N}^*.$$

(v) From Proposition 7, (ii), respectively (iv), it follows

$$H_n'(x) = 2nH_{n-1}(x),$$

respectively

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and then

$$H_{n+1}(x) + H_n'(x) - 2xH_n(x) = 2xH_n(x) - 2nH_{n-1}(x) + 2nH_{n-1}(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*.$$

□

Open problem: An important open problem is to define other orthogonal polynomials that fulfill similar conditions with those emphasized for the classical orthogonal polynomials in this article.

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