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Polynomial Stability in Average for Cocycles of Linear Operators

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Abstract

In the present paper we deal with the concept of polynomial stability in average. We obtain two characterization theorems that describe the concept mentioned above. In fact, we give a logarithmic criterion and a Datko type theorem for cocycles of linear operators.

Keywords: cocycles of linear operators, polynomial stability in average
2010 MSC: 34D05, 34D20 .

1. Introduction

The notion of exponential dichotomy in average was introduced by L.Barreira, D.Dragicevic and C. Valls in (Barreira *et al.*, 2016) and it includes the classical concept of uniform exponential dichotomy which appeared due to Perron (Perron, 1930).

A particular case of this dichotomy behavior was studied by Dragicevic in (Dragicevic, 2016). He obtained a version of a well known theorem of R. Datko for the notion of the exponential stability in average for cocycles over flows and also for cocycles over maps.

In (Hai, 2019) the author uses the theory of Banach function spaces to characterize polynomially bounded stochastic skew evolution semiflows. He talks about polynomial stability and polynomial instability in mean. He states and proves results which are continuous or discrete-time versions of the Datko type characterization theorems.

The objective of this paper is to find for cocycles of linear operators, similar approaches as in the exponential case, for the classical polynomial stability concept, that has been studied in many papers (Barreira & Valls, 2009), (Hai, 2015), (Megan *et al.*, 2003). In fact, we consider cocycles acting on functions from L^1 and we give a version of a logarithmic criterion for the concept of polynomial stability in average which is similar to the one obtained for the classical uniform polynomial stability concept. Also, we prove a Datko type characterization theorem for the concept mentioned.

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2. Preliminaries

Let $(\Omega, \mathrm{B}, \mu)$ be a probability space. We consider the sets $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$ and $T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$.

Definition 2.1. An application $\varphi : \mathbb{R}_+ \times \Omega \to \Omega$ is called a semiflow on Ω if:

(s_1) $\varphi(0,\omega) = \omega$, for all $\omega \in \Omega$.

(s_2) $\varphi(t+s,\omega) = \varphi(t, \varphi(s,\omega))$, for all $(t,s,\omega) \in \Delta \times \Omega$.

Let X be a Banach space and $\mathrm{B}(X)$ the Banach algebra of all bounded linear operators acting on X.

Definition 2.2. An application $\Phi : \mathbb{R}_+ \times \Omega \to \mathrm{B}(X)$ is called a cocycle over the semiflow φ if:

(c_1) $\forall x \in X$ the mapping $(t,\omega) \mapsto \Phi(t,\omega)x$ is Bochner measurable.

(c_2) $\forall (t,\omega) \in \mathbb{R}_+ \times \Omega,\ \exists\, \Phi(t,\omega)^{-1}$.

(c_3) $\Phi(0,\omega) = I, \forall \omega \in \Omega$, where I is the identity operator on X.

(c_4) $\Phi(t+s,\omega) = \Phi(t, \varphi(s,\omega))\, \Phi(s,\omega), \forall t,s \geq 0, \forall \omega \in \Omega$.

Let $L^1(\Omega, X, \mu)$ be the Banach space of all Bochner measurable functions $x : \Omega \to X$ such that

$$||x||_1 := \int_\Omega ||x(\omega)||\, d\mu(\omega) < \infty.$$

In what follows, we will denote by

$$\Phi_\omega(t,s) = \Phi(t,\omega)\, \Phi(s,\omega)^{-1}, \forall t,s \geq 0, \forall \omega \in \Omega.$$

Remark 2.1. It is easy to see that an evolution property holds:

$$\Phi_\omega(t,t_0) = \Phi_\omega(t,s)\, \Phi_\omega(s,t_0),\ \forall t \geq s \geq t_0 \geq 0,\ \forall \omega \in \Omega.$$

Indeed, $\Phi_\omega(t,s)\, \Phi_\omega(s,t_0) = \Phi(t,\omega)\, \Phi(s,\omega)^{-1}\, \Phi(s,\omega)\, \Phi(t_0,\omega)^{-1} = \Phi(t,\omega)\, \Phi(t_0,\omega)^{-1} = \Phi_\omega(t,t_0)$.

3. Polynomial stability in average

Definition 3.1. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathrm{B}(X)$ is polynomially stable in average if there exist $N > 1$ and $\nu > 0$ such that

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)|| d\mu(\omega) \leq M \left(\frac{s+1}{t+1}\right)^\nu \int_\Omega ||x(\omega)||\, d\mu(\omega)$$

for all $(t,s,\omega) \in \Delta \times \Omega$.

Remark 3.1. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathrm{B}(X)$ is polynomially stable in average if and only if there exist $N > 1$ and $\nu > 0$ such that

$$\int_\Omega ||\Phi_\omega(t,t_0)x(\omega)|| d\mu(\omega) \leq N \left(\frac{s+1}{t+1}\right)^\nu \int_\Omega ||\Phi_\omega(s,t_0)x(\omega)||\, d\mu(\omega)$$

for all $(t,s,t_0,\omega) \in T \times \Omega$.

Definition 3.2. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is uniformly stable in average if there exists $N > 1$ such that

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)||d\mu(\omega) \leq N \int_\Omega ||x(\omega)||\, d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Remark 3.2. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ is uniformly stable in average if and only if there exist $N > 1$ such that

$$\int_\Omega ||\Phi_\omega(t,t_0)x(\omega)||d\mu(\omega) \leq N \int_\Omega ||\Phi_\omega(s,t_0)x(\omega)||\, d\mu(\omega)$$

for all $(t, s, t_0, \omega) \in T \times \Omega$.

Definition 3.3. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ has polynomial growth in average if there exist $M > 1$ and $\alpha > 0$ such that

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)||d\mu(\omega) \leq M\left(\frac{t+1}{s+1}\right)^\alpha \int_\Omega ||x(\omega)||\, d\mu(\omega)$$

for all $(t, s, \omega) \in \Delta \times \Omega$.

Remark 3.3. The cocycle $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{B}(X)$ has polynomial growth in average if and only if there exist $M > 1$ and $\alpha > 0$ such that

$$\int_\Omega ||\Phi_\omega(t,t_0)x(\omega)||d\mu(\omega) \leq M\left(\frac{t+1}{s+1}\right)^\alpha \int_\Omega ||\Phi_\omega(s,t_0)x(\omega)||\, d\mu(\omega)$$

for all $(t, s, t_0\omega) \in T \times \Omega$.

In what follows, we will present a logarithmic criterion for the concept of polynomial stability in average.

Theorem 3.1. *Let Φ be a cocycle with polynomial growth in average. Then Φ in polynomially stable in average if and only if there exist a constant $L > 1$ such that:*

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)||d\mu(\omega) \ln\frac{t+1}{s+1} \leq L \int_\Omega ||x(\omega)||\, d\mu(\omega)$$

for all $t, s \geq 1, \omega \in \Omega$.

Proof. Necessity. We suppose that Φ is polynomially stable in average. Then, there exist $N \geq 1$ and $\nu > 0$ such that

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)||d\mu(\omega) \ln\frac{t+1}{s+1} \leq N\left(\frac{t+1}{s+1}\right)^{-\nu} \ln\frac{t+1}{s+1} \int_\Omega ||x(\omega)||d\mu(\omega).$$

We consider the application $f : [1, \infty) \to \mathbb{R},\ f(u) = u^{-\nu}\ln u$, where $u = \dfrac{t+1}{s+1}$. Then, we obtain $f(u) \leq \dfrac{N}{\nu e}$. It results that

$$\int_\Omega ||\Phi_\omega(t,s)x(\omega)||d\mu(\omega) \ln\frac{t+1}{s+1} \leq L \int_\Omega ||x(\omega)||d\mu(\omega), \text{ where } L = 1 + \frac{N^2}{\nu e}.$$

Sufficiency. We denote by $S = \sup_{t \geq s \geq 1} \int_{\Omega} \|\Phi_\omega(t,s)x(\omega)\| \, d\mu(\omega) \ln \frac{t+1}{s+1}$.

Step 1. Let $n \overset{d}{=} \left[\frac{\ln\left(\frac{t}{s}\right)}{4S} \right]$, with $(t,s) \in \Delta$, $s \geq 1$. Then, the following inequalities hold:

(1) $se^{4nS} \leq t < se^{4(n+1)S}$

(2) $\left(\frac{t+1}{s+1}\right)^{\frac{\ln 2}{4S}} \leq 2^{n+1}$, for all $t \geq s \geq 1$.

Indeed, the first relation is a simple computation which uses the property of the whole part of a number and the second inequality results immediately from the first one.

Step 2. We prove that

$$\int_{\Omega} \|\Phi_\omega(se^{4S}, s)x(\omega)\| \, d\mu(\omega) \leq \frac{1}{2} \int_{\Omega} \|x(\omega)\| \, d\mu(\omega), \ \forall s \geq 1, \forall \omega \in \Omega$$

From $\frac{1+s^{4S}}{1+s} \geq e^{2S}$, it results that

$$\int_{\Omega} \|\Phi_\omega(se^{4S}, s)x(\omega)\| \, d\mu(\omega) \leq \frac{S}{\ln \frac{1+se^{4S}}{1+s}} \int_{\Omega} \|x(\omega)\| \, d\mu(\omega) \leq \frac{S}{\ln e^{2S}} \int_{\Omega} \|x(\omega)\| d\mu(\omega) = \frac{1}{2} \int_{\Omega} \|x(\omega)\| \, d\mu(\omega).$$

Step 3. We show that

$$\int_{\Omega} \|\Phi_\omega(se^{4nS}, s)x(\omega)\| \, d\mu(\omega) \leq \frac{1}{2^n} \int_{\Omega} \|x(\omega)\| d\mu(\omega), \forall s \geq 1, \forall \omega \in \Omega.$$

Indeed, using step 2, we have

$$\begin{aligned} \int_{\Omega} \|\Phi_\omega(se^{4nS}, s)x(\omega)\| \, d\mu(\omega) &= \int_{\Omega} \|\Phi_\omega(se^{4nS}, se^{4(n-1)S})\Phi(se^{4(n-1)S}, s)\|x(\omega)\| \, d\mu(\omega) \leq \\ &\leq \frac{1}{2} \int_{\Omega} \|\Phi_\omega(se^{4(n-1)S}, s)x(\omega)\| \, d\mu(\omega) \leq \cdots \leq \frac{1}{2^n} \int_{\Omega} \|x(\omega)\| d\mu(\omega). \end{aligned}$$

Step 4. We prove that Φ is polynomially stable in average using the evolution property proved in Remark [2.1] and the previous steps.

$$\begin{aligned} \int_{\Omega} \|\Phi_\omega(t,s)x(\omega)\| d\mu(\omega) &= \int_{\Omega} \|\Phi_\omega(t, se^{4nS}) \cdot \Phi_\omega(se^{4nS}, s)x(\omega)\| d\mu(\omega) \leq \\ &\leq M \left(\frac{1+t}{1+se^{4nS}}\right)^{\alpha} \cdot \frac{1}{2^n} \int_{\Omega} \|x(\omega)\| d\mu(\omega) \leq \frac{Me^{4\alpha S}}{2^n} \int_{\Omega} \|x(\omega)\| d\mu(\omega) \leq \\ &\leq 2Me^{4\alpha S} \left(\frac{t+1}{s+1}\right)^{\frac{-\ln 2}{4S}} \int_{\Omega} \|x(\omega)\| d\mu(\omega) \leq N \left(\frac{t+1}{s+1}\right)^{-\nu} \int_{\Omega} \|x(\omega)\| d\mu(\omega), \end{aligned}$$

where $N = 2Me^{4\omega S} > 1$ and $\nu = \dfrac{\ln 2}{4S} > 0$.

□

Next, we will give a Datko type theorem for the polynomial stability in average concept.

Theorem 3.2. *Let Φ be a cocycle with polynomial growth in average. Then Φ is polynomially stable in average if and only if there exist the constants $D > 1$ and $d \geq 0$ such that*

$$\int_s^\infty (\tau+1)^{d-1}\left(\int_\Omega \|\Phi_\omega(\tau,s)x(\omega)\|\, d\mu(\omega)\right) d\tau \leq D(s+1)^d \int_\Omega \|x(\omega)\|\, d\mu(\omega), \forall s \geq 0, \forall \omega \in \Omega. \tag{3.1}$$

Proof. Neccesity. We suppose that Φ is polynomially stable in average. Then, we have that there exist $N \geq 1, \nu > 0$ such that for all $d \in [0, \nu)$:

$$\int_s^\infty (\tau+1)^{d-1}\left(\int_\Omega \|\Phi_\omega(\tau,s)x(\omega)\|\, d\mu(\omega)\right) d\tau \leq N\int_s^\infty (\tau+1)^{d-1}\left(\frac{s+1}{\tau+1}\right)^\nu \int_\Omega \|x(\omega)\|\, d\mu(\omega) d\tau =$$

$$= N(s+1)^\nu \int_\Omega \|x(\omega)\|\, d\mu(\omega) \int_s^\infty (\tau+1)^{d-\nu-1} d\tau = \frac{N}{\nu-d}(s+1)^d \int_\Omega \|x(\omega)\|\, d\mu(\omega) \leq$$

$$\leq D(s+1)^d \int_\Omega \|x(\omega)\|\, d\mu(\omega), \text{ where } D = 1 + \frac{N}{\nu-d}.$$

Sufficiency. We suppose that there exist $D > 1$ and $d \geq 0$ such that the integral inequality form the theorem states. First, we discuss the case when $d > 0$.
If $t \geq 2s+1$ we have

$$(t+1)^d \int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\, d\mu(\omega) = \frac{2}{t+1}\int_{\frac{t-1}{2}}^t (t+1)^d \cdot \int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\, d\mu(\omega) d\tau \leq$$

$$\leq 2\int_{\frac{t-1}{2}}^t \frac{(t+1)^d}{\tau+1} \int_\Omega \|\Phi_\omega(t,\tau)\Phi_\omega(\tau,s)x(\omega)\|\, d\mu(\omega) d\tau \leq 2M \int_{\frac{t-1}{2}}^t \frac{(t+1)^d}{\tau+1}\left(\frac{t+1}{\tau+1}\right)^{d+\alpha} \int_\Omega \|\Phi_\omega(\tau,s)x(\omega)\|\, d\mu(\omega) d\tau \leq$$

$$\leq 2^{\alpha+d+1} \cdot M \int_s^\infty (\tau+1)^{d-1} \int_\Omega \|\Phi_\omega(\tau,s)x(\omega)\|\, d\mu(\omega) d\tau \leq N_1(s+1)^d \int_\Omega \|x(\omega)\|\, d\mu(\omega) d\tau.$$

If $t \in [s, 2s+1)$, we obtain that $\dfrac{t+1}{s+1} \leq 2$. Then using the growth property we obtain

$$(t+1)^d \int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\, d\mu(\omega) \leq (t+1)^d M\left(\frac{t+1}{s+1}\right)^\alpha \int_\Omega \|x(\omega)\|\, d\mu(\omega) \leq N_2(s+1)^d \int_\Omega \|x(\omega)\|\, d\mu(\omega).$$

So, it results that Φ is polynomially stable in average. Next, we deal with the case when $d = 0$. From (3.2), for $d = 0$ it results

$$\int_s^\infty \frac{1}{\tau+1}\left(\int_\Omega \|\Phi_\omega(\tau,t_0)x(\omega)\|\,d\mu(\omega)\right)d\tau \le D\int_\Omega \|\Phi_\omega(s,t_0)x(\omega)\|\,d\mu(\omega), \forall (s,t_0)\in\Delta, \forall\omega\in\Omega.$$

Step 1. We prove that Φ is uniformly stable in average. Indeed, we consider firstly $t \ge 2s+1$ and then $t \in [s, 2s+1)$ and using similar techniques as in the first case we obtain the conclusion.

Step 2. We show that Φ is polynomially stable in average using the logarithmic criterion.

$$\int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\,d\mu(\omega)\ln\frac{t+1}{s+1} = \int_s^t \frac{1}{\tau+1}\int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\,d\mu(\omega)d\tau \le \int_s^\infty \frac{1}{\tau+1}\int_\Omega \|\Phi_\omega(t,s)x(\omega)\|\,d\mu(\omega)d\tau \le$$
$$\le D\int_\Omega \|x(\omega)\|\,d\mu(\omega)$$

From Theorem [3.1] it results that the cocycle Φ is polynomially stable in average. □

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