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Nonuniform Generalized Exponential Dichotomies Concepts for Skew-evolution Semiflows

Claudia Luminiţa Mihiţ[a,*], Ghiocel Moţ[a]

[a]*Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, 2 Elena Drăgoi Str., 310330 Arad, Romania*

Abstract

The aim of the paper is to prove characterizations for two concepts of nonuniform dichotomy in the general context of skew-evolution semiflows.

We use invariant, respectively strongly invariant projector families, to obtain the results.

Keywords: Skew-evolution semiflows, nonuniform generalized exponential dichotomy, Banach spaces.

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1. Introduction

One of the most representative asymptotic properties studied for dynamical systems is the dichotomy, treated from various perspectives in (Barreira & Valls, 2018), (Barreira & Valls, 2019), (Bento *et al.*, 2017), (Găină *et al.*, 2023), (Megan *et al.*, 2007), (Sasu *et al.*, 2013).

The sufficient criteria for the uniform exponential stability of evolution operators, obtained by S. Rolewicz in (Rolewicz, 1986) represented an important direction to give qualitative results for the asymptotic behaviours of dynamical systems, using integral conditions.
In this sense, we mention the integral characterizations proved in (Mihiţ & Megan, 2017), for a general property of splitting with growth rates and recently, in (Megan *et al.*, 2025), Zabczyk-Rolewicz type methods are used for the uniform exponential stability of nonautonomous dynamics. Also, in (Sasu *et al.*, 2012), the uniform exponential stability of variational discrete systems, respectively skew-product flows are treated through Zabczyk-Rolewicz techniques.

Concerning the notion of generalized exponential dichotomy, it is introduced by J. S. Muldowney in (Muldowney, 1984) and in (Lupa *et al.*, 2015), the authors approach this property in the case of evolution operators.

In this article, the concepts of generalized exponential dichotomy and nonuniform generalized exponential dichotomy of Rolewicz type are studied for skew-evolution semiflows in Banach spaces. Characterizations for these properties are established, considering invariant and strongly invariant projector families.

*Corresponding author

E-mail addresses: `claudia.mihit@uav.ro` (Claudia Luminiţa Mihiţ), `ghiocel.mot@uav.ro` (Ghiocel Moţ)

2. Definitions and notations

Let Θ be a metric space, X a Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$.

Also, we consider

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}, \quad T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

and $\Gamma = \Theta \times X$.

Definition 2.1. A continuous mapping $\sigma : \Delta \times \Theta \to \Theta$ is called *evolution semiflow* if:

(es_1) $\sigma(s, s, \theta) = \theta$, for all $(s, \theta) \in \mathbb{R}_+ \times \Theta$;

(es_2) $\sigma(t, s, \sigma(s, t_0, \theta)) = \sigma(t, t_0, \theta)$, for all $(t, s, t_0, \theta) \in T \times \Theta$.

Definition 2.2. A pair $C = (\sigma, \Phi)$ is said to be a *skew-evolution semiflow* on Γ if σ is an evolution semiflow on Θ and $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ satisfies the relations:

(ses_1) $\Phi(s, s, \theta) = I$ (the identity operator on X), for all $(s, \theta) \in \mathbb{R}_+ \times \Theta$;

(ses_2) $\Phi(t, s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta) = \Phi(t, t_0, \theta)$, for all $(t, s, t_0, \theta) \in T \times \Theta$;

(ses_3) $(t, s, \theta) \mapsto \Phi(t, s, \theta)x$ is continuous for every $x \in X$.

Example 2.1. We consider Θ a locally compact metric space, X a Banach space, σ an evolution semiflow on Θ and $A : \Theta \to \mathcal{B}(X)$ a continuous mapping. If $\Phi(t, s, \theta)$ is the solution of the problem

$$\dot{x}(t) = A(\sigma(t, s, \theta))x(t), \quad t \geq s \geq 0,$$

then the pair $C = (\sigma, \Phi)$ is a skew-evolution semiflow on Γ.

In what follows, we recall the notions of family of projectors and (strongly) invariant family of projectors.

Definition 2.3. A continuous mapping $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is *family of projectors* if

$$P^2(t, \theta) = P(t, \theta), \quad \text{for all} \quad (t, \theta) \in \mathbb{R}_+ \times \Theta.$$

Remark. If $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is a family of projectors for $C = (\sigma, \Phi)$, then $Q : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$, $Q(t, \theta) = I - P(t, \theta)$ is also a family of projectors for C and it is called the *complementary family* of P.

Definition 2.4. A family of projectors $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is said to be

(i) *invariant* for a skew-evolution semiflow $C = (\sigma, \Phi)$ if:

$$P(t, \sigma(t, s, \theta))\Phi(t, s, \theta) = \Phi(t, s, \theta)P(s, \theta), \quad \text{for all } (t, s, \theta) \in \Delta \times \Theta;$$

(ii) *strongly invariant* for $C = (\sigma, \Phi)$ if it is invariant for C and for all $(t, s, \theta) \in \Delta \times \Theta$, the restriction $\Phi(t, s, \theta)$ is an isomorphism from *Range* $Q(s, \theta)$ to *Range* $Q(t, \sigma(t, s, \theta))$.

3. Nonuniform generalized exponential dichotomy

We consider $C = (\sigma, \Phi)$ a skew-evolution semiflow and $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projectors for $C = (\sigma, \Phi)$.

Also, $\mathcal{F}$ represents the set of the continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with:

$$\int_s^t f(\tau)d\tau \xrightarrow[t\to+\infty]{} +\infty, \quad s \geq 0 \quad \text{fixed.}$$

Definition 3.1. The pair (C, P) is *nonuniformly generalized exponentially dichotomic* if there are $\varphi \in \mathcal{F}$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ such that:

$(nged_1)$ $\|\Phi(t, s, \theta)P(s, \theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|P(s, \theta)x\|$;

$(nged_2)$ $e^{\int_s^t \varphi(r)dr}\|Q(s, \theta)x\| \leq N(t)\|\Phi(t, s, \theta)Q(s, \theta)x\|$, for all $(t, s, \theta, x) \in \Delta \times \Gamma$.

Remark. As particular cases, we remark the following:

(i) if $N(s) = Be^{\int_0^s \xi(r)dr}$, with $B \geq 1$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function in Definition 3.1, then we recover the concept of *generalized exponential dichotomy in sense of Barreira and Valls*;

(ii) if there exists $c > 0$ such that $\varphi(s) \geq c$, for all $s \geq 0$ in Definition 3.1, then we have the notion of *nonuniform exponential dichotomy*.

Remark. The pair (C, P) admits a nonuniform generalized exponential dichotomy if and only if there are $\varphi \in \mathcal{F}$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, +\infty)$ with:

$(nged'_1)$ $\|\Phi(t, t_0, \theta)P(t_0, \theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|\Phi(s, t_0, \theta)P(t_0, \theta)x\|$;

$(nged'_2)$ $e^{\int_s^t \varphi(r)dr}\|\Phi(s, t_0, \theta)Q(t_0, \theta)x\| \leq N(t)\|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|$, for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Remark. We observe that if (C, P) has a nonuniform exponential dichotomy, then it also admits a nonuniform generalized exponential dichotomy. In general, the converse implication is not accomplished.

Example 3.1. We consider $\Theta = \mathbb{R}_+$ and $\sigma : \Delta \times \Theta \to \Theta$, $\sigma(t, s, \theta) = t - s + \theta$.

Also, $X = l^\infty(\mathbb{N}, \mathbb{R})$ represents the Banach space of bounded real-valued sequences, with the norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|, \quad x = (x_0, x_1, ...x_n, ...) \in X.$$

The families of projectors $P, Q : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ are given by

$$P(s, \theta)(x_0, x_1, x_2, ...) = (x_0, 0, x_2, 0, ...),$$

$$Q(s, \theta)(x_0, x_1, x_2, ...) = (0, x_1, 0, x_3, ...).$$

We define $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ by

$$\Phi(t,s,\theta)x = \left(\frac{s+1}{t+1} e^{-\int_s^t \frac{1}{r+1}dr} x_0, \frac{t+1}{s+1} e^{\int_s^t \frac{1}{r+1}dr} x_1, \frac{s+1}{t+1} e^{-\int_s^t \frac{1}{r+1}dr} x_2, \dots \right).$$

It is easy to verify that the pair (C,P) is nonuniformly generalized exponentially dichotomic with $N(s) = s+1$, $\varphi(s) = \frac{1}{s+1}$, $s \geq 0$.

Let us suppose that (C,P) has nonuniform exponential dichotomy. Then there exist $c > 0$ and a nondecreasing function $\tilde{N} : \mathbb{R}_+ \to [1,+\infty)$ with

$$\|\Phi(t,s,\theta)P(s,\theta)x\| \leq \tilde{N}(s)e^{-c(t-s)}\|P(s,\theta)x\|,$$

which implies

$$e^{-\int_s^t \frac{1}{r+1}dr}(s+1) \leq \tilde{N}(s)(t+1)e^{-c(t-s)},$$

for all $(t,s) \in \Delta$.

Considering $t = e^{2n\pi} - 1$ and $s = 0$, we obtain

$$e^{c(e^{2n\pi}-1)-4n\pi} \leq \tilde{N}(0),$$

which for $n \to +\infty$ leads to a contradiction.

Proposition 1. *If $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is a strongly invariant family of projectors for $C = (\sigma, \Phi)$, then there exists $\Psi : \Delta \times \Theta \to \mathcal{B}(X)$ isomorphism from Range $Q(t,\sigma(t,s,\theta))$ to Range $Q(s,\theta)$, with the properties:*

(Ψ_1) $\Phi(t,s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta)) = Q(t,\sigma(t,s,\theta))$;
(Ψ_2) $\Psi(t,s,\theta)\Phi(t,s,\theta)Q(s,\theta) = Q(s,\theta)$;
(Ψ_3) $\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta)) = Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))$;
(Ψ_4) $\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta)) = \Psi(s,t_0,\theta)\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))$,

for all $(t,s,t_0) \in T$, $\theta \in \Theta$.

Proof. See (Mihiţ *et al.*, 2017). □

In what follows, we will consider $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ a strongly invariant family of projectors for a skew-evolution semiflow $C = (\sigma, \Phi)$.

Theorem 3.2. *The pair (C,P) has a nonuniform generalized exponential dichotomy if and only if there exist $\varphi \in \mathcal{F}$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1,+\infty)$ such that:*

($nged_1$) $\|\Phi(t,s,\theta)P(s,\theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|P(s,\theta)x\|$;

($nged_2''$) $e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| \leq N(t)\|Q(t,\sigma(t,s,\theta))x\|$, for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Proof. Necessity.

For ($nged_2$) $\Rightarrow$ ($nged_2''$) we have:

$$e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| = e^{\int_s^t \varphi(r)dr}\|Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| \leq$$

$$\leq N(t)\|\Phi(t,s,\theta)Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| = N(t)\|Q(t,\sigma(t,s,\theta))x\|,$$

for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Sufficiency. We obtain:

$$e^{\int_s^t \varphi(r)dr}\|Q(s,\theta)x\| = e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)Q(s,\theta)x\| \leq$$

$$\leq N(t)\|Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)Q(s,\theta)x\| = N(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|,$$

for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Hence, $(nged_2)$ from Definition 3.1 is satisfied.

□

Proposition 2. *The pair* (C,P) *admits a nonuniform generalized exponential dichotomy if and only if there are* $\varphi \in \mathcal{F}$ *and a nondecreasing function* $N : \mathbb{R}_+ \to [1,+\infty)$ *with:*

$(nged_1')$ $\|\Phi(t,t_0,\theta)P(t_0,\theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|\Phi(s,t_0,\theta)P(t_0,\theta)x\|$;

$(nged_2''')$ $e^{\int_0^s \varphi(r)dr}\|\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta))x\| \leq N(s)\|\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\|$,

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$.

Proof. Necessity. Using the condition $(nged_2'')$ from Theorem 3.2, we deduce:

$$e^{\int_0^s \varphi(r)dr}\|\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta))x\| =$$

$$= e^{\int_0^s \varphi(r)dr}\|\Psi(s,t_0,\theta)Q(s,\sigma(s,t_0,\theta))\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\| \leq$$

$$\leq N(s)\|Q(s,\sigma(s,t_0,\theta))\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\| =$$

$$= N(s)\|\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\|,$$

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$.

Sufficiency. Considering $s = t$ in $(nged_2''')$, we obtain $(nged_2'')$ from Theorem 3.2. □

4. Nonuniform generalized exponential dichotomy of Rolewicz type

Further, $C = (\sigma,\Phi)$ is a skew-evolution semiflow, $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projectors for $C = (\sigma,\Phi)$ and $\mathcal{R}$ represents the set of continuous and nondecreasing functions $R : \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 4.1. We say that (C,P) admits a *nonuniform generalized exponential dichotomy of Rolewicz type* if there exist $R \in \mathcal{R}$, $\varphi \in \mathcal{F}$ and a nondecreasing function $\rho : \mathbb{R}_+ \to [1,+\infty)$ with:

$(Rnged_1)$ $$\int_s^{+\infty} R\left(e^{\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)P(s,\theta)x\|\right)d\tau \leq R\left(\rho(s)\|P(s,\theta)x\|\right), \text{ for all } (s,\theta,x) \in \mathbb{R}_+ \times \Gamma;$$

$(Rnged_2)$ $$\int_s^{t} R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau \leq R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right), \text{ for all } (t,s,\theta,x) \in \Delta \times \Gamma.$$

Remark. In particular, if $N(s) = Be^{\int_0^s \xi(r)dr}$, with $B \geq 1$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function in Definition 4.1, then we have the property of *generalized exponential dichotomy of Rolewicz type in sense of Barreira and Valls*.

Proposition 3. *The pair* (C, P) *has nonuniform generalized exponential dichotomy of Rolewicz type if and only if there exist* $R \in \mathcal{R}$, $\varphi \in \mathcal{F}$ *and a nondecreasing mapping* $\rho : \mathbb{R}_+ \to [1, +\infty)$*:*

$$(Rnged'_1) \quad \int_s^{+\infty} R\left(e^{\int_0^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)P(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right),$$

for all $(s, t_0, \theta, x) \in \Delta \times \Gamma$;

$$(Rnged'_2) \quad \int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_{t_0}^t \varphi(r)dr} \|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|\right),$$

for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$.

Proof. Necessity. $(Rnged'_1)$ For all $(s, t_0, \theta, x) \in \Delta \times \Gamma$, we have:

$$\int_s^{+\infty} R\left(e^{\int_0^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)P(t_0, \theta)x\|\right) d\tau =$$

$$= \int_s^{+\infty} R\left(e^{\int_0^s \varphi(r)dr} e^{\int_s^\tau \varphi(r)dr} \|\Phi(\tau, s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right) d\tau \leq$$

$$\leq R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|P(s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta)x\|\right) = R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right).$$

$(Rnged'_2)$ Similarly, for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$, we deduce:

$$\int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau =$$

$$= \int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} e^{-\int_\tau^t \varphi(r)dr} e^{\int_\tau^t \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau =$$

$$= \int_{t_0}^{t} R\left(e^{-\int_{t_0}^t \varphi(r)dr} e^{\int_\tau^t \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_{t_0}^t \varphi(r)dr} \|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|\right).$$

Sufficiency. Considering $t_0 = s$ in $(Rnged'_1)$, we obtain the condition $(Rnged_1)$ from Definition 4.1. For $t_0 = s$ in $(Rnged'_2)$, it follows

$$\int_s^{t} R\left(e^{-\int_s^\tau \varphi(r)dr} \|\Phi(\tau, s, \theta)Q(s, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_s^t \varphi(r)dr} \|\Phi(t, s, \theta)Q(s, \theta)x\|\right), \text{ for all } (t, s, \theta, x) \in \Delta \times \Gamma.$$

Thus,

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{-\int_s^\tau \varphi(r)dr}e^{\int_s^\tau \varphi(r)dr}e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_s^t \varphi(r)dr}e^{-\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau \leq$$

$$\leq R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right), \text{ for all } (t,s,\theta,x) \in \Delta\times\Gamma.$$

Hence, $(Rnged_2)$ from Definition 4.1 holds. □

Theorem 4.1. *Let $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ be a strongly invariant family of projectors for $C = (\sigma, \Phi)$. Then (C, P) admits nonuniform generalized exponential dichotomy of Rolewicz type if and only if there exist $R \in \mathcal{R}$, $\varphi \in \mathcal{F}$ and a nondecreasing mapping $\rho : \mathbb{R}_+ \to [1, +\infty)$ with:*

$$(Rnged_1) \quad \int_s^{+\infty} R\left(e^{\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)P(s,\theta)x\|\right)d\tau \leq R\left(\rho(s)\|P(s,\theta)x\|\right), \text{ for all } (s,\theta,x) \in \mathbb{R}_+\times\Gamma;$$

$$(Rnged_2'') \quad \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau \leq R\left(\rho(t)\|Q(t,\sigma(t,s,\theta))x\|\right),$$

for all $(t, s, \theta, x) \in \Delta \times \Gamma$.

Proof. We will prove that $(Rnged_2'')$ is equivalent with $(Rnged_2)$ from Definition 4.1.

Necessity. For all $(t, s, \theta, x) \in \Delta \times \Gamma$, it results

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)\Psi(\tau,s,\theta)\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau$$

$$\leq R(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\|) =$$

$$= R\left(\rho(t)\|Q(t,\sigma(t,s,\theta))x\|\right).$$

Sufficiency. For all $(t, s, \theta, x) \in \Delta \times \Gamma$, we obtain:

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))\Phi(t,\tau,\sigma(\tau,s,\theta))Q(\tau,\sigma(\tau,s,\theta))\Phi(\tau,s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr} \|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)x\| \right) d\tau \leq$$

$$\leq R(\rho(t)\|Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)x\|) = R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right).$$

□

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