Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 1–9

On Vector Valued Periodic Distributions

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Abstract

In this paper we consider vector valued (X-valued with X a Banach space) distributions on the euclidean space $\mathbb{R}^d$, extending the T-periodicity, and the T-periodic transform with $T = (T_1, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$ from the scalar case to the Banach space valued case.

Besides immediate basic properties of these concepts, a realization of the space of X-valued T-periodic distributions, up to a toplinear isomorphism, as the space of all bounded linear operators from the space of T-periodic test functions to the Banach space X is given.

Keywords: Periodic functions, periodic distributions, vector valued periodic distributions.
2000 MSC: 60E05, 58A30.

1. Introduction

It is well known the part played by the concept of "periodicity" in the mathematical description of the state of a "phenomenon" with some rhythmic evolutions, appearing in different particular sciences.

But in spite of fact that mathematical models are often well described in terms of vector valued periodic functions, there are many situations in which the ordinary concept of function is not satisfactory. Such situations are mainly determined by the absence of derivability of such functions, especially when the evolutions of the phenomena to be modeled must satisfy a law expressed by a differential equation. Such difficulties are well overcome in the more general setting of distributions, or, if we wish to describe a class of larger and more complex situations, of vector valued distributions.

It is the aim of this paper to enlarge the domains (the possibilities) of application of vector valued periodic functions, extending some important results on scalar periodic distributions to the vector valued case.

Let us mention that there is a very rich literature regarding distributions and even their periodicity in the scalar case (see (Schwartz, 1950), (Zemanian, 1965), (Kecs, 1978)), as well as the new developments connected especially to the theory of topological linear spaces, including some general aspects from the

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vector valued case (see (Schwartz, 1953*a*), (Schwartz, 1953*b*), (Gaşpar & Gaşpar, 2009), (Schwartz, 1957)), which we shall use elsewhere.

The content of the paper runs as follows.

In Section 2 we recall and complete some necessary basic results on the spaces of test functions and of locally r-summable functions on the euclidean space $\mathbb{R}^d$ with respect to the Lebesgue measure $m_d(\cdot)$, the T-periodicity with respect to a general period $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, the T-periodic transform taking a special place.

The Section 3 is devoted to the main results of the note.

Considering the class of X-valued T-periodic distributions as a subspace of the space of all X-valued distributions (X a Banach space), which is invariant to the multiplication operator with T-periodic test functions (Proposition 3.2) and to derivation (Proposition 3.3), the T-periodic transform is extended from the space of compactly supported test functions to the space of compactly supported X-valued distributions (Theorem 3.1).

It is also proved that the space of X-valued T-periodic distributions is isomorphic as linear topological space to the space of all bounded linear operators from space of T-periodic test functions on X (Theorem 3.2 and Theorem 3.3).

2. Periodic functions

In this section we define the T-periodicity for test and locally summable scalar functions, as well as the T-periodic transform on the space of scalar test functions.

Definition 2.1. (see (Zemanian, 1965), chap. 11, § 2, p. 314) An ordinary function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be periodic if there exists $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, such that $(L_T f)(t) = f(t)$, $t \in \mathbb{R}^d$, where L_τ, $\tau \in \mathbb{R}^d$ means the translation operator on $\mathbb{R}^d$. T is called a period of f. The set of all periods of f is kT ($kT = (k_1T_1, ..., k_dT_d)$, $k \in \mathbb{Z}^d$). The "smallest" period is called the fundamental period of f.

We will denote by $[0, T]$ the d-dimensional "parallelepiped" $[0, T_1] \times [0, T_2] \times ... \times [0, T_d]$, $T = (T_1, T_2, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, $i \in \mathbb{N}$.

Definition 2.2. (see (Zemanian, 1965), chap. 11, § 2, p. 314) A function $\theta : \mathbb{R}^d \to \mathbb{C}$ will be called T-periodic test function, if it is periodic of period T and infinitely smooth. The space of all such T-periodic test functions will be denoted by $\mathcal{D}_T(\mathbb{R}^d)$ or $\mathcal{D}_{d,T}$.

Let us recall the basic well known spaces of test functions used in distributions theory (see (Gaşpar & Gaşpar, 2009), (Schwartz, 1950)): $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{E}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$,$\dot{\mathcal{B}}(\mathbb{R}^d)$ and $\mathcal{O}_M(\mathbb{R}^d)$ which we shall briefly denote $\mathcal{D}_d$, $\mathcal{S}_d$, $\mathcal{E}_d$, $\mathcal{B}_d$, $\dot{\mathcal{B}}_d$ and $\mathcal{O}_{d,M}$. We also denote the Lebesgue spaces L^r_d of r-summable complex functions on $\mathbb{R}^d$ with respect to the Lebesgue measure m_d on $\mathbb{R}^d$ and $L^r_{d,loc}$ of locally r-summable complex valued functions on $\mathbb{R}^d$, while $L^r_{d,T}$ means the set of all elements from $L^r_{d,loc}$, which are T-periodic, where $1 \leq r \leq \infty$. For the space of all complex functions from $\mathcal{E}_d$ which together with all derivatives are in L^r_d we use the notation $\mathcal{D}_{d,L^r}$ $1 \leq r \leq \infty$ and $\mathcal{B}_d = \mathcal{D}_{d,L^\infty}$ (see (Schwartz, 1950), p. 55). For $r = 1$ we obtain the space of summable test functions $\mathcal{D}_{d,L^1}$.

These spaces satisfy the inclusions (with continuous embeddings):

$$\mathcal{D}_d \subset \mathcal{S}_d \subset \mathcal{D}_{d,L^1} \subset \mathcal{D}_{d,L^r} \subset \dot{\mathcal{B}}_d \subset \mathcal{B}_d \subset \mathcal{O}_{d,M} \subset \mathcal{E}_d \tag{2.1}$$

(see (Schwartz, 1950))

Remark. $\mathcal{D}_{d,T}$ is a linear space and following inclusions hold

$$\mathcal{D}_{d,T} \subset \mathcal{B}_d \subset \mathcal{O}_{d,M} \subset \mathcal{E}_d. \tag{2.2}$$

The space $\mathcal{D}_{d,T}$ will be endowed with the topology induced from $\mathcal{B}_d$, i.e. a sequence $\{\theta_k\}_{k\in\mathbb{N}}$ from $\mathcal{D}_{d,T}$ converges to zero, if the sequences of all derivatives $\{D^\alpha\theta_k\}_{k\in\mathbb{N}}$ $(\alpha \in \mathbb{N}^d)$ converge uniformly to zero.

Definition 2.3. The T-periodic transform on $\mathcal{D}_d$ denoted by ϖ_T (for T=(1,...,1) see (Schwartz, 1950), p. 85) is the $\mathcal{D}_{d,T}$-valued operator on $\mathcal{D}_d$ defined by

$$(\varpi_T\varphi)(t) = \sum_{n\in\mathbb{Z}^d} \varphi(t - nT) = \sum_{n\in\mathbb{Z}^d} (L_{nT}\varphi)(t),\ t \in \mathbb{R}^d,\ \varphi \in \mathcal{D}_d. \tag{2.3}$$

A function ξ from $\mathcal{D}_d$ is called a T-unitary function, or a T-partition of unity (see (Kecs, 1978), chap. 3, § 2, p. 133 and (Zemanian, 1965), chap. 11, § 2, p. 315), if $\varpi_T\xi = 1$. The space of all such functions ξ will be denoted by $\mathcal{U}_T(\mathbb{R}^d)$, or $\mathcal{U}_{d,T}$.

Remark. ϖ_T is a continuous linear operator from $\mathcal{D}_d$ onto $\mathcal{D}_{d,T}$.

Indeed, it is easy to see that ϖ_T is linear in φ and, if φ_j converges to zero $(j \to \infty)$ in $\mathcal{D}_d$, then $\varpi_T\varphi_j$ converges to zero in $\mathcal{D}_{d,T}$. Moreover ϖ_T is an onto mapping, since for any $\theta \in \mathcal{D}_{d,T}$ and a fixed $\xi \in \mathcal{U}_{d,T}$, we have $\xi\theta \in \mathcal{D}_d$ and $\varpi_T(\xi\theta) = \theta$. In this context it is obvious that the mapping

$$\mathcal{D}_{d,T} \ni \theta \mapsto \xi\theta \in \mathcal{D}_d, \tag{2.4}$$

is a linear continuous "inverse" of ϖ_T.

Remark. For each $\varphi \in \mathcal{D}_d$ the sum $\sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)(t)$ is finite and because $L_T(\sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)) = \sum\limits_{n\in\mathbb{Z}^d} (L_{nT}\varphi)$, it defines a function from $\mathcal{D}_{d,T}$.

Remark. ϖ_T can be extended in a natural way to the space $\mathcal{D}_{d,L^1}$ (compare with (Schwartz, 1950), p. 86).

Remark. It is immediately seen that

$$\mathcal{D}_d = \mathcal{U}_{d,T}\mathcal{D}_{d,T}, \tag{2.5}$$

holds.

Let us mention that this T-periodic transform on the space of test functions is used in the study of scalar periodic distributions by extending this transform from test functions to distributions. Namely such a T-periodic transform is extended to the space of compactly supported distributions, $\mathcal{E}'_d$ (see (Kecs, 1978), p. 138) and to the space $\mathcal{D}'_{d,L^1}$ of summable distributions (see (Schwartz, 1950), p. 86). We try to do that for the case of vector valued distributions in the next Section.

3. *T*-periodic transform of *X*-valued distributions

At the beginning let us recall some general facts.

Definition 3.1. (see (Schwartz, 1957), chap. II, § 2) Let X be a Banach space. Any linear and continuous operator $U : \mathcal{D}_d \to X$ is an X-valued distribution on $\mathbb{R}^d$. The set of all X-valued distributions on $\mathbb{R}^d$ will be denoted by $\mathcal{D}'_d(X)$.

Analogously, we can introduce the spaces $\mathcal{S}'_d(X)$ of X-valued tempered distributions, $\mathcal{E}'_d(X)$ of X-valued "compactly" supported distributions and $\mathcal{B}'_d(X)$ of X-valued bounded distributions.

Remark. $\mathcal{D}'_d(X) = \mathcal{D}'_d \varepsilon X$, $\mathcal{S}'_d(X) = \mathcal{S}'_d \varepsilon X$, $\mathcal{E}'_d(X) = \mathcal{E}'_d \varepsilon X$, $\mathcal{B}'_d(X) = \mathcal{B}'_d \varepsilon X$, where by ε we have denoted the ε - product (see (Schwartz, 1957), chap. I, § 2).

Considering also X-valued test functions and the corresponding spaces the following inclusions hold with continuous embeddings:

$$\begin{array}{cccccccc} \mathcal{D}_d(X) \subset & \mathcal{S}_d(X) \subset & \mathcal{D}_{d,L^1}(X) \subset & \mathcal{D}_{d,L^r}(X) \subset & \dot{\mathcal{B}}_d(X) \subset & \mathcal{B}_d(X) \subset & \mathcal{O}_{d,M}(X) \subset & \mathcal{E}_d(X) \\ \cap & \cap & \cap & \cap & \cap & \cap & \cap & \cap \\ \mathcal{E}'_d(X) \subset & \mathcal{O}'_{d,c}(X) \subset & \mathcal{D}'_{d,L^1}(X) \subset & \mathcal{D}'_{d,L^r}(X) \subset & \dot{\mathcal{B}}'_d(X) \subset & \mathcal{B}'_d(X) \subset & \mathcal{S}'_d(X) \subset & \mathcal{D}'_d(X), \end{array} \tag{3.1}$$

(see (Schwartz, 1950), (Popa, 2007)).

Analogously to the Lebesque type spaces L^r_d, $L^r_{d,loc}$, $L^r_{d,T}$ of complex valued functions, we associate in an obvious way the corresponding spaces $L^r_d(X)$, $L^r_{d,loc}(X)$, $L^r_{d,T}(X)$ $(1 \le r \le \infty)$ of X-valued functions.

Let us consider now $F \in L^1_{d,loc}(X)$. The operator U_F defined by

$$U_F(\varphi) := \int_{\mathbb{R}^d} \varphi(t)F(t)dt, \ \varphi \in \mathcal{D}_d \tag{3.2}$$

is clearly linear and continuous on $\mathcal{D}_d$, hence $U_F \in \mathcal{D}'_d(X)$.

Identifying F with U_F, the following continuous embeddings holds

$$L^r_d(X) \subseteq L^r_{d,loc}(X) \subseteq L^1_{d,loc}(X) \subseteq \mathcal{D}'_d(X). \tag{3.3}$$

For any $\varphi \in \mathcal{D}_d$ we recall the definition of the following operators on the spaces of X-valued distributions defined with the help of corresponding operators on the spaces of test functions:

- The translations $(\boldsymbol{L}_\tau U)(\varphi) := U(L_{-\tau}\varphi)$, $\tau \in \mathbb{R}^d$;
- Multiplications with functions $(\boldsymbol{M}_\psi U)(\varphi) := U(M_\psi \varphi)$, $\psi \in \mathcal{E}_d$;
- The derivation $(\boldsymbol{D}^\alpha U)(\varphi) := (-1)^{|\alpha|} U(D^\alpha \varphi)$, $\alpha \in \mathbb{Z}^d$, $|\alpha| = \alpha_1 + ... + \alpha_d$.

Now we try to extend to the vector valued case and $d > 1$ some results regarding the scalar periodic distributions treated in (Schwartz, 1950), (Zemanian, 1965), (Kecs, 1978).

Definition 3.2. A vector valued distribution $U \in \mathcal{D}'_d(X)$ is said to be T-periodic, where $T = (T_1, ..., T_d) \in \mathbb{R}^d$, $T_i > 0$, when $\boldsymbol{L}_T U = U$. T is called a period of U. The set of all periods of the distribution U is kT, $k \in \mathbb{Z}^d$. The "smallest" period is called the fundamental period of U (see (Zemanian, 1965) for $d = 1$)

By $\mathcal{D}'_T(\mathbb{R}^d, X)$, or $\mathcal{D}'_{d,T}(X)$, we shall denote the space of all such X-valued T-periodic distributions having the same period $T \in \mathbb{R}^d$, $T_i > 0$ (T - fixed).

In the next Theorem we extend the T-periodic transform from the space of compactly supported test functions, to the space of compactly supported X-valued distributions.

Theorem 3.1. *If $V \in \mathcal{E}'_d(X)$, then $\sum_{n \in \mathbb{Z}^d} \boldsymbol{L}_{nT} V$ defines an X-valued T-periodic distribution U. Conversely, any X-valued T-periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ can be written as follows*

$$U = \sum_{n \in \mathbb{Z}^d} \boldsymbol{L}_{nT} V, \tag{3.4}$$

where $V \in \mathcal{E}'_d(X)$.

Proof. Since X is a Banach space, $\mathcal{E}'_d(X)$ consists just of the compactly supported X-valued distributions (see (Schwartz, 1957), p. 62), hence the sum $\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT} V$ contains a finite nonzero terms. Denoting by U this X-valued distribution, we successively have

$$\mathbf{L}_T U = L_T \sum_{n\in\mathbb{Z}^d} \mathbf{L}_{nT} V = \sum_{n\in\mathbb{Z}^d} \mathbf{L}_T(\mathbf{L}_{nT} V) = \sum_{k\in\mathbb{Z}^d} \mathbf{L}_{kT} V = U,$$

i.e. $U \in \mathcal{D}'_{d,T}(X)$.

Conversely, let us consider $U \in \mathcal{D}'_{d,T}(X)$ an X-valued T-periodic distribution and $\xi \in \mathcal{U}_{d,T}$. Now the X-valued distribution $V = \boldsymbol{M}_\xi U$ (which is obvious from $\mathcal{E}'_d(X)$) satisfies

$$\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT} V = \sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}(\xi U) = U.$$

$\square$

From Theorem 3.1 it results that the operator ϖ_T defined by $U = \varpi_T V$ given by (3.4) is an onto mapping from $\mathcal{E}'_d(X)$ onto $\mathcal{D}'_{d,T}(X)$. It will be called the T*-periodic transform* on X-valued distributions.

Remark. It is a simple matter to observe that an analog of (2.5) also holds:

$$\mathcal{E}'_d(X) = \mathcal{U}_{d,T}\mathcal{D}'_{d,T}(X). \tag{3.5}$$

Remark. When $V \in \mathcal{D}'_{d,L^r}(X)$, then is not difficult to see that $\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT} V$ also makes sense, meaning that ϖ_T can be naturally extended to $\mathcal{D}'_{d,L^1}(X)$.

Remark. Regarding the mapping ϖ_T, from the successive equalities

$$(\varpi_T V)(\varphi) = \sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT} V)(\varphi) = V(\sum_{n\in\mathbb{Z}^d} L_{-nT}\varphi) = V(\varpi_T\varphi),\ \varphi \in \mathcal{D}_d,$$

we see that ϖ_T is the restrictions to $\mathcal{E}'_d(X)$ of the "adjoint" operator $\varpi'_T \in \mathcal{B}(\mathcal{B}'_d(X), \mathcal{D}'_d(X))$.

This transformation enables us to identify, up to an isomorphism, the space of X-valued T-periodic distributions on $\mathbb{R}^d$ with the "dual" of $\mathcal{D}_{d,T}$, i.e. with the space $\mathcal{B}(\mathcal{D}_{d,T}, X)$ of all bounded linear operators from $\mathcal{D}_{d,T}$ to X. Indeed now we can prove

Theorem 3.2. *(a) For each* $U \in \mathcal{D}'_{d,T}(X)$*, the operator* U_T *defined by*

$$U_T(\theta) := (U, \xi\theta),\ \theta \in \mathcal{D}_{d,T}, \tag{3.6}$$

is correctly defined, being independent of the choice of $\xi \in \mathcal{U}_{d,T}$.
(b) $U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$.

Proof. (a) For any $U \in \mathcal{D}'_d(X)$ we have that $\eta U \in \mathcal{E}'_d(X)$, $\eta \in \mathcal{U}_{d,T}$, where $\eta U(\varphi) = U(\eta\varphi)$, $\varphi \in \mathcal{D}_d$. Also, because

$$\sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT}\eta U)(\varphi) = U(\sum_{n\in\mathbb{Z}^d} L_{nT}\eta)(\varphi) = U(\varphi),\ \varphi \in \mathcal{D}_d,$$

we have

$$U = \sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}\eta U.$$

For any ξ and $\eta \in \mathcal{U}_{d,T}$, assuming that U is T-periodic, we can write $U = \boldsymbol{L}_{nT}U$ and for any $\theta \in \mathcal{D}_{d,T}$ we have

$$(U, \xi\theta) = (\sum_{n\in\mathbb{Z}^d} \boldsymbol{L}_{nT}\eta U, \xi\theta) = \sum_{n\in\mathbb{Z}^d} (U L_{nT}\eta\xi, \theta) = \sum_{n\in\mathbb{Z}^d} (\boldsymbol{L}_{nT}\eta U, \xi\theta) = \sum_{n\in\mathbb{Z}^d} (\eta U, L_{-nT}\xi\theta)$$

$$= (U, \eta(\sum_{n\in\mathbb{Z}^d} L_{-nT}\xi)\theta) = (U, \eta\theta),$$

for any ξ, $\eta \in \mathcal{U}_{d,T}$ and $\theta \in \mathcal{D}_{d,T}$.

(b) We show that U_T from (3.6) is a linear and continuous operator between $\mathcal{D}_{d,T}$ and X, i.e. $U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X)$.

For linearity, let us consider the functions θ_1, $\theta_2 \in \mathcal{D}_{d,T}$ and $\alpha, \beta \in \mathbb{C}$. Then

$$U_T(\alpha\theta_1 + \beta\theta_2) = (U, \xi(\alpha\theta_1 + \beta\theta_2)) =$$

$$= \alpha(U, \xi\theta_1) + \beta(U, \xi\theta_2) = \alpha U_T(\theta_1) + \beta U_T(\theta_2).$$

For continuity of U_T we consider the sequence $\{\theta_k\}_{k=1}^{\infty}$ converging to 0 in $\mathcal{D}_{d,T}$. Because, in this case, $\xi\theta_k \to 0$, $(k \to \infty)$ in $\mathcal{D}_d$ it results

$$U_T(\theta_k) = (U, \xi\theta_k) \to 0,\ (k \to \infty).$$

□

In this way $\mathcal{D}'_{d,T}(X)$ is linearly continuously embedded in $\mathcal{B}(\mathcal{D}_{d,T}, X)$.

Before proving that $U \mapsto U_T$ is a toplinear isomorphism let us put in evidence the embedding of vector valued T-periodic summable functions in $\mathcal{D}'_{d,T}(X)$.

Proposition 3.1. *If U_F is a distribution corresponding to the locally integrable X-valued T-periodic function F, then $(U_F)_T$ from (3.2) will be expressed by the integral on a parallelepiped of the form*

$$[a, a+T] = [a_1, a_1 + T_1] \times [a_2, a_2 + T_2] \times \ldots \times [a_d, a_d + T_d],\ a \in \mathbb{R}^d.$$

Proof. Let $F \in L^1_T(\mathbb{R}^d, X) \subset L^1_{loc}(\mathbb{R}^d, X)$. Then for the distribution $U_F \in \mathcal{D}'(\mathbb{R}^d, X)$ from (3.2) and $\theta \in \mathcal{D}_{d,T}$, $\xi \in \mathcal{U}_{d,T}$, a, $T \in \mathbb{R}^d$, $T_i > 0$, we have

$$(U_F)_T(\theta) = (U_F, \xi\theta) = \int_{\mathbb{R}^d} F(t)\xi(t)\theta(t)dt = \sum_{n\in\mathbb{Z}^d} \int_{[a+nT, a+nT+T]} F(t)\xi(t)\theta(t)dt =$$

$$= \sum_{n\in\mathbb{Z}^d} \int_{[a,a+T]} F(t+nT)\xi(t+nT)\theta(t+nT)dt = \int_{[a,a+T]} F(t)\theta(t) \sum_{n\in\mathbb{Z}^d} \xi(t+nT)dt = \int_{[a,a+T]} F(t)\theta(t)dt,$$

because F and θ are T-periodic, and $\sum_{n\in\mathbb{R}^d} \xi(t+nT) = 1$. □

Remark. The map $L^1_T(X)(\subset L^1_{loc}(X)) \ni F \mapsto U_F \in \mathcal{D}'_{d,T}(X)$ being linear and injective, the space $L^1_T(X)$ is linear continuous embedded in $\mathcal{D}'_{d,T}(X)$ through $F \equiv U_F$, where (compare with (3.2) and (3.3))

$$(U_F)_T(\theta) = \int_{[0,T)} F(t)\theta(t)dt, \quad \theta \in \mathcal{D}_{d,T}. \tag{3.7}$$

Proposition 3.2. *The multiplication of a vector valued T-periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ with a T-periodic test function $\psi \in \mathcal{D}_{d,T}$ is also a vector valued T-periodic distribution, i.e.*

$$\boldsymbol{M}_\psi U \in \mathcal{D}'_{d,T}(X).$$

Proof. We consider the vector valued periodic distribution $U \in \mathcal{D}'_{d,T}(X)$ and the periodic test function $\psi \in \mathcal{D}_{d,T}$.

We show that $\boldsymbol{M}_\psi U \in \mathcal{D}'_{d,T}(X)$. Because $(\boldsymbol{M}_\psi U)(\varphi) = U(\varphi\psi)$, $\varphi \in \mathcal{D}_d$ is easy to see that $\boldsymbol{M}_\psi U$ is linear and continuous as operator from $\mathcal{D}_d$ to X. It remains to show that $\boldsymbol{M}_\psi U$ is an X-valued periodic distribution of period T. Applying $\boldsymbol{L}_T U = U$ and $L_T\psi = \psi$, we have:

$$(\boldsymbol{L}_T\psi U)(\varphi) = (\psi U)(L_{-T}\varphi) = U(\psi L_{-T}\varphi) = U(L_{-T}(L_T\psi)\varphi) =$$

$$= \boldsymbol{L}_T U((L_T\psi)\varphi) = U(\varphi\psi) = (\psi U)(\varphi),\ \varphi \in \mathcal{D}_d.$$

□

Remark. $\mathcal{D}'_{d,T}(X) = \mathcal{D}'_{d,T}\varepsilon X.$

Indeed, $\mathcal{D}_{d,T}$ have the topology γ, i.e. $((\mathcal{D}_{d,T})'_c)'_c = \mathcal{D}_{d,T}$, where $(\mathcal{D}_{d,T})'_c$ is the dual of $\mathcal{D}_{d,T}$ endowed with the uniform convergence topology on the absolutely convex and compact sets from $\mathcal{D}_{d,T}$, and

$$(\mathcal{D}_{d,T}(X))'_c \approx \mathcal{L}_c(\mathcal{D}_{d,T}, X) \approx \mathcal{L}_\varepsilon(X'_c, (\mathcal{D}_{d,T})'_c) \approx (\mathcal{D}_{d,T})'_c \widehat{\otimes}_\varepsilon X$$

(compare with (Schwartz, 1953*a*), (Schwartz, 1953*b*) and (Schwartz, 1957))

Proposition 3.3. *The subspace $\mathcal{D}'_{d,T}(X)$ of $\mathcal{D}'_d(X)$ is invariant to the derivation operators $\boldsymbol{D}^\alpha$, $\alpha \in \mathbb{N}^d$.*

Proof. We successively have

$$U \in \mathcal{D}'_{d,T}(X) \Rightarrow \boldsymbol{L}_T U = U \Rightarrow$$

$$\Rightarrow (\boldsymbol{D}^\alpha U)(\varphi) = (-1)^{|\alpha|}U(D^\alpha\varphi) = (-1)^{|\alpha|}(\boldsymbol{L}_T U)(D^\alpha\varphi) = (-1)^{|\alpha|}U(L_{-T}D^\alpha\varphi)$$

and

$$(\boldsymbol{L}_T\boldsymbol{D}^\alpha U)(\varphi) = (\boldsymbol{D}^\alpha U)(L_{-T}\varphi) = (-1)^{|\alpha|}U(D^\alpha L_{-T}\varphi),$$

respectively.
Because $L_{-T}D^\alpha\varphi = D^\alpha L_{-T}\varphi$ it follows that $\boldsymbol{L}_T(\boldsymbol{D}^\alpha U) = \boldsymbol{D}^\alpha U$. □

Finally we shall prove that the map constructed in Theorem 3.2,

$$\mathcal{D}'_{d,T}(X) \ni U \mapsto U_T \in \mathcal{B}(\mathcal{D}_{d,T}, X) \tag{3.8}$$

is a toplinear isomorphism.

By applying the properties of the T-periodic transform ϖ_T on $\mathcal{D}_d$, because of (3.6), for any $\varphi \in \mathcal{D}_d$, we have $(U, \varphi) = U_T(\varpi_T \varphi) = (\boldsymbol{\varpi}'_T U_T)(\varphi)$, i.e.

$$U = \boldsymbol{\varpi}'_T U_T. \tag{3.9}$$

Therefore, for each $\lambda_1, \lambda_2 \in \mathbb{C}$ and every $\theta \in \mathcal{D}_{d,T}$, we have

$$(\lambda_1 U_1 + \lambda_2 U_2)_T(\theta) = (\lambda_1 U_1 + \lambda_2 U_2)(\xi\theta) =$$

$$= \lambda_1 U_1(\xi\theta) + \lambda_2 U_2(\xi\theta) = (\lambda_1 (U_1)_T + \lambda_2 (U_2)_T)(\theta).$$

The injectivity results from the successive implications

$$U_T = 0 \Rightarrow \boldsymbol{\varpi}'_T U_T = 0 \Rightarrow U = 0.$$

For continuity we have that

$$U_n \overset{\mathcal{D}'_{d,T}}{\longrightarrow} 0 \Rightarrow U_n(\varphi) \longrightarrow 0,\ \varphi \in \mathcal{D}_d \Rightarrow \boldsymbol{\varpi}'_T (U_n)_T(\theta_\varphi) \longrightarrow 0 \Rightarrow$$

$$(U_n)_T(\varpi_T \varphi) \longrightarrow 0,\ \varphi \in \mathcal{D}_d \Rightarrow (U_n)_T(\theta) \longrightarrow 0,\ \theta \in \mathcal{D}_{d,T} \Rightarrow (U_n)_T \overset{\mathcal{B}(\mathcal{D}_{d,T}, X)}{\longrightarrow} 0.$$

Let us consider an element V from $\mathcal{B}(\mathcal{D}_{d,T}, X)$ and define U by $U(\varphi) = V(\varpi_T \varphi)$, $\varphi \in \mathcal{D}_d$. So U is an X-valued T-periodic distribution from $\mathcal{D}'_d(X)$, i.e. $U \in \mathcal{D}'_{d,T}(X)$. Indeed U satisfies $\boldsymbol{L}_T U = U$, because, from

$$\varpi_T(L_{-T}\varphi) = \varpi_T(\varphi),\ \varphi \in \mathcal{D}_d,$$

we have:

$$(\boldsymbol{L}_T U)(\varphi) = U(L_{-T}\varphi) = V(\varpi_T \varphi) = U(\varphi),\ \varphi \in \mathcal{D}_d.$$

Hence we have constructed just the inverse of (3.8), which is easy to see that it is also continuous. Thus we obtain

Theorem 3.3. *The mapping (3.8) is a toplinear isomorphism from $\mathcal{D}'_{d,T}(X)$ onto $\mathcal{B}(\mathcal{D}_{d,T}, X)$.*

Proof. It only remains to prove that $\{(U_k)_T\}_{k=1}^{\infty}$ converges in $\mathcal{B}(\mathcal{D}_{d,T}, X)$ to zero then $\{(U_k)\}_{k=1}^{\infty}$ converges in $\mathcal{D}'_d(X)$ to zero. Indeed, for ξ in $\mathcal{U}_{d,T}$ and θ in $\mathcal{D}_{d,T}$, we have

$$(U_k)_T(\theta) = (U_k, \xi\theta) \to 0,\ \ (k \to \infty),$$

which means $U_k \to 0$ $(k \to \infty)$ in $\mathcal{D}'_d(X)$. □

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Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 10–14

The Reduced Differential Transform Method for the Exact Solutions of Advection, Burgers and Coupled Burgers Equations

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Abstract

Reduced differential transform method (RDTM) is employed to obtain the solution of simple homogeneous advection, Burgers and coupled Burgers equations exactly. The RDTM produces a solution with few and easy computation. The method is simple, accurate and efficient.

Keywords: Reduced differential transform method, advection equation, Burgers and coupled Burgers equations.
2000 MSC: 35Qxx.

1. Introduction

The concept of differential transform method has been introduced to solve linear and non linear initial value problems in electric circuit analysis, It was first introduced by (Zhou, 1986). Burgers equation generally appears in fluid mechanics. This equation incorporates both convection and diffision in fluid dynamics, and is used to describe the structure of shock waves. Coupled Burgers equation is a simple model of sedimentation or evalution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity. Reseachers have used other methods such as tanh method, HAM, VIM in (Hassan, 2009), (Alomari *et al.*, 2008) and (Abdou & Soliman, 2005) respectively. In this letter, RDTM is used to obtain the exact solution of simple homogeneous advection equation, Burgers equation and coupled Burgers equation.

2. Analysis of the method

The basic definitions of reduced differential transform method are introduced as follows:

Definition 2.1. If the function $u(x,t)$ is analytic and differentiated continously with respect to time t and space X in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0} \tag{2.1}$$

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where the t-dimensional spectrum function $U_k(x)$ is the transformed function.

Definition 2.2. The differential inverse transform of $U_k(x)$ is defined as follows

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{2.2}$$

The fundamental mathematical operations performed by RDTM as given by (Keskin & c, 2010*a*) and (Keskin & c, 2010*b*) are provided in Table1:

Table 1
The fundamental mathematical operations performed by RDTM.

Functional form	Transformed form
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k} u(x,t)]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k = \alpha U_k(x)\ \alpha$ *is a constant*
$w(x,t) = x^m t^n$	$W_k = x^m \delta(k-n), \delta(k) = \begin{cases} 1, & \text{k} = 0 \\ 0 & \text{k} \neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r U_{k-r}(x) = \sum_{r=0}^{k} U_r V k - r(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$	$W_k(x) = \frac{\partial^2}{\partial x^2} U_k(x)$

3. Applications

Example1: Consider the homogeneous advection equation given by (Alomari *et al.*, 2008) as,

$$u_t + u u_x = 0, \quad u(x,0) = -x. \tag{3.1}$$

Here $u_t = -u u_x$. Now taking the reduced differential transform of 3.1 we have

$$(k+1)U_{k+1} = -\sum_{r=0}^{k} U_r \frac{\partial}{\partial x} U_{k-r}, \tag{3.2}$$

with $U_0(x) = -x$ we can then obtain $U_k(x)$ values successively as $U_1(x) = U_2(x) = U_3(x) = ... = U_k(x) = -x$.

Using the differential inverse transform 2.2 we have:

$$u(x,t) = -x \sum_{n=0}^{\infty} t^n \tag{3.3}$$

equation 3.3 is a taylor series that converges to

$$u(x,t) = \frac{x}{t-1} \tag{3.4}$$

under $|t| < 1$ which is the exact solution.

Example2: Consider the one dimensional Burgers equation given by (Alomari *et al.*, 2008), that has the form

$$u_t + uu_x - \nu u_{xx} = 0 \tag{3.5}$$

subject to the boundary condition

$$u(x,0) = \frac{\alpha + \beta + (\beta - \alpha)e^{\gamma}}{1 + e^{\gamma}}, \tag{3.6}$$

where $\gamma = \alpha(\frac{x}{\nu})$ and the parameters α, β, ν are arbitrary constants.
Taking the reduced differential transform of 3.5 we have

$$(k+1)U_{k+1}(x) = -\sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}U_{k-r}(x) + \nu\frac{\partial^2}{\partial x^2}U_k(x) \tag{3.7}$$

$U_0 = \frac{\alpha+\beta+(\beta-\alpha)e^{\gamma}}{1+e^{\gamma}}$ we then obtain $U_k(x)$ values successively as
$U_1 = -U_0\frac{\partial}{\partial x}U_0 + \nu\frac{\partial^2}{\partial x^2}U_0(x)$

$$= \frac{1\alpha^2\beta e^{\gamma}}{\nu(1+e^{\gamma})^2}$$

$U_2 = -\frac{1}{2}(U_0(x)\frac{\partial}{\partial x}U_1(x) + U_1(x)\frac{\partial}{\partial x}U_0(x) + \nu\frac{\partial^2}{\partial x^2}U_1(x))$

$$= \frac{\alpha^3\beta^2(e^{\gamma}-1)e^{\gamma}}{\nu^2(1+e^{\gamma})^3}$$

$$U_3 = \frac{\alpha^4\beta^3e^{\gamma}(1-4e^{\gamma}-e^{2\gamma})}{3\nu^3(1+e^{\gamma})^4}$$

.
.
.

Using the differential inverse transform 2.2 we have:

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{\gamma}}{1+e^{\gamma}} + \frac{1\alpha^2\beta e^{\gamma}}{\nu(1+e^{\gamma})^2}t + \frac{\alpha^3\beta^2(e^{\gamma}-1)e^{\gamma}}{\nu^2(1+e^{\gamma})^3}t^2 + \frac{\alpha^4\beta^3e^{\gamma}(1-4e^{\gamma}-e^{2\gamma})}{3\nu^3(1+e^{\gamma})^4}t^3 + ... \tag{3.8}$$

which in its closed form gives

$$u(x,t) = \frac{\alpha+\beta+(\beta-\alpha)e^{\frac{\alpha}{\nu}(x-\beta t)}}{1+e^{\frac{\alpha}{\nu}(x-\beta t)}}. \tag{3.9}$$

Example3: Consider the following system of coupled Burgers equation given in (Alomari *et al.*, 2008) as

$$u_t - uu_{xx} - 2uu_x + (uv)_x = 0, \tag{3.10}$$

$$v_t - v_{xx} - 2vv_x + (uv)_x = 0, \tag{3.11}$$

subject to the initial conditions

$$u(x,0) = sin(x), \quad v(x,0) = sin(x). \tag{3.12}$$

Taking the reduced differential differential transform of 3.10 and 3.11, we have

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + 2\sum_{r=0}^{k} U_r(x)\frac{\partial}{\partial x}U_{k-r} - \frac{\partial}{\partial x}\sum_{r=0}^{k} U_r V_{k-r}, \tag{3.13}$$

$$(k+1)V_{k+1}(x) = \frac{\partial^2}{\partial x^2}V_k(x) + 2\sum_{r=0}^{k} V_r(x)\frac{\partial}{\partial x}V_{k-r} - \frac{\partial}{\partial x}\sum_{r=0}^{k} U_r V_{k-r}. \tag{3.14}$$

Using equation 3.13 and 3.14 with
$U_0 = V_0 = sin(x)$
we recursively obtain
$U_1 = V_1 = -sin(x)$,
$U_2 = V_2 = \frac{1}{2}sin(x)$,
$U_3 = V_3 = -\frac{1}{6}sin(x)$,
.
.
.

Using the differential inverse transform 2.2 we have

$$u(x,t) = sin(x) - sin(x)t + \frac{1}{2!}sin(x)t^2 - \frac{1}{3!}sin(x)t^3 + ..., \tag{3.15}$$

$$v(x,t) = sin(x) - sin(x)t + \frac{1}{2!}sin(x)t^2 - \frac{1}{3!}sin(x)t^3 + ..., \tag{3.16}$$

which is

$$u(x,t) = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...), \tag{3.17}$$

$$v(x,t) = sin(x)(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + ...), \tag{3.18}$$

and finally in its closed form gives

$$u(x,t) = e^{-t}sin(x) \tag{3.19}$$

and

$$v(x,t) = e^{-t}sin(x), \tag{3.20}$$

which are the exact solution of the coupled Burgers equation.

4. Conclusion

Exact solutions of simple homogeneous advection equation, Burgers equation and Coupled Burgers equation were presented via the reduced differential transform method (RDTM). The method is applied in a direct way without any linearization or descretization. The computational size of this method is small compared with those of DTM, HAM, HPM and Adomian decomposition method. Hence, this method is a powerful and an efficient technique in finding the exact solutions for wide classes of problems, also the speed of the convergence is very fast.

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Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 15–22

$\mathcal{I}$-limit Points in Random 2-normed Spaces

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Abstract

In this article we introduce the notion $\mathcal{I}$-cluster points, and investigate the relation between $\mathcal{I}$-cluster points and limit points of sequences in the topology induced by random 2-normed spaces and prove some important results.

Keywords: t-norm, random 2-normed space, ideal convergence, $\mathcal{I}$-cluster points, F-topology.
2000 MSC: 40A35, 46A70, 54E70.

1. Introduction

Probabilistic metric (PM) spaces were first introduced by Menger (Menger, 1942) as a generalization of ordinary metric spaces and further studied by Schweizer and Sklar (Schweizer & Sklar, 1960, 1983). The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric. In this theory, the notion of distance has a probablistic nature. Namely, the distance between two points x and y is represented by a distribution function F_{xy}; and for $t > 0$, the value $F_{xy}(t)$ is interpreted as the probability that the distance from x to y is less than t. After that it was developed by many authors. Using this concept, Śerstnev (Śerstnev, 1962) introduced the concept of probabilistic normed space. It provides an important method of generalizing the deterministic results of linear normed spaces. It has also very useful applications in various fields, e.g., continuity properties (Alsina *et al.*, 1997), topological spaces (Frank, 1971), linear operators (Golet, 2005), study of boundedness (Guillén *et al.*, 1999), convergence of random variables (Guillén & Sempi, 2003), statistical and ideal convergence of probabilistic normed space or 2-normed space (Karakus, 2007), (Mohiuddine & Savaş, 2012), (Mursaleen, 2010), (Mursaleen & Mohiuddine, 2010), (Mursaleen & Mohiuddine, 2012), (Mursaleen & Alotaibi, 2011), (Rahmat & Harikrishnan, 2009), (Tripathy *et al.*, 2012) etc.

The concept of 2-normed spaces was initially introduced by Gähler (Gähler, 1963), (Gähler, 1964) in the 1960's. Since then, many researchers have studied these subjects and obtained various results (Gunawan & Mashadi, 2001), (Gürdal & Pehlivan, 2004), (Gürdal, 2006), (Gürdal & Açik, 2008), (Gürdal *et al.*, 2009), (Savaş, 2011), (Siddiqi, 1980), (Şahiner *et al.*, 2007).

P. Kostyrko et al (cf. (Kostyrko *et al.*, 2000); a similar concept was invented in (Katětov, 1968)) introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such

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convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. The notion of statistical convergence of sequences of real numbers was introduced by H. Fast in (Fast, 1951) and H. Steinhaus in (Steinhaus, 1951).

There are many pioneering works in the theory of $\mathcal{I}$-convergence. The aim of this work is to introduce and investigate the relation between $\mathcal{I}$-cluster points and ordinary limit points of sequence in random 2-normed spaces.

2. Definitions and Notations

First we recall some of the basic concepts, which will be used in this paper.

Definition 2.1. ((Freedman & Sember, 1981), (Fast, 1951)) A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if $\delta(E) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\chi_E(k)$ exists. A number sequence $(x_n)_{n\in\mathbb{N}}$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$. If $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to L we write st-lim $x_n = L$, which is necessarily unique.

Definition 2.2. ((Kelley, 1955), (Kostyrko *et al.*, 2000)) A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if **(i)** $\emptyset \in \mathcal{I}$; **(ii)** $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; **(iii)** $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in Y is called an admissible ideal if it is different from $P(\mathbb{N})$ and it contains all singletons, i.e., $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Let $\mathcal{I} \subset P(Y)$ be a non-trivial ideal. A class $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \backslash A\}$ is a filter on Y, called the filter associated with the ideal $\mathcal{I}$.

Definition 2.3. ((Kostyrko *et al.*, 2000), (Kostyrko *et al.*, 2005)) Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - L\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Definition 2.4. ((Gähler, 1963) (Gähler, 1964)) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies **(i)** $\|x, y\| = 0$ if and only if x and y are linearly dependent; **(ii)** $\|x, y\| = \|y, x\|$; **(iii)** $\|\alpha x, y\| = |\alpha|\,\|x, y\|$, $\alpha \in \mathbb{R}$; **(iv)** $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2).$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|x, y\| \geq 0$ and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Also, if x, y and z are linearly dependent, then $\|x, y + z\| = \|x, y\| + \|x, z\|$ or $\|x, y - z\| = \|x, y\| + \|x, z\|$. Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: a sequence (x_n) in X is said to be convergent to x in X if $\lim_{n\to\infty}\|x_n - x, y\| = 0$ for every $y \in X$.

All the concepts listed below are studied in depth in the fundamental book by Schweizer and Sklar (Schweizer & Sklar, 1983).

Definition 2.5. Let $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = [0, 1]$ the closed unit interval. A mapping $f : \mathbb{R} \to S$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We denote the set of all distribution functions by D^+ such that $f(0) = 0$. If $a \in \mathbb{R}_+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.6. A triangular norm (t-norm) is a continuous mapping $* : S \times S \to S$ such that $(S, *)$ is an abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$ for all $a, b, c, d \in S$. A triangle function τ is a binary operation on D^+ which is commutive, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

Definition 2.7. Let X be a linear space of dimension greater than one, τ is a triangle, and $F : X \times X \to D^+$. Then F is called a probabilistic 2-norm and (X, F, τ) a probabilistic 2-normed space if the following conditions are satisfied:

(2.2.1) $F(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $F(x, y; t)$ denotes the value of $F(x, y)$ at $t \in \mathbb{R}$,

(2.2.2) $F(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

(2.2.3) $F(x, y; t) = F(y, x; t)$ for all $x, y \in X$,

(2.2.4) $F(\alpha x, y; t) = F(x, y; \frac{t}{|\alpha|})$ for every $t > 0, \alpha \neq 0$ and $x, y \in X$,

(2.2.5) $F(x + y, z; t) \geq \tau(F(x, z; t), F(y, z; t))$ whenever $x, y, z \in X$ and $t > 0$,

If (2.2.5) is replaced by

$(2.2.5)'$ $F(x + y, z; t_1 + t_2) \geq F(x, z; t_1) * F(y, z; t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_+$;

then $(X, F, *)$ is called a random 2-normed space (for short, RTN space).

Remark. Note that every 2-norm space $(X, \|., .\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $F(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a, b\}, a, b \in S$;

(ii) $F(x, y; t) = \frac{t}{t + \|x, y\|}$ for every $x, y \in X, t > 0$ and $a * b = ab$ for $a, b \in S$.

Let $(X, F, *)$ be an RTN space. Since $*$ is a continuous t-norm, the system of (ε, λ)-neighborhoods of θ (the null vector in X)

$$\{\mathcal{N}_\theta(\varepsilon, \lambda) : \varepsilon > 0, \ \lambda \in (0, 1)\},$$

where

$$\mathcal{N}_\theta(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) > 1 - \lambda\}.$$

determines a first countable Hausdorff topology on X, called the F-topology. Thus, the F-topology can be completely specified by means of F-convergence of sequences. It is clear that $x - y \in \mathcal{N}_\theta$ means $y \in \mathcal{N}_x$ and vice versa.

A sequence $x = (x_n)$ in X is said to be F-convergence to $L \in X$ if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and for each nonzero $z \in X$ there exists a positive integer N such that

$$x_n, z - L \in \mathcal{N}_\theta(\varepsilon, \lambda) \text{ for each } n \geq N$$

or equivalently,

$$x_n, z \in \mathcal{N}_L(\varepsilon, \lambda) \text{ for each } n \geq N.$$

In this case we write $F\text{-}\lim x_n, z = L$.

3. The Relation Between $\mathcal{I}$-cluster Points and Ordinary Limit Points in Random 2-Normed Spaces

It is known (see (Fridy, 1993)) that statistical cluster Γ_x and statistical limit points set Λ_x of a given sequence (x_n) are not altered by changing the values of a subsequence the index set of which has density zero. Moreover, there is a strong connection between statistical cluster points and ordinary limit points of a given sequence. We will prove that these facts are satisfied for $\mathcal{I}$-cluster and $\mathcal{I}$-limit point sets of a given sequences in the topology induced by random 2-normed spaces

Definition 3.1. Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n) \in X$.

(i) An element $L \in X$ is said to be an $\mathcal{I}$-limit point of the sequence x with respect to the random 2-norm F (or $\mathcal{I}_F^2(x)$-limit point) if there is a set $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $F\text{-}\lim_{k\to\infty} x_{n_k}, z = L$ for each nonzero z in X. The set of all $\mathcal{I}_F^2$-limit points of x is denoted by $\mathcal{I}\left(\Lambda_F^2(x)\right)$.

(ii) An element $L \in X$ is said to be an $\mathcal{I}$-cluster point of x with respect to the random 2-norm F (or $\mathcal{I}_F^2$-cluster point) if for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}.$$

The set of all $\mathcal{I}_F^2$-cluster points of x is denoted by $\mathcal{I}\left(\Gamma_F^2(x)\right)$.

Proposition 3.1. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal. Then for each sequence $x = (x_n)_{n\in\mathbb{N}}$ of X we have $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right)$ and the set $\mathcal{I}\left(\Gamma_F^2(x)\right)$ is a closed set.*

Proof. Let $L \in \mathcal{I}\left(\wedge_F^2(x)\right)$. Then there exists a set $M = \{n_1 < n_2 < ...\} \notin \mathcal{I}$ such that

$$F\text{-}\lim_{k\to\infty} x_{n_k}, z = L \tag{3.1}$$

for each nonzero z in X. According to 3.1, for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X there exists a positive integer k_0 such that for $k > k_0$ we have $x_{n_k}, z \in \mathcal{N}_L(\varepsilon, \lambda)$. Hence

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \supset M \backslash \{n_1, ..., n_{k_0}\}$$

and so

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I},$$

which means that $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$.

Let $y \in \overline{\mathcal{I}\left(\Gamma_F^2\right)}$. Take $\varepsilon > 0$ and $\lambda \in (0,1)$. There exists $L \in \mathcal{I}\left(\Gamma_F^2(x)\right) \cap \mathcal{N}_\theta(y, \varepsilon, \lambda)$. Choose $\eta > 0$ such that $\mathcal{N}_\theta(L, \eta, \lambda) \subset \mathcal{N}_\theta(y, \varepsilon, \lambda)$. We obviously have

$$\{n \in \mathbb{N} : y - x_n, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \supset \{n \in \mathbb{N} : L - x_n, z \in \mathcal{N}_\theta(\eta, \lambda)\}.$$

Hence $\{n \in \mathbb{N} : y - x_n, z \in \mathcal{N}_\theta(\varepsilon, \lambda)\} \notin \mathcal{I}$ and $y \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. □

Definition 3.2. Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n)_{n\in\mathbb{N}}$ be a sequence in X.

If $K = \{k_1 < k_2 < ...\} \in \mathcal{I}$, then the subsequence $x_K = (x_k)_{n\in\mathbb{N}}$ in X is called $\mathcal{I}_F^2$-thin subsequence of the sequence x in X.

If $M = \{m_1 < m_2 < ...\} \notin \mathcal{I}$, then the subsequence $x_M = (x_m)_{n\in\mathbb{N}}$ in X is called $\mathcal{I}_F^2$-nonthin subsequence of the sequence x in X.

It is clear that if L is a $\mathcal{I}_F^2$-limit point of $x \in X$, then there is a $\mathcal{I}_F^2$-nonthin subsequence x_M that convergent to L with respect to the random 2-norm F.

Definition 3.3. Let $(X, F, *)$ be an RTN space and $x = (x_n)_{n \in \mathbb{N}} \in X$. An element $L \in X$ is said to be limit point of the sequence $x = (x_n)$ with respect to the random 2-norm F if there is subsequence of the sequence x which converges to L with respect to the random 2-norm F. By $L_F^2(x)$, we denote the set of all limit points of the sequence $x = (x_n)$ with respect to the random 2-norm F.

It is obvious $\mathcal{I}\left(\Lambda_F^2(x)\right) \subseteq L_F^2(x)$, $\mathcal{I}\left(\Gamma_F^2(x)\right) \subseteq L_F^2(x)$: Take $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$, then $\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$ for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero z in X. If $L \notin L_F^2(x)$, then there is $\varepsilon' > 0$ such that $\mathcal{N}_L(\varepsilon', \lambda)$ contains only a finite number of elements of x in X. Then $\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon', \lambda)\} \in \mathcal{I}$, but it contradicts to $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Hence $x \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Thus $x \in L_F^2(x)$, and so $\mathcal{I}\left(\Gamma_F^2(x)\right) \subseteq L_F^2(x)$.

Lemma 3.1. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal. For a sequence $x = (x_n) \in X$, if x is $\mathcal{I}_F$-convergent with respect to the random 2-norm F, then $\mathcal{I}\left(\Lambda_F^2(x)\right)$ and $\mathcal{I}\left(\Gamma_F^2(x)\right)$ are both equal to the singleton set $\{\mathcal{I}_F\text{-}\lim x_n, z\}$ for each nonzero z in X.*

Proof. Let $\mathcal{I}_F$-$\lim_n x_n, z = L$. Show that $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$. By definition of $\mathcal{I}_F$-convergence we have

$$A(\varepsilon, \lambda) = \{n \in \mathbb{N} : x_n, z \notin \mathcal{N}_L(\varepsilon, \lambda)\} \in \mathcal{I}$$

for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. Since $\mathcal{I}$ is an admissible ideal we can choose the set $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ such that $n_k \notin A\left(\frac{1}{k}, \lambda\right)$ and $x_{n_k}, z \in \mathcal{N}_L\left(\frac{1}{k}, \lambda\right)$ for all $k \in \mathbb{N}$. That is F-$\lim_{k\to\infty} x_{n_k}, z = L$. Suppose $M \in \mathcal{I}$. Since $M \subset \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\}$,

$$(\mathbb{N} \backslash M) \cap \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\} = \emptyset,$$

but $\mathbb{N} \backslash M \in \mathcal{F}(\mathcal{I})$ and

$$\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(1, \lambda)\} \in \mathcal{F}(\mathcal{I}).$$

This contradiction gives $M \notin \mathcal{I}$. Hence we get $M = \{n_1 < n_2 < ...\} \subset \mathbb{N}$ and $M \notin \mathcal{I}$ such that F-$\lim_{k\to\infty} x_{n_k}, z = L$, i.e., $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$. Since $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right)$, $\xi \in \mathcal{I}\left(\Gamma_F^2(x)\right)$.

Now we suppose there is $\eta \in \mathcal{I}\left(\Gamma_F^2(x)\right)$ such that $\eta \neq L$. It is clear that

$$A = \left\{n \in \mathbb{N} : x_n, z \notin \mathcal{N}_L\left(\frac{|\eta - L|}{2}, \lambda\right)\right\} \in \mathcal{I}$$

and

$$B = \left\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L\left(\frac{|\eta - L|}{2}, \lambda\right)\right\} \notin \mathcal{I}$$

for $\lambda \in (0,1)$ and each nonzero $z \in X$. We have $B \subset A \in \mathcal{I}$. This contradiction shows $\mathcal{I}\left(\Gamma_F^2(x)\right) = \{L\}$. Hence from inclusion $\mathcal{I}\left(\Lambda_F^2(x)\right) \subset \mathcal{I}\left(\Gamma_F^2(x)\right) = \{L\}$, we have $\mathcal{I}\left(\Lambda_F^2(x)\right) = \mathcal{I}\left(\Gamma_F^2(x)\right) = L$. The lemma is proved. $\square$

Theorem 3.2. *Let $(X, F, *)$ be an RTN space, $\mathcal{I}$ be an admissible ideal and $x = (x_n)$, $y = (y_n)$ are sequences in X such that*

$$M = \{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}.$$

Then $\mathcal{I}\left(\Lambda_F^2(x)\right) = \mathcal{I}\left(\Lambda_F^2(y)\right)$ and $\mathcal{I}\left(\Gamma_F^2(x)\right) = \mathcal{I}\left(\Gamma_F^2(y)\right)$.

Proof. Let $M = \{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$. If $L \in \mathcal{I}\left(\Lambda_F^2(x)\right)$, then there is a set $K = \{n_1 < n_2 < ...\} \notin \mathcal{I}$ such that $F\text{-}\lim_k x_{n_k}, z = L$. Given $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $N \in \mathbb{N}$ such that $x_{n_k}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for $k > N$ and nonzero $z \in X$. Since $K_1 = \{n \in \mathbb{N} : n \in K \wedge x_n \neq y_n\} \subset M \in \mathcal{I}$,

$$K_2 = \{n \in \mathbb{N} : n \in K \wedge x_n = y_n\} \notin \mathcal{I}.$$

Indeed, if $K_2 \in \mathcal{I}$, then $K = K_1 \cup K_2 \in \mathcal{I}$, but $K \notin \mathcal{I}$. Hence the sequence $y_{K_2} = (y_n)_{n \in K_2}$ is an $\mathcal{I}_F^2$-nonthin subsequence of $y = (y_n)_{n \in \mathbb{N}}$ and y_{K_2} convergent to L with respect to the random 2-norm F. This implies that $L \in \mathcal{I}\left(\Lambda_F^2(y)\right)$. Similarly we can show that $\mathcal{I}\left(\Lambda_F^2(y)\right) \subset \mathcal{I}\left(\Lambda_F^2(x)\right)$. Hence $\mathcal{I}\left(\Lambda_F^2(y)\right) = \mathcal{I}\left(\Lambda_F^2(x)\right)$. Now let $L \in \mathcal{I}\left(\Gamma_F^2(x)\right)$. Then

$$B_1 = \{n \in \mathbb{N} : x_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$$

for each $\varepsilon > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$ and

$$B_2 = \{n \in \mathbb{N} : n \in B_1 \wedge x_n = y_n\} \notin \mathcal{I}.$$

Therefore, $B_2 \subset \{n \in \mathbb{N} : y_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\}$. It shows that $\{n \in \mathbb{N} : y_n, z \in \mathcal{N}_L(\varepsilon, \lambda)\} \notin \mathcal{I}$, i.e., $L \in \mathcal{I}\left(\Gamma_F^2(y)\right)$. The theorem is proved. □

The next theorem proves a strong connection between $\mathcal{I}_F^2$-cluster and limit points of a given sequence with respect to the random 2-norm F.

Definition 3.4. (Kostyrko *et al.*, 2000) An admissible ideal $\mathcal{I} \subset P(\mathbb{N})$ is said to satisfy the property (AP) if for every sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets of $\mathcal{I}$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$, such that the symmetric difference $A_n \Delta B_n$ is a finite set for every $n \in \mathbb{N}$ and $\cup_{n \in \mathbb{N}} B_n \in \mathcal{I}$.

Theorem 3.3. *Let $(X, F, *)$ be an RTN space and $\mathcal{I}$ be an admissible ideal with property (AP) and $x = (x_n)$ be a sequence in X. Then there is a sequence $y = (y_n) \in X$ such that $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(x)\right)$ and $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$.*

Proof. If $\mathcal{I}\left(\Gamma_F^2(x)\right) = L_F^2(x)$, then $y = x$ and this case is trivial. Let $\mathcal{I}\left(\Gamma_F^2(x)\right)$ be a proper subset of $L_F^2(x)$. Then $L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right) \neq \emptyset$ for each $L \in L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right)$. There is an $\mathcal{I}_F^2$-thin subsequence $\left(x_{j_k}\right)_{k \in \mathbb{N}}$ of x such that $\lim_k x_{j_k}, z = L$, i.e., given $\varepsilon > 0$, $\lambda \in (0,1)$ there exists a positive integer N such that $x_{j_k}, z \notin \mathcal{N}_L(\varepsilon, \lambda)$ for $k > N$ and nonzero $z \in X$. Hence there exists an $\mathcal{N}_L(\varepsilon, \lambda)$ such that $\{k \in \mathbb{N} : x_k, z \in \mathcal{N}_L = \mathcal{N}_L(\delta, \lambda)\} \in \mathcal{I}$ for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$.

It is obvious that the collection of all $\mathcal{N}_L$'s is an open cover of $L_F^2(x) \setminus \mathcal{I}\left(\Gamma_F^2(x)\right)$. So by Covering Theorem there is a countable and mutually disjoint subcover $\left\{\mathcal{N}_j\right\}_{j=1}^{\infty}$ such that each $\mathcal{N}_j$ contains an $\mathcal{I}_F^2$-thin subsequence of $(x_n) \in X$.

Now let

$$A_j = \left\{n \in \mathbb{N} : x_n, z \in \mathcal{N}_j = \mathcal{N}_j(\delta, \lambda),\ j \in \mathbb{N}\right\}$$

for each $\delta > 0$, $\lambda \in (0,1)$ and nonzero $z \in X$. It is clear that $A_j \in \mathcal{I}$ $(j = 1, 2, ...)$ and $A_i \cap A_j = \emptyset$. Then by (AP) property of $\mathcal{I}$ there is a countable collection $\left\{B_j\right\}_{j=1}^{\infty}$ of subsets of $\mathbb{N}$ such that $B = \cup_{j=1}^{\infty} B_j \in \mathcal{I}$ and $A_j \diagdown B$ is a finite set for each $j \in \mathbb{N}$. Let $M = \mathbb{N} \backslash B = \{m_1 < m_2 < ...\} \subset \mathbb{N}$. Now the sequence $y = (y_k) \in X$ is defined by $y_k = x_{m_k}$ if $k \in B$ and $y_k = x_k$ if $k \in M$. Obviously, $\{k \in \mathbb{N} : x_k \neq y_k\} \subset B \in \mathcal{I}$, and so by Theorem 3.2, $\mathcal{I}\left(\Gamma_F^2(y)\right) = \mathcal{I}\left(\Gamma_F^2(x)\right)$. Since $A_j \backslash B$ is a finite set, the sequence $y_B = (y_k)_{k \in B}$ has no limit point with respect to the random 2-norm F that is not also an $\mathcal{I}_F^2$-limit point of y, i.e., $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(y)\right)$. Therefore, we have proved $L_F^2(y) = \mathcal{I}\left(\Gamma_F^2(x)\right)$. □

Acknowledgements

We thank to the referees for their very carefully reading of the paper and making useful remarks that improved the presentation of the paper.

This work is supported by Süleyman Demirel University with Project 2947-YL-11.

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Theory and Applications of
Mathematics & Computer Science
(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 23–28

On λ-Zweier Convergent Sequence Spaces

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Abstract

In this paper we introduce a new concept of λ-Zweier convergence and λ-statistical Zweier convergence and give some relations between these two kinds of convergence.

Keywords: Strong summable sequences, Zwier Space, Satistical convergence, Banach space.
2000 MSC: 40C05, 40J05, 46A45.

1. Preliminaries

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ and l_∞, c and c_0 for the sets of all bounded, convergent sequences and null sequences, respectively.

A sequence space X with linear topology is called a K -space if each of the maps $P_i : X \to \mathbb{C}$ defined by $P_i(x) = x_i$ is continuous for $i = 1, 2, \cdots$.

A Fréchet space is a complete linear metric space, or equivalently , a complete totally paranormed space. In otherwords a locally convex space is called a Fréchet space if it is metrizable paranormed space and the Fréchet space is complete.

K -space X is called an FK-space if X is complete linear metric space. In otherwords we say that X is an FK-space if X is Fréchet space with continuous coordinate projection, we mean if $x^{(n)} \to x$ $(n \to \infty)$ in the metric of X then $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$, the linear functional $P_k(x) = x_k$ is such that P_k is continuous on X, i.e. X is K-space. Note that ω is a locally convex FK space with its usual metric. A BK-space is a normed FK-space (Choudhry & Nanda, 1989).

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers and $x \in \omega$. We write

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n = 0, 1, 2, \cdots)$$

and

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$$A(x) = (A_n(x))_{n=0}^{\infty}.$$

For any subset X of ω, the set

$$X_A = \{x = (x_k) \in \omega : A(x) \in X\}$$

is called the matrix domain of A in X.

Let $\lambda = (\lambda_n)$ be a non decereasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The generalized de la Vallee - Poussin means is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V,λ)-summable to a number l if $t_n(x) \to l$ as $n \to \infty$ (Leindler, 1965). We write

$$[V, \lambda]^0 = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0\},$$

$$[V, \lambda] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - le| = 0, \text{for some } l \in \mathbb{C}\},$$

$$[V, \lambda]^\infty = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty\}.$$

For the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded by the de la Valle- Poussin method.In the special case when $\lambda_n = n$ for $n = 1, 2, 3, \cdots$ the sets $[V, \lambda]^0$, $[V, \lambda]$ and $[V, \lambda]^\infty$ reduce the sets w_0 , w and w_∞ introduced and studied by Maddox (Maddox, 1986).

In (Sengönül, 2007), Sengönül introduced Z and Z_0 spaces as the set of all sequences such that £ - transforms of them are in the spaces c and c_0 , respectively , i.e.

$$Z = \{x = (x_k) \in \omega : £ \in c\},$$

$$Z_0 = \{x = (x_k) \in \omega : £ \in c_0\},$$

where $£ = (z_{nk})$, $(n, k = 0, 1, 2, \cdots)$ denotes by the matrix

$$z_{nk} = \begin{cases} \frac{1}{2}, & k \leq n \leq k + 1, \quad (n, k \in \mathbb{N}) \\ 0, & otherwise \end{cases}$$

This matrix is called Zweier matrix.

The concept of statistical convergence was first introduced by Fast (Fast, 1951) and further studied by Salat in (Salat, 1980), Fridy in (Fridy, 1985), Connor in (Connor, 1988), Kolk in (Kolk, 1996), (Kolk, 1993), M. K. Khan and C. Orhan in (Khan & Orhan, 2007), Fridy and Orhan in (Fridy & Orhan, 1997), (Fridy & Orhan, 1993) and many others. Let $\mathbb{N}$ be the set of natural numbers and E$\subset$ $\mathbb{N}$. Then the natural density of E is denoted by

$$\delta(E) = \lim_{n \to \infty} n^{-1} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set.
The sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every $\epsilon > 0$, the set $\{k : |x_k - l| \geq \epsilon\}$ has natural density 0, and we write $l = \mathrm{st} - \lim x$. We shall also write S to denote the set of all statistically convergent sequences.

2. Main Results

We introduce the sequence spaces $[V,\lambda]^0[Z], [V,\lambda][Z]$ and $[V,\lambda]^\infty[Z]$ as the set of all sequences such that Z-transforms of them are in the $[V,\lambda]^0, [V,\lambda]$ and $[V,\lambda]^\infty$ respectively i.e

$$[V,\lambda]^0[Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\frac{1}{2}(x_k + x_{k-1})| = 0\},$$

$$[V,\lambda][Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\frac{1}{2}(x_k + x_{k-1}) - l| = 0, \quad \text{for some l} \in \mathbb{C}\},$$

$$[V,\lambda]^\infty[Z] = \{x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\frac{1}{2}(x_k + x_{k-1})| < \infty\},$$

The $\pounds = (z_{nk})_{n,k \geq 0}$ matrix is well known as a regular matrix (Boos, 2000).Define the sequence y which will be frequently used, as £ - transform of the sequence x i.e.,

$$y_k = \frac{1}{2}(x_k + x_{k-1}), \quad (k \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *The sets* $[V,\lambda]^0[Z], [V,\lambda][Z]$ *and* $[V,\lambda]^\infty[Z]$ *are the linear spaces with the coordinatewise addition and scalar multiplication with the norm*

$$\|x\|_{[V,\lambda]^0[Z]} = \|x\|_{[V,\lambda][Z]} = \|x\|_{[V,\lambda]^\infty[Z]} = \|\pounds x\|_\lambda.$$

Proof. Suppose that x,y $\in [V,\lambda]^0[Z]$ and α, β are complex numbers. Then

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\frac{1}{2}[\alpha(x_k + x_{k-1}) + \beta(y_k + y_{k-1})]| \leq \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} (|\frac{1}{2}\alpha(x_k + x_{k-1})| + |\frac{1}{2}\beta(y_k + y_{k-1})|)$$

$$= \lim_n \frac{\alpha}{\lambda_n} \sum_{k \in I_n} (|\frac{1}{2}(x_k + x_{k-1})| + \lim_n \frac{\beta}{\lambda_n} \sum_{k \in I_n} (|\frac{1}{2}(x_k + x_{k-1})| = 0, \quad \text{as} \quad r \to \infty$$

Furthermore , since for any subset X of ω , the set

$$X_A = \{x = (x_k) \in \omega : A(x) \in X\} \qquad \text{(is called matrix domian of A in X)},$$

holds and $[V,\lambda]^0, [V,\lambda]$ are BK-spaces with respect to the norm defined by

$$\|x\|_{[V,\lambda]} = \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k|$$

and the matrix $\pounds = (z_{nk})$ is normal, that is $z_{nk} \neq 0$, for $0 \leq k \leq n$ and $z_{nk} = 0$ for $k > n$ for all $n, k \in \mathbb{N}$ and also by Theorem 4.3.2 of Wilansky (Wilansky, 1984) gives the fact that the spaces $[V,\lambda]^0[Z]$ and $[V,\lambda][Z]$ are BK spaces. □

Theorem 2.2. *The sequence spaces* $[V,\lambda]^0[Z], [V,\lambda][Z]$ *and* $[V,\lambda]^\infty[Z]$ *are linearly isomorphic to the spaces* $[V,\lambda]^0, [V,\lambda]$ *and* $[V,\lambda]^\infty$ *respectively.*

Proof. We want to show the existance of the linear bijection between the spaces $[V,\lambda]^0[Z]$ and $[V,\lambda]^0$.Consider the transformation £ defined by (1), from $[V,\lambda]^0[Z]$ to $[V,\lambda]^0$ by

$$£ : [V,\lambda]^0[Z] \to [V,\lambda]^0$$

$$x \to £x = y, \quad y = (y_k), \quad y_k = \frac{1}{2}(x_k + x_{k-1}), \quad (k \in \mathbb{N}).$$

The linearity of £ is clear. Further it is trivial that $x = 0$ when $£x = 0$ and hence £ is injective. Let $y \in [V,\lambda]^0$ and define the sequence $x = (x_k)$ by

$$x_k = 2\sum_{i=0}^{k}(-1)^{k-i}y_i \qquad (n \in \mathbb{N}).$$

Then

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|\frac{1}{2}(x_k + x_{k-1})|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{k-i}y_i + 2\sum_{k=0}^{i}(-1)^{(k-1)-i}y_i)|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}|y_i|.$$

This implies that $x \in [V,\lambda]^0$[Z]. Also

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}|\frac{1}{2}(x_k + x_{k-1})|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}|\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{i-k}y_k + 2\sum_{k=0}^{i}(-1)^{i-k-1}y_k)|$$

$$\|x\|_{[V,\lambda]^0[Z]} = \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}|y_k| = \|y\|^0_{[V,\lambda]}.$$

Thus we have that $x \in [V,\lambda]^0$[Z] and consequently £ is surjective. Hence £ is linear bijection which therefore says us that the spaces $[V,\lambda]^0[Z]$ and $[V,\lambda]^0$ are linearly isomorphic. It is clear here that if the spaces $[V,\lambda]^0[Z]$ and $[V,\lambda]^0$ replaced by the spaces $[V,\lambda][Z]$ and $[V,\lambda]$ or $[V,\lambda]^\infty[Z]$and $[V,\lambda]^\infty$, respectively. Then

$$[V,\lambda][Z] \cong [V,\lambda]^\infty[Z] \quad \text{or} \quad [V,\lambda]^\infty[Z] \cong [V,\lambda]^\infty.$$

This completes the proof. □

A sequence $x = (x_k)$ is said to be λ-statistical Zweir convergent to a number l if for $\epsilon > 0$.

$$R_\lambda[Z] = \{\frac{1}{\lambda_n}\sum_{k\in I_n}|£M_\lambda(\epsilon)| = 0\},$$

where

$$£M_\lambda(\epsilon) = \{[n - \lambda_n + 1, n] : |\frac{1}{2}(x_k + x_{k-1}) - l \geq \epsilon|\}.$$

Let

$$[n-\lambda_n+1,n]^* = \{[n-\lambda_n+1,n] : |\frac{1}{2}(x_k+x_{k-1}) - l \geq \epsilon|\} = CM_\lambda(\epsilon)$$

and

$$[n-\lambda_n+1,n]^{**} = \{[n-\lambda_n+1,n] : |\frac{1}{2}(x_k+x_{k-1}) - l| < \epsilon\}.$$

Theorem 2.3. *If* $x_k \to l[V,\lambda][Z] \Longrightarrow x_k \to l(R_\lambda[Z])$.

Proof. Let $\epsilon > 0$ and $x_k \to l[V,\lambda][Z]$, then

$$\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]}|\frac{1}{2}(x_k+x_{k-1}) - l|$$

$$\geq \frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^*}|\frac{1}{2}(x_k+x_{k-1}) - l|$$

$$\geq \frac{1}{\lambda_n}|£M_\lambda(\epsilon)|.$$

This implies that $x_k \to l(R_\lambda[Z])$. □

Theorem 2.4. *If* $x \in [V,\lambda]^\infty[Z]$ *and* $x_k \to l[V,\lambda][Z] \Longrightarrow x_k \to l(R_\lambda[Z])$.

Proof. Suppose that $x \in [V,\lambda]^\infty[Z]$ and $x_k \to l[V,\lambda][Z]$. Since $sup_k|\frac{1}{2}(x_k + x_{k-1} - l)| < \infty$, there exists a constant $T > 0$ such that $|\frac{1}{2}(x_k + x_{k-1}) - l| < T$ for all k. Then we have, for every $\epsilon > 0$ that

$$\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]}|\frac{1}{2}(x_k+x_{k-1}) - l|$$

$$= \frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^*}|\frac{1}{2}(x_k+x_{k-1}) - l|$$

$$+\frac{1}{\lambda_n}\sum_{k\in[n-\lambda_n+1,n]^{**}}|\frac{1}{2}(x_k+x_{k-1}) - l|$$

$$\leq \frac{T}{\lambda_n}|£M_\lambda(\epsilon)| + \epsilon,$$

taking limit as $\epsilon \to 0$. Thus $x_k \to l([V,\lambda]^\infty[Z])$. □

Theorem 2.5. *If* $x \in [V,\lambda]^\infty[Z]$ *then* $[V,\lambda][Z] = R_\lambda[Z]$.

Proof. Proof follows from Theorem 2.3 and Theorem 2.4. □

Acknowledgments. The author would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

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Theory and Applications of Mathematics & Computer Science
(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 29–36

On H–Dichotomy for Skew-Evolution Semiflows in Banach Spaces

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Abstract

The aim of this paper is to define and characterize a particular case of dichotomy for skew-evolution semiflows, called the H–dichotomy, as a useful tool in describing the behaviors for the solutions of evolution equations that describe phenomena from engineering or economics. The paper emphasizes also other asymptotic properties, as ω–growth and ω–decay, H–stability and H–instability, as well as the classic concept of exponential dichotomy.

Keywords: Evolution semiflow, evolution cocycle, skew-evolution semiflow, ω–growth, ω–decay, H–dichotomy.
2000 MSC: 34D05, 34D09, 93D20.

1. Preliminaries

The study of the behaviors of the solutions of evolution equations by means of associated operator families has allowed to obtain answers to some previously open problems by involving techniques of functional analysis and operator theory.

In the qualitative theory of evolution equations, the exponential dichotomy is one of the most important asymptotic properties, and in the last years it was treated from various perspectives.

The notion of exponential dichotomy for linear differential equations was introduced by O. Perron in 1930. The classic paper (Perron, 1930) of Perron served as a starting point for many works on the stability theory. The property of exponential dichotomy for linear differential equations has gained prominence since the appearance of two fundamental monographs due to J.L. Daleckii and M.G. Krein (see (Daleckii & Krein, 1974)) and J.L. Massera and J.J. Schäffer (see (Massera & Schäffer, 1966)).

Diverse and important concepts of dichotomy for linear skew-product semiflows were studied by C. Chicone and Y. Latushkin in (Chicone & Latushkin, 1999), S.N. Chow and H. Leiva in (Chow & Leiva, 1995), R.J. Sacker and G.R. Sell in (Sacker & Sell, 1994) as well as G.R. Sell and Y.You in (Sell & You, 2002).

The exponential stability and exponential instability for nonautonomous differential equations are studied by L. Barreira and C. Valls in (Barreira & Valls, 2008), and, in particular, for linear skew-product semiflows, by M. Megan, A.L. Sasu and B. Sasu in (Megan *et al.*, 2004).

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We have reconsidered the definitions of asymptotic properties by means of skew-evolution semiflow on a Banach space, introduced in (Megan & Stoica, 2008*a*), as an important tool in the stability theory and as a natural generalization for semigroups of operators, evolution operators and skew-product semiflows.

A skew-evolution semiflow depends on three variables t, t_0 and x, while the classic concept of cocycle depends only on t and x, thus justifying a further study of asymptotic behaviors for skew-evolution semiflows in a more general case, the nonuniform setting (relative to the third variable).

The notion of linear skew-evolution semiflow arises naturally when considering the linearization along an invariant manifold of a dynamical system generated by a nonlinear differential equation. The notion has proved itself of interest in the development of the stability theory, in a uniform as well as in a nonuniform setting, being already adopted by some researchers, as, for example, P. Viet Hai in (Viet Hai, 2010) and A.J.G. Bento and C.M. Silva in (Bento & Silva, 2012). Some results concerning the asymptotic properties for skew-evolution semiflows were published in (Megan & Stoica, 2008*b*), (Megan & Stoica, 2010), (Stoica & Megan, 2010) and (Stoica, 2010).

In what follows, we will consider a more general case for asymptotic behaviors that does not involve necessarily exponentials, but, instead, properly defined functions, which allows a non restrained approach. The aim of this paper is to define and characterize a more general case of dichotomy for skew-evolution semiflows, called the H–dichotomy, as a tool in the study the behaviors for the solutions of differential equations that describe processes from engineering, physics or economics, and to emphasize connections with the classic concept.

The motivation for the approach of the H–dichotomy is due to the fact that the characterizations in this case do not impose restrictions neither on the matrix A, which defines the system of differential equations, nor on the solutions, such as bounded growth or decay.

2. Notations. Definitions

Let us denote by X a metric space, by V a Banach space, by V^* its dual, and by $\mathcal{B}(V)$ the space of all bounded linear operators from V into itself. We consider the set $T = \left\{(t, t_0) \in \mathbb{R}^2_+,\ t \geq t_0\right\}$. Let I be the identity operator on V. We denote $Y = X \times V$ and $Y_x = \{x\} \times V$, where $x \in X$.

Let us define the sets

$$\mathcal{H} = \{H : \mathbb{R}_+ \to \mathbb{R}^*_+ |\ H \text{ continuous}\}$$

and

$$\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ |\ \exists \mu \in \mathbb{R} \text{ such that } f(t) = e^{\mu t}, \forall t \geq 0\}$$

with the subsets $\mathcal{F}_+$ and $\mathcal{F}_-$ for positive, respectively negative values of μ.

We will denote by $\mathcal{K}$ the set of all continuous functions $h : \mathbb{R}_+ \to [1, \infty)$ such that, for all $H \in \mathcal{H}$, there exist a function $f \in \mathcal{F}$ and a constant $k > 0$ with the properties

$$h(s) \leq kf(t - s)H(t), \text{ and } h(2t)h(2s) \leq H(t + s),\ \forall t, s \geq 0.$$

Remark. As we can consider $h(t) = f(t) = e^{\nu t}$ and $H(t) = e^{2\nu t}$, $\nu > 0$, $t \geq 0$, it follows that the set $\mathcal{K}$ is not empty.

Definition 2.1. The mapping $C : T \times Y \to Y$ defined by the relation

$$C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v),$$

where $\varphi : T \times X \to X$ has the properties

(s_1) $\varphi(t,t,x) = x$, $\forall (t,x) \in \mathbb{R}_+ \times X$;
(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$
and $\Phi : T \times X \to \mathcal{B}(V)$ satisfy
(c_1) $\Phi(t,t,x) = I, \forall (t,x) \in \mathbb{R}_+ \times X$;
(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x), \forall (t,s,t_0) \in T, x \in X$,
is called *skew-evolution semiflow* on Y.

Remark. φ is called *evolution semiflow* and Φ *evolution cocycle* over the evolution semiflow φ.

Remark. If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$\Phi_\alpha : T \times X \to \mathcal{B}(V),\ \Phi_\alpha(t,t_0,x) = e^{\alpha(t-t_0)}\Phi(t,t_0,x), \tag{2.1}$$

is the *α-shifted skew-evolution semiflow*.

Example 2.1. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t,s,x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V),\ \Phi_E(t,s,x) = E(t-s+x,x)$$

is an evolution cocycle on V over the evolution semiflow φ. Hence, an evolution operator on V is generating a skew-evolution semiflow on Y.

Example 2.2. Let $f : \mathbb{R}_+ \to (0,\infty)$ be a decreasing function. We denote by X the closure in C, the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, of the set $\{f_t,\ t \in \mathbb{R}_+\}$, where $f_t(\tau) = f(t+\tau), \forall \tau \in \mathbb{R}_+$. The mapping $\varphi_0 : \mathbb{R}_+ \times X \to X,\ \varphi_0(t,x) = x_t$, where $x_t(\tau) = x(t+\tau), \forall \tau \geq 0$, is a semiflow on X. Let $V = \mathcal{L}^2(0,1)$ be a separable Hilbert space with the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ defined by $e_0 = 1$ and $e_n(y) = \sqrt{2}\cos n\pi y$, where $y \in (0,1)$ and $n \in \mathbb{N}$. Let us consider the Cauchy problem

$$\begin{cases} \dot{v}(t) = A(\varphi_0(t,x))v(t), & t > 0 \\ v(0) = v_0. \end{cases} \tag{2.2}$$

where $A : X \to \mathcal{B}(V)$ is a continuous mapping. We consider a C_0-semigroup S given by the relation

$$S(t)v = \sum_{n=0}^{\infty} e^{-n^2\pi^2 t}\langle v, e_n\rangle e_n,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in V. The mapping

$$\Phi_0 : \mathbb{R}_+ \times X \to \mathcal{B}(V),\ \Phi_0(t,x)v = S\left(\int_0^t x(s)ds\right)v$$

is a cocycle over the semiflow φ_0 and $C_0 = (\varphi_0, \Phi_0)$ is a linear skew-product semiflow on Y. Also, for all $v_0 \in D(A)$, we have that $v(t) = \Phi_0(t,x)v_0$, $t \geq 0$, is a strong solution of system (2.2). Then the mapping

$$C : T \times Y \to Y,\ C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where

$$\varphi(t,s,x) = \varphi_0(t-s,x) \text{ and } \Phi(t,s,x) = \Phi_0(t-s,x),\ \forall (t,s,x) \in T \times X$$

is a skew-evolution semiflow on Y. Hence, the skew-evolution semiflows are generalizations of skew-product semiflows.

Other examples of skew-evolution semiflows are given in (Stoica & Megan, 2010).

Definition 2.2. $C = (\varphi, \Phi)$ has *ω–growth* if there exists a nondecreasing function $\omega : \mathbb{R}_+ \to [1, \infty)$ with the property $\lim\limits_{t\to\infty} \omega(t) = \infty$ such that:

$$\|\Phi(t, t_0, x)v\| \leq \omega(t-s) \|\Phi(s, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. If C has ω–growth, then the $-\alpha$-shifted skew-evolution semiflow $C_{-\alpha} = (\varphi, \Phi_{-\alpha})$, $\alpha > 0$, has also ω–growth.

Remark. The property of ω–growth is equivalent with the property of exponential growth (see (Stoica, 2010)).

Definition 2.3. $C = (\varphi, \Phi)$ has *ω–decay* if there exists a nondecreasing function $\omega : \mathbb{R}_+ \to [1, \infty)$ with the property $\lim\limits_{t\to\infty} \omega(t) = \infty$ such that:

$$\|\Phi(s, t_0, x)v\| \leq \omega(t-s) \|\Phi(t, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. If C has ω–decay, then the α-shifted skew-evolution semiflow $C_{\alpha} = (\varphi, \Phi_{\alpha})$, $\alpha > 0$, has also ω–decay.

Remark. The property of ω–decay is equivalent with the property of exponential decay (see (Stoica, 2010)).

3. Concepts of dichotomy

Definition 3.1. A continuous mapping $P : Y \to Y$ defined by

$$P(x, v) = (x, P(x)v), \ \forall (x, v) \in Y, \tag{3.1}$$

where $P(x)$ is a linear projection on Y_x, is called *projector* on Y.

Definition 3.2. A projector P on Y is called *invariant* relative to a skew-evolution semiflow $C = (\varphi, \Phi)$ if following relation holds:

$$P(\varphi(t, s, x))\Phi(t, s, x) = \Phi(t, s, x)P(x), \tag{3.2}$$

for all $(t, s) \in T$ and all $x \in X$.

Definition 3.3. Two projectors P and Q, defined by (3.1), are said to be *compatible* with a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

(t_1) each of the projectors is invariant on Y, as in (3.2);

(t_2) $\forall x \in X$, the projections $P(x)$ and $Q(x)$ verify the relations

$$P(x) + Q(x) = I \text{ and } P(x)Q(x) = 0.$$

Definition 3.4. $C = (\varphi, \Phi)$ is *exponentially dichotomic* relative to the compatible projectors P and Q if there exist $\alpha > 0$ and two nondecreasing mappings $N_1, N_2 : \mathbb{R}_+ \to [1, \infty)$ such that:

(ed_1)

$$e^{\alpha(t-s)} \|\Phi_P(t, t_0, x)v\| \leq N_1(s) \|\Phi_P(s, t_0, x)v\| ; \tag{3.3}$$

(ed_2)

$$e^{\alpha(t-s)} \left\|\Phi_Q(s, t_0, x)v\right\| \leq N_2(t) \left\|\Phi_Q(t, t_0, x)v\right\| , \tag{3.4}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$.

Remark. In Definition 3.4, relation (3.3) is the definition for the exponential stability and relation (3.4) for the exponential instability.

A more general concept of dichotomy is given by

Definition 3.5. $C = (\varphi, \Phi)$ is *H–dichotomic* relative to the compatible projectors P and Q if there exist two nondecreasing mappings $N_1, N_2 : \mathbb{R}_+ \to [1, \infty)$ such that:

(Hed_1)

$$H(t) \|\Phi_P(t, t_0, x)v\| \leq N_1(t_0) \|P(x)v\| ; \tag{3.5}$$

(Hed_2)

$$H(s) \left\|\Phi_Q(s, t_0, x)v\right\| \leq N_2(t) \left\|\Phi_Q(t, t_0, x)v\right\| , \tag{3.6}$$

for all $(t, s), (s, t_0) \in T$, all $(x, v) \in Y$ and all $H \in \mathcal{H}$.

Remark. For $H(t) = e^{\nu t},\ t \geq 0,\ \nu > 0$ the exponential dichotomy for skew-evolution semiflows is obtained.

Example 3.1. Let us consider the system of differential equations

$$\begin{cases} \dot{u} = (-2t \sin t - 3)u \\ \dot{w} = (t \cos t + 2)w \end{cases}$$

Let $X = \mathbb{R}_+$ and $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|,\ v = (v_1, v_2) \in \mathbb{R}^2$. Then the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi(t, s, x) = x_{t-s}$$

is an evolution semiflow and the mapping $\Phi : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}^2)$ given by

$$\Phi(t, s, x)(v_1, v_2) = (U(t, s)v_1, W(t, s)v_2) =$$

$$= (e^{2t \cos t - 2s \cos s - 2 \sin t + 2 \sin s - 3t + 3s} v_1, e^{t \sin t - s \sin s + \cos t - \cos s + 2t - 2s} v_2),$$

where $U(t, s) = u(t)u^{-1}(s)$, $W(t, s) = w(t)w^{-1}(s)$, $(t, s) \in T$, and $u(t)$, $w(t)$, $t \in \mathbb{R}_+$, are the solutions of the given differential equations, is an evolution cocycle. We obtain that the skew-evolution semiflow $C = (\varphi, \Phi)$ is H–dichotomic relative to the compatible projectors P, $Q : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $P(x, v) = (v_1, 0)$ and $Q(x, v) = (0, v_2)$, where $v = (v_1, v_2)$, with

$$H(u) = e^u,\ N_1(s) = e^{5s+4} \text{ and } N_2(s) = e^{-t+2}.$$

In what follows, if P is a given projector, we will denote for every $(t, s, x) \in T \times X$

$$\Phi_P(t, s, x) = \Phi(t, s, x)P(x) \text{ and } C_P = (\varphi, \Phi_P).$$

We remark that

(i) $\Phi_P(t, t, x) = P(x)$, for all $(t, x) \in \mathbb{R}_+ \times X$;

(ii) $\Phi_P(t, s, \varphi(s, t_0, x))\Phi_P(s, t_0, x) = \Phi_P(t, t_0, x)$, for all $(t, s), (s, t_0) \in T$, $x \in X$.

The following result is an integral characterization for the concept of H-dichotomy.

Theorem 3.2. *Let P, $Q : \mathbb{R}_+ \to \mathcal{B}(V)$ be two projectors compatible with $C = (\varphi, \Phi)$ with the property that C_P has ω–growth and C_Q has ω–decay. Let $H \in \mathcal{H}$ and $h \in \mathcal{K}$. Then C is H–dichotomic if and only if there exist two mappings $M_1, M_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:*

(i)

$$\int_{t_0}^{t} h(\tau) \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \leq M_1(t_0)H(t) \left\|P(x)v^*\right\|, \tag{3.7}$$

(ii)

$$h(t_0) \int_0^t \frac{1}{H(\tau)} \left\|\Phi_Q(\tau, t_0, x)v\right\| d\tau \leq M_2(t) \left\|\Phi_Q(t, t_0, x)v\right\|, \tag{3.8}$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$, $v^* \in V^*$ with $\|v^*\| \leq 1$.

Proof. *Necessity.* (i) As the skew-evolution semiflow C is H-dichotomic, it implies that the relation (3.5) of Definition 3.5 holds. There exist a function $f \in \mathcal{F}_-$ and a constant $k > 0$ such that

$$h(s) \leq kf(t - s)H(t),\ \forall (t, s) \in T.$$

Let us denote $f(t) = e^{-\nu t}$, $\nu > 0$. We obtain the inequalities

$$\|\Phi_P(t, t_0, x)v\| \leq \frac{N_1(t)}{H(t)} \|\Phi_P(s, t_0, x)v\| \leq k\frac{N_1(s)}{h(s)} e^{-\nu(t-s)} \|\Phi_P(s, t_0, x)v\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. Further we have

$$\int_{t_0}^{t} h(\tau) \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \leq kH(t) \int_{t_0}^{t} h(\tau)e^{-\nu(t-\tau)} \left\|\Phi_P(t, \tau, x)^* v^*\right\| d\tau \leq M_1(t_0)H(t) \left\|P(x)v^*\right\|,$$

where we have denoted $M_1(t) = k\nu^{-1}N_1(t)$, $t \geq 0$.

(ii) We have that the relation (3.6) of Definition 3.5 takes place. There exist a function $f \in \mathcal{F}_-$ and a constant $k > 0$ such that

$$h(t_0) \leq kf(s - t_0)H(s),\ \forall (s, t_0) \in T.$$

Let us consider $f(t) = e^{-\nu t}$, $\nu > 0$. We have

$$\left\|\Phi_Q(s, t_0, x)v\right\| \leq \frac{N_2(t)}{H(s)} \left\|\Phi_Q(t, t_0, x)v\right\| \leq k\frac{N_2(t)}{h(t_0)} e^{-\nu(s-t_0)} \left\|\Phi_Q(t, t_0, x)v\right\| \leq$$

$$\leq k\frac{N_2(t)}{h(t_0)} e^{\nu t} e^{-\nu(s-t_0)} e^{-\nu(2s-t_0)} \left\|\Phi_Q(t, t_0, x)v\right\| \leq kN_2(t)e^{\nu t} e^{-\nu(t-s)} \left\|\Phi_Q(t, t_0, x)v\right\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. Further we have

$$h(t_0) \int_{t_0}^{t} \frac{1}{H(\tau)} \left\| \Phi_Q(t, t_0, x)v \right\| d\tau \le kM \int_{t_0}^{t} e^{-\nu(\tau - t_0)} e^{\delta(t-\tau)} \left\| \Phi_Q(t, t_0, x)v \right\| d\tau \le M_2(t) \left\| P(x)v \right\|,$$

where we have denoted $M_2(t) = \dfrac{kM}{\nu + \delta} e^{(\nu+\delta)t}$, $t \ge 0$, and where we have defined in Definition 2.3 the function $\omega(t) = Me^{\delta t}$, $M \ge 1$ and $\delta > 0$.

Sufficiency. (i) We suppose that relation (3.7) takes place. Let us first consider the case $t \in [t_0, t_0 + 1)$. We have, as $0 \le t - t_0 < 1$,

$$\left\| \Phi_P(t, t_0, x)v \right\| \le Me^{\alpha+\delta} e^{-\alpha(t-t_0)} \left\| P(x)v \right\|,$$

for all $(x, v) \in Y$, where we have considered in Definition 2.2 the function $\omega(t) = Me^{\delta t}$, $M \ge 1$ and $\delta > 0$.

On the second hand, we consider the case $t \ge t_0 + 1$ and $s \in [t_0, t_0 + 1]$. As $H \in \mathcal{H}$ and $h \in \mathcal{K}$, there exists a constant $\alpha > 0$ such that $h(s) \ge e^{-\alpha(t-s)} H(t)$, for all $(t, s \in T)$. We have

$$e^{-(\alpha+\delta)} \left| \langle v^*, e^{\alpha(t-t_0)} \Phi_P(t, t_0, x)v \rangle \right| \le e^{-(\alpha+\delta)(\tau-t_0)} \left| \langle v^*, e^{\alpha(t-t_0)} \Phi_P(t, t_0, x)v \rangle \right| =$$

$$= e^{-(\alpha+\delta)(\tau-t_0)} \int_{t_0}^{t_0+1} \left| \langle \Phi_P(t, \tau, \varphi(\tau, t_0, x))^* v^*, e^{\alpha(t-t_0)} \Phi_P(\tau, t_0, x)v \rangle \right| d\tau \le$$

$$\le \int_{t_0}^{t_0+1} e^{\alpha(t-\tau)} \left\| \Phi_P(t, \tau, \varphi(\tau, t_0, x))^* v^* \right\| e^{-\delta(\tau-t_0)} \left\| \Phi_P(\tau, t_0, x)v \right\| d\tau \le$$

$$\le M \left\| P(x)v \right\| \int_{t_0}^{t} e^{\alpha(t-\tau)} \left\| \Phi_P(t, \tau, \varphi(\tau, t_0, x))^* v^* \right\| d\tau \le$$

$$\le MM_1(t_0) \left\| P(x)v \right\| \left\| P(x)v^* \right\|.$$

By taking supremum relative to $\|v^*\| \le 1$ it follows that

$$\left\| \Phi_P(t, t_0, x)v \right\| \le Me^{\alpha+\delta} M_1(t_0) e^{-\alpha(t-t_0)} \left\| P(x)v \right\|$$

Thus, we obtain

$$\left\| \Phi_P(t, t_0, x)v \right\| \le Me^{\alpha+\delta} \left[M_1(t_0) + 1 \right] e^{-\alpha(t-t_0)} \left\| P(x)v \right\|,$$

for all $(t, t_0) \in T$ and $(x, v) \in Y$. Let us now define $H(t) = e^{\alpha t}$ and $N_1(t_0) = Me^{\alpha+\delta} \left[M_1(t_0) + 1 \right] e^{\alpha t_0}$. We obtain thus relation (3.5).

(ii) For $H \in \mathcal{H}$ and $h \in \mathcal{K}$, there exists a constant $\beta > 0$ such that $h(s) \le e^{-\beta(t-s)} H(t)$, $\forall (t, s) \in T$. Let us denote

$$K = \int_0^1 e^{-\beta\tau} \omega(\tau) d\tau,$$

where the function ω is given by Definition 2.3. We have

$$K \left\| Q(x)v \right\| = \int_{t_0}^{t_0+1} e^{-\beta(\tau-t_0)} \omega(\tau - t_0) \left\| \Phi_Q(t_0, t_0, x)v \right\| d\tau \le$$

$$\le \int_{t_0}^{t_0+1} e^{-\beta(\tau-t_0)} \left\| \Phi_Q(\tau, t_0, x)v \right\| d\tau \le M_2(t) e^{\beta(t-t_0)} \left\| \Phi_Q(t, t_0, x)v \right\|,$$

for all $(t, t_0) \in T$ and all $(x, v) \in Y$. This relation implies

$$\left\| \Phi_Q(s, t_0, x)v \right\| \leq \frac{1}{K} M_2(t) e^{\beta(t-s)} \left\| \Phi_Q(t, t_0, x)v \right\|,$$

for all $(t, s), (s, t_0) \in T$ and all $(x, v) \in Y$. Let us define $H(s) = e^{\beta s}$ and $N_2(t) = \dfrac{1}{K} M_2(t) e^{\beta t}$. Relation (3.6) is thus obtained.

$\square$

Remark. In Definition 3.5, relation (3.5) gives the definition for the H-stability and relation (3.6) for the H-instability, characterized, respectively, by the relations (3.7) and (3.8) of Theorem 3.2.

Acknowledgements

The authors gratefully acknowledge helpful suggestions and support from Professor Mihail Megan. Also, we would like to express our gratitude to the referees for the valuable comments which have improved the final version of the paper.

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Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 37–52

On Multiset Topologies

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Abstract

In this paper an attempt is made to extend the concept of topological spaces in the context of multisets (mset, for short). The paper begins with basic definitions and operations on msets. The mset space $[X]^w$ is the collection of msets whose elements are from X such that no element in the mset occurs more than finite number (w) of times. Different types of collections of msets such as power msets, power whole msets and power full msets which are submsets of the mset space and operations under such collections are defined. The notion of M-topological space and the concept of open msets are introduced. More precisely, an M-topology is defined as a set of msets as points. Furthermore the notions of basis, sub basis, closed sets, closure and interior in topological spaces are extended to M-topological spaces and many related theorems have been proved. The paper concludes with the definition of continuous mset functions and related properties, in particular the comparison of discrete topology and discrete M-topology are established.

Keywords: Multisets, Power Multisets, Multiset Relations, Multiset Functions, M-Topology, M-Basis and Sub M-Basis, Continuous Mset Functions.
2000 MSC: 00A05, 03E70, 03E99, 06A99.

1. Introduction

The notion of a multiset (bag) is well established both in mathematics and computer science (Clements, 1988; Conder *et al.*, 2007; Galton, 2003; Singh *et al.*, 2011; Skowron, 1988; Šlapal, 1993). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained (Singh, 1994; Singh *et al.*, 2007; Singh & Singh, 2003; Wildberger, 2003). In various counting arguments it is convenient to distinguish between a set like $\{a, b, c\}$ and a collection like $\{a, a, a, b, c, c\}$. The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer.

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Classical set theory states that a given element can appear only once in a set, it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different. The situation in science and in ordinary life is not like this. In the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate. This leads to three possible relations between any two physical objects; they are different, they are the same but separate or they coincide and are identical. For the sake of definiteness we say that two physical objects are the same or equal, if they are indistinguishable, but possibly separate, and identical if they physically coincide.

Topology, as a branch of mathematics, can be formally defined as the study of qualitative properties of certain objects called topological spaces that are invariant under certain kinds of transformations called continuous maps (Galton, 2003; Skowron, 1988; Šlapal, 1993). There are many occasions, however, when one encounters collections of non-distinct objects. In such situations the term 'multiset' is used instead of 'set'. In this paper topologies on multisets are provided and they can be useful for measuring the similarities and dissimilarities between the universes of the objects which are multisets. Moreover, topologies on multisets can be associated to IC-bags or n^k-bags introduced by K. Chakrabarthy (Chakrabarty, 2000; Chakrabarty & Despi, 2007) with the help of rough set theory. The association of rough seth theory and topologies on multisets through bags with interval counts (Chakrabarty & Despi, 2007) can be used to develop theoretical study of covering based rough sets with respect to universe as multisets.

The mset space $[X]^w$ is the collection of finite msets whose elements are from X such that no member of an element of $[X]^w$ occurs more than finite number (w) of times. i.e., every msets in the collection $[X]^w$ are finite cardinality with each element having multiplicity atmost w. Different types of collections of msets such as power msets, power whole msets and power full msets which are submsets of the mset space and operations under such collections of msets are defined. The notion of M-topological space and the concept of open multisets are introduced. More precisely, a multiset topology is defined as a set of multisets as points. The notion of basis, sub basis, closed sets, closure and interior in topological spaces are extended to M-topological spaces and many related theorems have been proved. The paper concludes with the definition of continuous mset functions and related properties.

2. Preliminaries and Basic Definitions

In this section some basic definitions, results and notations as introduced by V. G. Cerf et al. (Gostelow *et al.*, 1972) in 1972, J. L. Peterson (Peterson, 1976) in 1976, R. R. Yager (Yager, 1987, 1986) in 1986, W. D. Blizard (Blizard, 1989*a*, 1990, 1989*b*, 1991) in 1989, K. Chakrabarty et al. (Chakrabarty & Despi, 2007; Chakrabarty *et al.*, 1999*b*,*a*; Chakrabarty & Despi, 2007) in 1999, S. P. Jena et al. (Jena *et al.*, 2001) in 2001 and the authors concepts in (Girish & John, 2009*a*, 2012, 2009*b*; Girish & Jacob, 2011; Girish & John, 2011) are presented.

Definition 2.1. (Girish & John, 2012) An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, \ldots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, \ldots, n$ in the mset M. However those elements which are not included in the mset M have zero count.

Example 2.1. (Girish & John, 2012) Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Let M and N be two msets drawn from a set X. Then, the following are defined in (Girish & John, 2012):

(i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
(ii) $M \subseteq N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.
(iii) $P = M \cup N$ if $C_P(x) = \text{Max}\ \{C_M(x), C_N(x)\}$ for all $x \in X$.
(iv) $P = M \cap N$ if $C_P(x) = \text{Min}\ \{C_M(x), C_N(x)\}$ for all $x \in X$.
(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.
(vi) $P = M \ominus N$ if $C_P(x) = \text{Max}\{C_M(x) - C_N(x), 0\}$ for all $x \in X$ where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an empty mset if for all $x \in X$, $C_M(x) = 0$.

The cardinality of an mset M drawn from a set X is denoted by Card (M) or M and is given by Card $M = \sum_{x \in X} C_M(x)$.

Definition 2.2. (Girish & John, 2012) A domain X, is defined as a set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times.

The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset.

If $X = \{x_1, x_2, \ldots, x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, \ldots, m_k/x_k\} : \text{for } i = 1, 2, \ldots k;\ m_i \in \{0, 1, 2, \ldots w\}\}$.

Definition 2.3. (Girish & John, 2012) Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C^c_M(x) = w - C_M(x)$ for all $x \in X$.

Remark 2.1. Using Definition 2.3, the mset sum can be modified as follows:

$$C_{M_1 \oplus M_2}(x) = \min\{w, C_{M1}(x) + C_{M2}(x)\} \text{ for all } x \in X.$$

Notation 2.1. *(Girish & John, 2012) Let M be an mset from X with x appearing n times in M. It is denoted by $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ where M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[M]_x$ denotes that the element x belongs to the mset M and $|[M]_x|$ denotes the cardinality of an element x in M.*

A new notation can be introduced for the purpose of defining Cartesian product, Relation and its domain and co-domain. The entry of the form $(m/x, n/y)/k$ denotes that x is repeated m-times, y is repeated n-times and the pair (x, y) is repeated k times. The counts of the members of the domain and co-domain vary in relation to the counts of the x co-ordinate and y co-ordinate in $(m/x, n/y)/k$. For this purpose we introduce the notation $C_1(x, y)$ and $C_2(x, y)$. $C_1(x, y)$ denotes the count of the first co-ordinate in the ordered pair (x, y) and $C_2(x, y)$ denotes the count of the second co-ordinate in the ordered pair (x, y).

Throughout this paper M stands for a multiset drawn from the multiset space $[X]^w$. We can define the following types of submets of M and collection of submsets from the mset space $[X]^w$.

Definition 2.4. (Girish & John, 2012) (Whole submset) A submset N of M is a whole submset of M with each element in N having full multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for every x in N.

Definition 2.5. (Girish & John, 2012) (Partial Whole submset) A submset N of M is a partial whole submset of M with at least one element in N having full multiplicity as in M. i.e., $C_N(x) = C_M(x)$ for some x in N.

Definition 2.6. (Girish & John, 2012) (Full submset) A submset N of M is a full submset of M if each element in M is an element in N with the same or lesser multiplicity as in M. i.e., $M^* = N^*$ with $C_N(x) \leq C_M(x)$ for every x in N.

Note 2.1. *(Girish & John, 2012) Empty set $\emptyset$ is a whole submset of every mset but it is neither a full submset nor a partial whole submset of any nonempty mset M.*

Example 2.2. (Girish & John, 2012) Let $M = \{2/x, 3/y, 5/z\}$ be an mset. Following are the some of the submsets of M which are whole submsets, partial whole submsets and full submsets.

(a) A submset $\{2/x, 3/y\}$ is a whole submset and partial whole submset of M but it is not full subset of M.
(b) A submset $\{1/x, 3/y, 2/z\}$ is a partial whole submset and full submset of M but it is not a whole submset of M.
(c) A submset $\{1/x, 3/y\}$ is partial whole submset of M which is neither whole submset nor full submset of M.

Definition 2.7. (Girish & John, 2012) (Power Whole Mset) Let $M \in [X]^w$ be an mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M. i. e., for constructing power whole submsets of M, every element of M with its full multiplicity behaves like an element in a classical set. The cardinality of $PW(M)$ is 2^n where n is the cardinality of the support set (root set) of M.

Definition 2.8. (Girish & John, 2012) (Power Full Mset) Let $M \in [X]^w$ be an mset. Then the power full mset of M, $PF(M)$, is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

Note 2.2. *$PW(M)$ and $PF(M)$ are ordinary sets whose elements are msets.*

If M is an ordinary set with n distinct elements, then the power set $P(M)$ of M contains exactly 2^n elements. If M is a multiset with n elements (repetitions counted), then the power set $P(M)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(M)$. In the classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 2.9. (Girish & John, 2012) (Power Mset) Let $M \in [X]^w$ be an mset. The power mset $P(M)$ of M is the set of all sub msets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \Phi$, then $N \in^1 P(M)$; and if $N = \Phi$, then $N \in^k P(M)$ where $k = \prod_z \binom{[M]_z}{[N]_z}$, the product $\prod_z$ is taken over by distinct elements of z of the mset N and $|[M]_z| = m$ iff $z \in^m M$, $|[N]_z| = n$ iff $z \in^n N$, then

$$\left(\begin{array}{c} |[M]_z| \\ |[N]_z| \end{array} \right) = \left(\begin{array}{c} m \\ n \end{array} \right) = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$. The following theorem shows the cardinality of the power set of an mset.

Theorem 2.1 (23). *Let $P(M)$ be a power mset drawn from the mset $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ and $P^*(M)$ be the power set of an mset M. Then $Card(P^*(M)) = \prod_{i=1}^{n}(1 + m_i)$.*

Example 2.3. (Girish & John, 2012) Let $M = \{2/x, 3/y\}$ be an mset.
The collection $PW(M) = \{\{2/x\}, \{3/y\}, M, \emptyset\}\}$ is a power whole submset of M.
The collection $PF(M) = \{\{2/x, 1/y\}, \{2/x, 2/y\}, \{2/x, 3/y\}, \{1/x, 1/y\}, \{1/x, 2/y\}, \{1/x, 3/y\}\}$ is a power full submset of M.
The collection $P(M) = \{3/\{2/x, 1/y\}, 3/\{2/x, 2/y\}, 6/\{1/x, 1/y\}, 6/\{1/x, 2/y\}, 2/\{1/x, 3/y\}, 1/\{2/x\}, 1/\{3/y\}, 2/\{1/x\}, 3/\{1/y\}, 3/\{2/y\}, M, \emptyset\}\}$ is the power mset of M.
The collection $P^*(M) = \{\{2/x, 1/y\}, \{2/x, 2/y\}, \{1/x, 1/y\}, \{1/x, 2/y\}, \{1/x, 3/y\}, \{2/x\}, \{3/y\}, \{1/x\}, \{1/y\}, \{2/y\}, M, \emptyset\}\}$ is the support set of $P(M)$.

Note 2.3. *Power mset is an mset but its support set is an ordinary set whose elements are msets.*

Definition 2.10. (Girish & John, 2012) The maximum mset is defined as Z where $C_Z(x) = \text{Max}\ \{C_M(x) : x \in^k M, M \in [X]^w$ and $k \leq w\}$.

Operations under collection of msets. (Girish & John, 2012) Let $[X]^w$ be an mset space and $\{M_1, M_2, \ldots\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of msets.

(i) The union
$$\prod_{i\in I} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = \max\{C_{M_i}(x) : x \in X\}\}.$$

(ii) The intersection
$$\cap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = \min\{C_{M_i}(x) : x \in X\}\}.$$

(iii) The mset addition
$$\oplus_{i\in I} M_i = \{C_{\oplus M_1}(x)/x : C_{\oplus M_i}(x) = \sum_{i\in I} C_{M_i}(x), x \in X\}.$$

(iv) The mset complement
$$M^c = Z \ominus M = \{C_M c(x)/x : C_M c(x) = C_Z(x) - C_M(x), x \in X\}.$$

Remark 2.2. Every nonempty set of real numbers that has an upper bound has a supremum and that have a lower bound has an infimum. Thus, the arbitrary union and arbitrary intersection defined in 2.20 are closed under the collection $\{M_i\}_{i\in I}$, because the collection $\{M_i\}_{i\in I}$ drawn from $[X]^m$ contains elements with finite cardinality and multiplicity of each element x_i in M_i is always less than or equal to m.

Definition 2.11. (Girish & John, 2012) Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1, y \in^n M_2\}$.

We can define the Cartesian product of three or more nonempty msets by generalizing the definition of the Cartesian product of two msets.

Definition 2.12. (Girish & John, 2012) A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/x\ R\ n/y$.

The Domain and Range of the mset relation R on M is defined as follows:

Dom $R = \{x \in^r M : \exists y \in^s M$ such that $r/xRs/y\}$ where $C_{DomR}(x) = \sup\{C_1(x, y) : x \in^r M\}$.
Ran $R = \{y \in^s M : \exists x \in^r M$ such that $r/xRs/y\}$ where $C_{RanR}(y) = \sup\{C_2(x, y) : y \in^s M\}$.

Example 2.4. (Girish & John, 2012) Let M=$\{8/x, 11/y, 15/z\}$ be an mset. Then $R = \{(2/x, 4/y)/8, (5/x, 3/x)/15, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$ is an mset relation defined on M. Here Dom $R = \{7/x, 11/y, 14/z\}$ and Ran $R = \{6/x, 10/y, 13/z\}$. Also $S = \{(2/x, 4/y)/5, (5/x, 3/x)/10, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$ is a submset of $M \times M$ but S is not an mset relation on M because $C_S((x, y)) = 5 \neq 2 \times 4$ and $C_S((x, x)) = 10 \neq 5 \times 3$, i.e., count of some elements in S is not a product of $C_1(x, y)$ and $C_2(x, y)$.

Definition 2.13. (Girish & John, 2012)

(i) An mset relation R on an mset M is reflexive if $m/xRm/x$ for all m/x in M.
(ii) An mset relation R on an mset M is symmetric if $m/xRn/y$ implies $n/yRm/x$.
(iii) An mset relation R on an mset M is transitive if $m/xRn/y$, $n/yRk/z$ then $m/xRk/z$.

An mset relation R on an mset M is called an equivalence mset relation if it is reflexive, symmetric and transitive.

Example 2.5. (Girish & John, 2012) Let $M = \{3/x, 5/y, 3/z, 7/r\}$ be an mset. Then the mset relation given by $R = \{(3/x, 3/x)/9,$
$(3/z, 3/z)/9, (3/x, 7/r)/21, (7/r, 3/x)/21, (5/y, 5/y)/25, (3/z, 3/z)/9, (7/r, 7/r)/49, (3/z, 3/x)/9, (3/z, 7/r)/21, (7/r, 3/z)/21\}$ is an equivalence mset relation.

Definition 2.14. (Girish & John, 2012) An mset relation f is called an mset function if for every element m/x in Dom f, there is exactly one n/y in Ran f such that $(m/x, n/y)$ is in f with the pair occurring as the product of $C_1(x, y)$ and $C_2(x, y)$.

For functions between arbitrary msets it is essential that images of indistinguishable elements of the domain must be indistinguishable elements of the range but the images of the distinct elements of the domain need not be distinct elements of the range.

Example 2.6. (Girish & John, 2012) Let $M_1 = \{8/x, 6/y\}$ and $M_2 = \{3/a, 7/b\}$ be two msets. Then an mset function from M_1 to M_2 may be defined as $f = \{(8/x, 3/a)/24, (6/y, 7/b)/42\}$.

3. Multiset Topology

This section gives the basic definitions and examples introduced in (Girish & John, 2012).

Definition 3.1. (Girish & John, 2012) Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties.

1. The mset M and the empty mset $\emptyset$ are in τ.
2. The mset union of the elements of any sub collection of τ is in τ.
3. The mset intersection of the elements of any finite sub collection of τ is in τ.

Mathematically a multiset topological space is an ordered pair (M, τ) consisting of an mset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Note that τ is an ordinary set whose elements are msets. Multiset Topology is abbreviated as an M-topology.

General topology is defined as a set of sets but multiset topology is defined as a set of multisets. Moreover in general topology τ is a subset of the power set but in M-topology τ is a subset of support set of the power mset. If M is an M-topological space with M-topology τ, we say that a submset U of M is an open mset of M if U belongs to the collection τ. Using this terminology, one can say that an M-topological space is an mset M together with a collection of submsets of M, called open msets, such that $\emptyset$ and M are both open and the arbitrary mset unions and finite mset intersections of open msets are open.

Example 3.1. (Girish & John, 2012) Let M be any mset in $[X]^w$. The collection $P^*(M)$, the support set of the power mset of M is an M-topology on M and is called the discrete M-topology.

In general topology, discrete topology is the power set but in M-topology, discrete M-topology is the support set of the power mset.

Example 3.2. (Girish & John, 2012) The collection consisting of M and $\emptyset$ only, is an M-topology called indiscrete M-topology, or trivial M-topology.

Example 3.3. (Girish & John, 2012) If M is any mset in $[X]^w$, then the collection $PW(M)$ is an M-topology on M.

Example 3.4. (Girish & John, 2012) The collection $PF(M)$ is not an M-topology on M, because $\emptyset$ does not belong to $PF(M)$, but $PF(M) \cup \{\emptyset\}$ is an M-topology on M.

Example 3.5. (Girish & John, 2012) The collection τ of partial whole submsets of M is not an M-topology. Let $M = \{2/x, 3/y\}$. Then $A = \{2/x, 1/y\}$ and $B = \{1/x, 3/y\}$ are partial whole submsets of M. Now $A \cap B = \{1/x, 1/y\}$, but it is not a partial whole submset of M. Thus τ is not closed under finite intersection.

4. *M*-Basis and Sub *M*-Basis

Definition 4.1. (Girish & John, 2012) If M is an mset, then the M-basis for an M- topology on M in $[X]^w$ is a collection $\mathcal{B}$ of submsets of M (called M basis elements) such that

1. For each $x \in^m M$, for some $m > 0$, there is at least one M-basis element $B \in \mathcal{B}$ containing m/x. i.e., for each indistinguishable element in M, there is at least one M-basis element in $\mathcal{B}$ having that element with same multiplicity as in M.
2. If m/x belongs to the intersection of two M-basis elements M and N, then there exists an M-basis element P containing m/x such that $P \subseteq M \cap N$ with $C_P(x) = C_M \cap N(x)$ and $C_P(y) \leq C_{M \cap N}(y)$ for all $y \neq x$.

Remark 4.1. (Girish & John, 2012) If a collection $\mathcal{B}$ satisfies the conditions of M-basis, then the M-topology τ generated by $\mathcal{B}$ can be defined as follows. A submset U of M is said to be an open mset in M (i.e., to be an element of τ) if for each $x \in^k U$, there is an M-Basis element $B \in \mathcal{B}$ such that $x \in^k B$ and $C_B(y) \leq C_U(y)$ for all $y \neq x$.

Note that each M-basis element is itself an element of τ.

Theorem 4.1. *The collection τ generated by an M-basis* **B** *is an M-topology on M in $[X]^w$.*

Proof. 1. Clearly $\emptyset$ and M are in τ.

2. Let $\{U_\alpha\}_{\alpha\in J}$ be an indexed family of elements of τ. Then $* = \coprod_{\alpha\in J} U_\alpha$ belongs to τ. For, given $x \in^m \mathcal{U}$, $m = \max_\alpha\{C_{U_\alpha}(x)\}$, there is an index α such that U_α containing m/x. Since U_α is an open mset, there is an M-basis element B containing m/x such that $B \subseteq U_\alpha$. Then $x \in^m B$ and $B \subseteq \mathcal{U}$, so that $\mathcal{U}$ is an open mset, by definition.
3. If U_1 and U_2 are two elements of τ, to prove $U_1 \cap U_2$ belongs to τ. Given $x \in^k U_1 \cap U_2$, $k = \min\{C_{U_1}(x), C_{U_2}(x)\}$. By definition of M-basis, there exists an element B_1 containing k/x, such that $B_1 \subseteq U_1$ and another M-basis element B_2 containing k/x such that $B_2 \subseteq U_2$. The second condition for an M-basis enables us to choose an M-basis element B_3 containing k/x such that $B_3 \subseteq B_1 \cap B_2$. Then $x \in^k B_3$ and $B_3 \subseteq U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to τ, by definition.

Finally, by induction it follows that any finite intersection $U_1 \cap U_2 \cap \cdots \cap U_k$ of elements of τ is in τ. This fact is trivial for $k = 1$ and to be proved for $k = n$. Now $U_1 \cap U_2 \cap \cdots \cap U_n = (U_1 \cap U_2 \cdots \cap U_{n-1}) \cap U_n$. By hypothesis, $U_1 \cap U_2 \cap \cdots \cap U_{n-1}$ belongs to τ and by the result proved above, the intersection of $U_1 \cap U_2 \cap \cdots \cap U_{n-1}$ and U_n also belongs to τ. Thus the collection of open msets generated by an M-basis $\mathcal{B}$ is, in fact, an M-topology. □

Theorem 4.2. *Let M be an mset in $[X]^w$ and $\mathcal{B}$ be an M-basis for an M-topology τ on M. Then τ equals the collection of all mset unions of elements of the M-basis $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, which are also elements of τ, because τ is an M-topology, their union is in τ. Conversely, given $U \in \tau$, for each m/x in U, there is an element B of $\mathcal{B}$ containing m/x, denoted by $B_{m/x}$, such that $B_{m/x} \subseteq U$. Then $U = \cup B_{m/x}$, so U equals a union of elements of $\mathcal{B}$. □

Lemma 4.3. *Let $M \in [X]^w$ be an M-topological space. Suppose $\mathcal{M}$ is a collection of open msets of M such that for each open mset U of M and each element m/x in U, there is an element N of $\mathcal{M}$ containing m/x such that $C_N(x) \leq C_U(x)$. Then $\mathcal{M}$ is an M-basis for the M-topology of M.*

Proof. Given $x \in^m M$, since M itself is an open mset, by hypothesis there is an element N of $\mathcal{M}$ containing m/x such that $N \subseteq M$. To check the second condition, let m/x be in $N_1 \cap N_2$, N_1 and N_2 are elements of $\mathcal{M}$. Since N_1 and N_2 are open msets, so is its intersection $N_1 \cap N_2$. Therefore, by hypothesis there exists an element N_3 in C containing m/x such that $N_3 \subseteq N_1 \cap N_2$. Hence the collection $\mathcal{M}$ is an M-basis.

Let τ be the collection of open msets of M. Then the M-topology τ' generated by $\mathcal{M}$ equals the M-topology τ. If U belongs to τ and $x \in^m U$, then by hypothesis there is an element N of $\mathcal{M}$ containing m/x such that $N \subseteq U$. By definition, it follows that U belongs to the M-topology τ'. Conversely, if W belongs to the M-topology τ', then W equals a union of elements of $\mathcal{M}$, by theorem 4.4. Since each element $\mathcal{M}$ belongs to τ and τ is an M-topology, W also belongs to τ. Thus the M-topology generated by the M-basis and M-topology on M are the same. □

Definition 4.2. Suppose τ and τ' are two M-topologies on a given mset M in $[X]^w$. If $\tau' \subset \tau$, then we say that τ' is finer than τ or τ is coarser than τ'. If $\tau' \subset \tau$, then τ' is strictly finer than τ or τ is strictly coarser than τ'. Thus τ is comparable with τ' if either $\tau' \supseteq \tau$ or $\tau \supseteq \tau'$.

The next theorem gives a criterion for determining whether an M-topology on M is finer than another in terms of M-basis.

Theorem 4.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ are M-basis for the M-topologies τ and τ' on M in $[X]^w$ respectively. Then the following are equivalent:*

1. *τ' is finer than τ.*

2. *For each $x \in^m M$ and each M-basis element $B \in \mathcal{B}$ containing m/x, there is an M-basis element $B' \in \mathcal{B}'$ containing m/x such that $C_{B'}(x) \leq C_B(x)$.*

Proof. (1) ⇒ (2). Given an element m/x in M and $B \in \mathcal{B}$ containing m/x, B belongs to τ by definition and $\tau \subseteq \tau'$ by (1). Therefore $B \in \tau'$. Since τ' is generated by $\mathcal{B}'$, there is an M-basis element $B' \in \mathcal{B}$ containing m/x such that $C_{B'}(x) \leq C_B(x)$.

(2) ⇒ (1). Given an element U of τ, we show that $U \in \tau'$. Let $x \in^m U$, since $\mathcal{B}$ generates τ, there is an M-basis element $B \in \mathcal{B}$ containing m/x such that $B \subseteq U$. From (2), there exists an M-basis element $B' \in \mathcal{B}'$ containing m/x such that $B' \subseteq B$. Then $B' \subseteq U$ and $U \in \tau'$. □

Example 4.1. The collection $\{\{m/x\} : x \in^m M\}$ is an M-basis for the M-topology $PW(M)$.

In general topology $\{\{x\} : x \in X\}$ is a basis for the discrete topology, but in the case of M-topology the collection $\{\{m/x\} : x \in^m M\}$ is not an M-basis for the discrete M-topology.

Definition 4.3. Let (M, τ) be an M-topological space and N is a submset of M. The collection $\tau_N = \{U' = N \cap U; U \in \tau\}$ is an M-topology on N, called the subspace M-topology. With this M-topology, N is called a subspace of M and its open msets consisting of all mset intersections of open msets of M with N.

Theorem 4.5. *If $\mathcal{B}$ is an M-basis for the M-topology of M in $[X]^w$, then the collection $\mathcal{B}_N = \{B \cap N : B \in \mathcal{B}\}$ is an M-basis for the subspace M-topology on a submset N of M.*

Proof. Given U open in M and $y \in^m U \cap N$, we can choose an element B of $\mathcal{B}$ such that $y \in^m B \subseteq U$. Then, $y \in^m B \cap N \subseteq U \cap N$. It follows from Lemma 4.5 that $\mathcal{B}_N$ is an M-basis for the subspace M-topology on N. □

Example 4.2. Let $M = \{3/a, 4/b, 2/c, 5/d\}$ and $\tau = \{\emptyset, M, \{2/c\}, \{2/a\}, \{3/a, 2/b\}, \{2/a, 3/d\}, \{2/a, 2/c\}, \{3/a, 3/b, 3/d\}, \{3/a, 4/b, 2/c\}, \{2/a, 2/c, 3/d\}\}$ is an M-topology on M. If $N = \{2/a, 2/b, 3/d\} \subseteq M$, then $\tau' = \{\emptyset, \{2/a, 2/b, 3/d\}, \{2/a\}, \{2/a, 2/b\}, \{2/a, 3/d\}\}$ is an M-topology on N and it is the subspace M-topology on N.

Definition 4.4. A sub collection $\mathcal{P}$ of τ on M is called a sub M-basis for τ, if the collection of all finite mset intersections of elements of $\mathcal{P}$ is an M-basis for τ. The M-topology generated by the sub M-basis $\mathcal{P}$ is defined to be the collection τ of mset union of all finite mset intersections of elements of $\mathcal{P}$.

Note 4.1. *The empty mset intersection of the members of sub M-basis is the universal mset.*

Theorem 4.6. *Let (M, τ) be an M-topological space and $\mathcal{P}$ be a collection of submsets of M. Then $\mathcal{P}$ is a sub M-basis for τ if and only if $\mathcal{P}$ generates τ.*

Proof. Let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{P}$ and $\mathcal{P}$ be a sub M-basis for τ. It can be shown that τ is the smallest M-topology on M containing $\mathcal{P}$. Since $\mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \tau, \subseteq \tau$. Suppose τ^* is some other M-topology on M such that $\mathcal{P} \subseteq \tau^*$. We have to show that $\tau \subseteq \tau^*$. Since $\mathcal{P} \subseteq \tau^*$, τ^* contains all finite intersections of members of $\mathcal{P}$, i.e., $\mathcal{B} \subseteq \tau^*$. Since $\mathcal{B}$ is an M-basis, each member of τ can be written as the union of some members of $\mathcal{B}$ and it follows that $\tau \subseteq \tau^*$.

Conversely suppose τ is the smallest M-topology containing $\mathcal{P}$. We have to show that $\mathcal{P}$ is a sub M-basis for τ. i.e., $\mathcal{B}$ is an M-basis for τ. Suppose there is an M-topology τ^* on M such that $\mathcal{B}$ is an M-basis for τ^*. Then every member of τ^* can be expressed as a union of the sub family of $\mathcal{B}$ and so it is in τ since $\mathcal{B} \subseteq \tau$. This means $\tau^* \subseteq \tau$ and consequently $\tau^* = \tau$. Since τ is the smallest M-topology containing $\mathcal{P}$, it can be shown that $\mathcal{B}$ is an M-basis for τ and $\mathcal{P}$ is a sub M-basis for τ. □

Example 4.3. Let $M = \{3/a, 5/b, 4/c\}$. If the collection $\mathcal{P} = \{\{3/a, 5/b\}, \{5/b, 4/c\}\}$ is a sub M-basis, then the collection $\mathcal{B} = \{\{5/b\}, \{3/a, 5/b\}, \{5/b, 4/c\}\}$ is the corresponding M-basis and $\tau = \{M, \emptyset, \{5/b\}, \{3/a, 5/b\}, \{5/b, 4/c\}\}$ is the M-topology generated by the M-basis.

If we assume the empty mset intersection of the members of sub M-basis is the universal mset, then we can give the following example.

Example 4.4. Let $M = \{3/a, 4/b, 2/c, 5/d\}$. If the collection
$\mathcal{P} = \{\{3/a, 3/b\}, \{4/d\}, \{2/a\}\}$ is a sub M-basis, then the collection
$\mathcal{B} = \{\{3/a, 3/b\}, \{4/d\}, \{2/a\}, \emptyset, M\}$ is the corresponding M-basis and
$\tau = \{\emptyset, M, \{2/a\}, \{4/d\}, \{3/a, 3/b\}, \{2/a, 4/d\}, \{3/a, 3/b, 4/d\}\}$
is the M-topology generated by the M-basis.

5. Closed Multisets

Definition 5.1. A sub mset N of an M-topological space M in $[X]^w$ is said to be closed if the mset $M \ominus N$ is open.

In discrete M-topology every mset is an open mset as well as a closed mset. In the M-topology $PF(M) \cup \{\emptyset\}$, every mset is an open mset as well as a closed mset.

Theorem 5.1. *Let (M, t) be an M-topological space. Then the following conditions hold:*

1. *The mset M and the empty mset $\emptyset$ are closed msets.*
2. *Arbitrary mset intersection of closed msets is a closed mset.*
3. *Finite mset union of closed msets is a closed mset.*

Proof. 1. $\emptyset$ and M are closed msets because they are the complements of the open msets M and $\emptyset$ respectively.

2. Given a collection of closed msets $\{N_\alpha\}_{\alpha \in J}$, we have

$$C_{M \oplus \cap_\alpha N_\alpha}(x) = C_M(x) - \min_{\alpha \in J}\{C_{N_\alpha}(x)\} = \max_{\alpha \in J}\{C_M(x) - C_{N_\alpha}(x)\}$$
$$= C_{\cap_\alpha (M \ominus) N_\alpha)}(x)$$

From this

$$M \ominus \cap_{\alpha \in J} N_\alpha = cap_\alpha (M \ominus) N_\alpha)$$

By definition the msets $M \ominus N_\alpha$'s are open. Since the arbitrary union open of msets are open, $M \ominus \cap_{\alpha \in J} N_\alpha$ is an open mset and therefore $\cap_{\alpha \in J} N_\alpha$ is a closed mset.

3. Similarly, if N_i is closed, for $i = 1, 2, \ldots, n$, consider

$$C_{M \ominus \prod_i N_i}(x) = C_M(x) - \max_i\{C_{N_i}(x)\} = \min_i\{C_M(x) - C_{N_i}(x)\} = C_{\cap_i (M \oplus N_i)(x)}.$$

Thus

$$M \ominus \prod_{i=1}^{n} N_i = \cap_{i=1}^{n} (M \ominus N_i).$$

Since finite mset intersections of open msets are open, $\prod_{i=1}^{n} N_i$ is a closed mset. □

Theorem 5.2. *Let N be a subspace of an M-topological space M in $[X]^w$. Then an mset A is a closed mset in N if and only if it equals the intersection of a closed mset of M with N.*

Proof. Assume $A = C \cap N$ where C is a closed mset in M. By the definition of subspace M-topology, $M \ominus C$ is an open mset in M, so that $(M \ominus C) \cap N$ is an open mset in N. But $(M \ominus C) \cap N = N \ominus A$. Hence $N \ominus A$ is an open mset in N, so that A is a closed mset in N. Conversely, assume that A is closed mset in N. Then $N \ominus A$ is open mset in N, so that by definition it equals the intersection of an open mset U of M with N. The mset $M \ominus U$ is a closed mset in M and $A = N \cap (M \ominus U)$, so that A equals the intersection of the closed mset of M with N, as desired. □

Theorem 5.3. *Let N be a subspace of an M-topological space M in $[X]^w$. If A is a closed mset in N and N is a closed mset in M, then A is a closed mset in M.*

Proof. Proof directly follows from Theorem 5.3. □

6. Closure, Interior and Limit Point

Definition 6.1. Given a submset A of an M-topological space M in $[X]^w$, the interior of A is defined as the mset union of all open msets contained in A and is denoted by Int (A).

i.e., Int $(A) = \cup\{G \subseteq M : G$ is an open mset and $G \subseteq A\}$ and $C_{\mathrm{Int}(A)}(x) = \max\{C_G(x) : G \subseteq A\}$.

Definition 6.2. Given a submset A of an M-topological space M in $[X]^w$, the closure of A is defined as the mset intersection of all closed msets containing A and is denoted by Cl (A).

i.e., $\mathrm{Cl}(A) = \cap\{K \subseteq M : K$ is a closed mset and $A \subseteq K\}$ and $C_{\mathrm{Cl}}(A)(x) = \min\{C_K(x) : A \subseteq K\}$.

Definition 6.3. Let (M, τ) be an M-topological space, let $x \in^k M$ and $N \subseteq M$. Then N is said to be a neighborhood of k/x if there is an open mset V in τ such that $x \in^k V$ and $C_V(y) \leq C_N(y)$ for all $y \neq x$.

i.e., a neighborhood of k/x in M means any open mset containing k/x. Here k/x is said to be an interior point of N.

Definition 6.4. Let A be a submset of the M-topological space M in $[X]^w$. If k/x is an element of M, then k/x is a limit point of an mset A when every neighborhood of k/x intersects A in some point (point with non zero multiplicity) other than k/x itself. A' denotes the mset of all limit points of A.

Theorem 6.1. *Let N be a subspace of an M-topological space M in $[X]^w$ and A be a submset of an mset N and $Cl(A)$ denote the closure of an mset A in M. Then the closure of an mset A in N equals $Cl(A) \cap N$.*

Proof. Let B denote the closure of an mset A in N. If mset $\mathrm{Cl}(A)$ is a closed mset in M, then by Theorem 5.3 $\mathrm{Cl}(A) \cap N$ is a closed mset in N. Since $\mathrm{Cl}(A) \cap N$ contains A, and since by definition, B equals the intersection of all closed submsets of N containing A, we get $B \subseteq \mathrm{Cl}(A) \cap N$.

On the other hand, B is a closed mset in N. Hence by Theorem 4.4.3, $B = C \cap N$ for some mset C, a closed mset in M. Then C is a closed mset of M containing A, because $\mathrm{Cl}(A)$ is the intersection of all such closed msets. We conclude that $\mathrm{Cl}(A) \subseteq C$. Therefore $\mathrm{Cl}(A) \cap N \subseteq C \cap N = B$. □

Theorem 6.2. *Let (M, τ) be an M-topological space, $x \in^k M$ and $A \subseteq M$, then*
1. $x \in^k cl(A)$ if and only if every open mset U containing k/x intersects A.
2. If the M-topology (M, τ) is given by an M-basis $\mathcal{B}$, then, $x \in^k Cl(A)$ if and only if every M-basis element $B \in \mathcal{B}$ containing k/x intersects A.

Proof. 1. If k/x is not in Cl(A), then the mset $U = M \ominus \mathrm{Cl}(A)$ is an open mset containing k/x that does not intersect A. Conversely, if there exists an open mset U containing k/x which does not intersect A, then the mset $M \ominus U$ is a closed mset containing A. By the definition of the closure Cl(A), the mset $M \ominus U$ must contain Cl(A). Therefore k/x cannot be in Cl(A).

2. If every open mset containing k/x intersects A, so does every M-basis element B containing k/x, because B is an open mset.

Conversely, if every M-basis element containing k/x intersects A, so does every open mset U containing k/x, because U contains an M-basis element that contains k/x. □

Theorem 6.3. *A submset of an M-topological space is an open mset if and only if it is a neighborhood of each of its elements with some multiplicity.*

Proof. Let M be an M-topological space and $N \subseteq M$. First suppose N is an open mset. Then clearly N is a neighborhood of each of its points with some multiplicity. Conversely suppose N is a neighborhood of each of its points, then for each k/x in N, there is an open mset $V_{k/x}$ such that $x \in^k V_{k/x}$ and $V_{k/x} \subseteq N$. Clearly,

$$N = \prod_{x \in^k N} V_{k/x},\ k = \max\{C_{V_{k/x}(x)}\}.$$

Since each $V_{k/x}$ is an open mset so is N. □

Theorem 6.4. *Let A be a submset of the M-topological space M and A' be the mset of all limit points of A. Then $C_{Cl\,(A)}(x) = \max\{C_A(x), C_{A'}(x)\}$.*

Proof. If k/x is in A', then every neighborhood of k/x intersects A. Therefore, by Theorem 4.5.6 k/x belongs to Cl(A). Hence $A' \subseteq \mathrm{Cl}\,(A)$. Since by definition $A \subseteq \mathrm{Cl}\,(A)$, it follows that $A \cup A' = \mathrm{Cl}\,(A)$.

Conversely suppose k/x is a point of Cl(A), then $x \in^k A \cup A'$. If k/x is in A, it is clear that $x \in^k A \cup A'$. Suppose k/x does not belonging to A, since $x \in^k \mathrm{Cl}\,(A)$, we know that every neighborhood U of k/x intersects A. Thus the mset U must intersect A in a point different from k/x. Hence $x \in^k A'$ and $x \in^k A \cup A'$. □

Corollary 6.5. *A submset of an M-topological space is a closed mset if and only if it contains all its limit points.*

Proof. The mset A is a closed mset:

- if and only if $A = \mathrm{Cl}\,(A)$,
- if and only if $A = A \cup A'$,
- if and only if $A' \subseteq A$.

□

Theorem 6.6. *If A and B are submsets of the M-topological space M in $[X]^w$, then the following properties hold:*

1. *If $C_A(x) \leq C_B(x)$, then $C_{A'}(x) \leq C_{B'}(x)$.*
2. *If $C_A(x) \leq C_B(x)$, then $C_{Int\,(A)}(x) \leq C_{Int\,(B)}(x)$.*
3. *If $C_A(x) \leq C_B(x)$, then $C_{Cl\,(A)}(x) \leq C_{Cl\,(B)}(x)$*

4. $C_{Int\,(A\cap B)}(x) = \min\{C_{Int\,(A)}(x), C_{Int\,(B)}(x)\}$.
5. $C_{Cl(A\cup B)}(x) = \max\{C_{Cl\,(A)}(x), C_{Cl\,(B)}(x)\}$.

Proof. 1. $x \in^k A'$ if and only if $(N \ominus \{k/x\}) \cap A \neq \emptyset$, for all open mset N containing k/x. Since $B \supseteq A$, $(N\ominus\{k/x\})\cap B \supseteq (N\ominus\{k/x\})\cap A \neq \emptyset$. So $x \in^k A'$ implies $x \in^k B'$. Thus $A' \subseteq B'$ and $C_{A'}(x) \leq C_{B'}(x)$.

2. We have $C_{\text{Int}\,(A)}(x) \leq C_A(x)$ and $C_{\text{Int}\,(B)}(x) \leq C_B(x)$. Since $A \subseteq B$ and $C_A(x) \leq C_B(x)$, we get $C_{\text{Int}\,(A)}(x) \leq C_B(x)$ and Int $(A) \subseteq B$. Thus Int (A) is an open mset contained in B, but Int (B) is the largest open mset contained in B. Hence $C_{\text{Int}\,(A)}(x) \leq C_{\text{Int}\,(B)}(x)$ and Int$(A) \subseteq$ Int (B).
3. We have

$$\begin{aligned} C_{\text{Cl}\,(A)}(x) &= \max\{C_A(x), C_{A'}(x)\}, \text{ from Theorem 6.8} \\ &\leq \max\{C_B(x), C_{B'}(x)\}, \text{ by (1)} \\ &= C_{\text{Cl}\,(B)}(x) \end{aligned}$$

Thus Cl$(A) \subseteq$ Cl (B).

4. We have $C_{\text{Int}\,(A\cap B)}(x) \leq C_{\text{Int}\,(A)}(x)$ and $C_{\text{Int}\,(A\cap B)}(x) \leq C_{\text{Int}\,(B)}(x)$.
Therefore $C_{textInt\,(A\cap B)}(x) \leq \min\{C_{\text{Int}\,(A)}(x), C_{\text{Int}\,(B)}(x)\}$. Thus

$$\text{Int}\,(A \cap B) \subseteq \text{Int}\,(A) \cap \text{Int}(B) \tag{i}$$

Also $C_{\text{Int}(A)}(x) \leq C_A(x)$ and $C_{\text{Int}\,(B)}(x) \leq C_B(x)$.
Therefore $\min\{C_{\text{Int}\,(A)}(x), C_{\text{Int}\,(B)}(x)\} \leq \min\{C_A(x), C_B(x)\}$.
Thus Int$(A)\cap$ Int$(B) \subseteq A \cap B$, but Int $(A \cap B)$ is the largest open mset contained in $A \cap B$, i.e., $C_{\text{Int}\,(A\cap B)}(x)$ is that largest integer which is less than or equal to $C_{A\cap B}(x)$.
Therefore $\min\{C_{\text{Int}\,(A)}(x), C_{\text{Int}\,(B)}(x)\} \leq C_{\text{Int}\,(A\cap B)}(x)$. Thus

$$\text{Int}\,(A) \cap \text{Int}\,(B) \subseteq \text{Int}\,(A \cap B) \tag{ii}$$

From (i) and (ii) it follows that Int $(A \cap B) =$ Int$(A)\cap$ Int(B).

5. We have $C_{\text{Cl}\,(A)}(x) \leq C_{\text{Cl}\,(A\cup B)}(x)$ and $C_{\text{Cl}\,(B)}(x) \leq C_{\text{Cl}\,(A\cup B)}(x)$. Therefore

$$\max\{C_{\text{Cl}\,(A)}(x), C_{\text{Cl}\,(B)}(x)\} \leq C_{\text{Cl}\,(A\cup B)}(x) \tag{i}$$

But $C_A(x) \leq C_{\text{Cl}\,(A)}(x)$ and $C_B(x) \leq C_{\text{Cl}\,(B)}(x)$.
Therefore $\max\{C_A(x), C_B(x)\} \leq \max\{C_{\text{Cl}\,(A)}(x), C_{\text{Cl}\,(B)}(x)\}$. Hence

$$C_{\text{Cl}(A\cup B)}(x) \leq \max\{C_{\text{Cl}\,(A)}(x), C_{\text{Cl}\,(B)}(x)\} \tag{ii}$$

From (i) and (ii) it follows that $C_{\text{Cl}\,(A\cup B)}(x) = \max\{C_{\text{Cl}\,(A)}(x), C_{\text{Cl}\,(B)}(x)\}$.

Thus Cl$(A \cup B) =$ Cl $(A) \cup$ Cl (B). □

7. Continuous Multiset Functions

Definition 7.1. Let M and N be two M-topological spaces. The mset function $f : M \to N$ is said to be continuous if for each open submset V of N, the mset $f^{-1}(V)$ is an open submset of M, where $f^{-1}(V)$ is the mset of all points m/x in M for which $f(m/x) \in^n V$ for some n.

Example 7.1. Let $M = \{5/a, 4/b, 4/c, 3/d\}$ and $N = \{7/x, 5/y, 6/z, 4/w\}$ be two msets, $\tau = \{M, \emptyset, \{5/a\}, \{5/a, 4/b\}, \{5/a, 4/b, 4/c\}\}$ and $\tau' = \{N, \emptyset, \{7/x\}, \{5/y\}, \{7/x, 5/y\}, \{5/y, 6/z, 4/w\}\}$ be two M-topologies on M and N respectively.

Consider the mset functions $f : M \to N$ and $g : M \to N$ are given by

$$f = \{(5/a, 5/y)/25, (4/b, 6/z)/24, (4/c, 4/w)/16, (3/d, 6/z)/18\},$$
$$g = \{(5/a, 7/x)/35, (4/b, 7/x)/28, (4/c, 6/z)/24, (3/d, 4/w)/12\}.$$

The mset function f is continuous since the inverse of each member of the M-topology τ' on N is a member of the M-topology τ on M. The mset function g is not continuous since $\{5/y, 6/z, 4/w\} \in \tau'$, i.e., an open mset of N, but its inverse image $g^{-1}(\{5/y, 6/z, 4/w\}) = \{4/c, 3/d\}$ is not an open submset of M, because the mset $\{4/c, 3/d\}$ does not belong to τ.

Example 7.2. Let $f : M \to N$ be an mset function and $\tau = P^*(M)$, the support set of the power mset of M, the M-topology on M. Then every mset function $f : M \to N$ is continuous for any M-topology on N.

Example 7.3. Let $f : M \to N$ be an mset function and $\tau' = PF(N) \cup \{\emptyset\}$ be an M-topology on N, then every mset function $f : M \to N$ is continuous for any M-topology τ on M, because open msets in τ' are submsets of N whose support set is N^*. Let H and $\emptyset$ be open in τ', then $f^{-1}(H) = M$ and $f^{-1}(\emptyset) = \emptyset$. Hence f is continuous for any τ.

Theorem 7.1. *Let M and N be two M-topological spaces and $f : M \to N$ be an mset function. Then the following are equivalent:*

1. *The mset function f is continuous,*
2. *For every submset A of M, $C_{f(Cl(A))}(x) \leq C_{Cl\,(f(A))}(x)$,*
3. *For every closed mset B of N, the mset $f^{-1}(B)$ is a closed mset in M,*
4. *For each $x \in^k M$ and each neighborhood V of $f(k/x)$, there is a neighborhood U of k/x such that $C_{f(U)}(x) \leq C_V(x)$.*

Proof. (1) ⇒ (2) Assume that the mset function f is continuous. Let A be a submset of M. We show that if $x \in^k \mathrm{Cl}\,(A)$, then $f(k/x) \in^r \mathrm{Cl}\,(f(A))$ for some r. If V is a neighborhood of $f(k/x)$, then $f^{-1}(V)$ is an open mset of M containing k/x which intersects A in some point n/y. Then V intersects $f(A)$ in the point $f(n/y)$ and $f(k/x) \in^r \mathrm{Cl}(f(A))$ for some r.

(2) ⇒ (3) Let B be a closed mset in N and let $A = f^{-1}(B)$. We wish to prove that A is a closed mset in M; we show that $\mathrm{Cl}(A) = A$. We have $f(A) = f(f^{-1}(B)) \subseteq B$. Therefore, if $x \in^k \mathrm{Cl}\,(A)$, then $f(k/x) \in^r f(\mathrm{Cl}\,(A)) \subseteq \mathrm{Cl}\,(f(A)) \subseteq \mathrm{Cl}\,(B) = B$. So that $x \in^k f^{-1}(B) = A$. Thus $\mathrm{Cl}(A) \subseteq A$, so that $\mathrm{Cl}\,(A) = A$.

(3) ⇒ (1) Let V be an open mset of N. Set $B = N \ominus V$. Then $f^{-1}(B) = f^{-1}(N) \ominus f^{-1}(V) = M \ominus f^{-1}(V)$. Now since B is a closed mset of N, $f^{-1}(B)$ is a closed mset in M by hypothesis so that $f^{-1}(V)$ is an open mset in M.

(1) ⇒ (4) Let $x \in^k M$ and let V be a neighborhood of $f(k/x)$. Then the mset $U = f^{-1}(V)$ is a neighborhood of k/x such that $f(U) \subseteq V$.

(4) ⇒ (1) Let V be an open mset of N and k/x be a point of $f^{-1}(V)$. Then $f(k/x) \in^r V$ for some r, so by hypothesis there is a neighborhood U_x of k/x such that $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open msets U_x. Thus $f^{-1}(V)$ is an open mset of M and f is continuous. □

Theorem 7.2. *If M, N and P are M-topological spaces and $f : M \to N$ and $g : N \to P$ are continuous mset functions, then its composition $g \circ f : M \to P$ is a continuous mset function.*

Proof. If H is an open mset in P, then $g^{-1}(H)$ is an open mset in N by continuity of g. Now again by continuity of $f, f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is an open mset in M. Thus $g \circ f$ is a continuous mset function. □

Remark 7.1.

1. In general topology, discrete topology is the set of all subsets of X and clearly it contains 2^n elements where n is the cardinality of X. But in an M-topology, discrete M-topology $P^*(M)$ is the support set of the power mset of M in $[X]^w$ and it contains $\prod_{i=1}^{n}(1 + m_i) < 2^n$ elements where m_i is the occurrence of an element x_i in the mset M and n is the cardinality of the mset M.
2. In general topology any function $f : X \to Y$ is continuous if X has the discrete topology and Y has any topology. But in the case of M-topological spaces, every mset function $f : M \to N$ is continuous whenever M-topology of M in $[X]^w$ contains $\prod_{i=1}^{n}(1 + m_i) < 2^n$ elements and for any M-topology of N in $[X]^w$ where m_i is the occurrence of an element x_i in the multiset M and n is the cardinality of the multiset M.

8. Conclusion and Future Work

In this paper the authors focus on topology of multisets. This work extends the theory of general topology on general sets to multisets. It begins with a brief survey of the notion of msets introduced by Yager, different types of collections of msets and operations under such collections. It also gives the definition of mset relation and mset function introduced by the authors. After presenting the preliminaries and basic definitions the authors introduced the notion of M-topological space. Basis, sub basis, closure, interior and limit points of multisets are defined and some of the existing theorems are proved in the context of multisets. Finally the authors have established the relationship between continuous function and discrete topology in the context of M-topological space.

The concept of topological structures and their generalizations is one of the most powerful notions in branches of science such as chemistry, physics and information systems. In most applications the topology is employed out of a need to handle the qualitative information. In any information system, some situations may occur, where the respective counts of objects in the universe of discourse are not single. In such situations we have to deal with collections of information in which duplicates are significant. In such cases multisets play an important role in processing the information. The information system dealing with multisets is said to be an information multisystem. Thus, information multisystems are more compact when compared to the original information system. In fact, topological structures on multisets are generalized methods for measuring the similarity and dissimilarity between the objects in multisets as universes. The theoretical study of general topology on general sets in the context of multisets can be a very useful theory for analyzing an information multisystem.

Most of the theoretical concepts of multisets come from combinatorics. Combinatorial topology is the branch of topology that deals with the properties of geometric figures by considering the figures as being composed of elementary geometric figures. The combinatorial method is used not only to construct complicated figures from simple ones but also to deduce the properties of the complicated from the simple. In combinatorial topology it is remarkable that the only machinery to make deductions is the elementary process of counting. In such situations we may deal with collections of elements with duplicates. The theory of M-topology may be useful for studying combinatorial topology with collections of elements with duplicates.

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Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 53–60

GPS Satellite Range and Relative Velocity Computation

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Abstract

In this work the estimation of a Global Positioning System satellite orbit is considered. The range and relative velocity of the satellite is computed in the observer's local reference frame (topocentric system of coordinates) by including the Earth gravitational perturbations (up to J_3 term) and the solar radiation pressure. Gauss perturbation equations are used to obtain the orbital elements as a function of time, from which the position vector is derived.

Keywords: GPS Satellite, Gauss Equations, Solar Radiation Pressure, Range.

1. Introduction

Global Positioning System (GPS) satellites are used in a variety of applications such as wireless locations, navigation, GPS/INS integrations, as well in attitude and orbit estimation (Mikhailov & Vasilév, 2011). GPS satellite orbits are at an altitude of $25,000$ km, with eccentricity ranging from 0.001 to 0.02, and inclined at 55°. At such high altitude the atmospheric drag can be disregarded and the dominant forces affecting the orbital motion are the gravitational and the Solar Radiation Pressure (SRP). Reference (Stelian, 2007) has used fourth-order Runge-Kutta algorithm to numerically integrate the GPS satellite perturbed orbit showing that the most dominant orbital perturbation is the Earth oblateness, namely the so called J_2 term of the Earth gravitational potential.

In this work the J_2 and J_3 orbital gravitational perturbations are considered as well as the solar radiation pressure. Gaussian planetary differential equations are integrated to quantify the effects of the perturbations in the orbital elements. The time-varying orbital elements are obtained by rewriting the Gaussian planetary equations in the orbital coordinate system. Then, from the ephemerides the GPS satellite position and velocity can be evaluated at any time and in any reference coordinate system. In particular, position and velocity vectors can be computed in a ground station reference frame, from where the satellite is observed. This transformation implies the evaluation of the geodetic latitude to consider the Earth an oblate spheroid. The GPS satellite position and velocity are then evaluated in the Earth-Centered-Inertial (ECI) reference

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frame and then transformed into the topo-centric ground station coordinate system. The final purpose of this study is to quantify the variation in the GPS satellite range (as seen by an observer in the ground station) due to the J_2 and J_3 orbital gravitational and solar pressure perturbations.

2. Coordinate system used

To quantify the range rate effect due to orbital perturbation in the ground reference frame, four coordinates systems are adopted. There are shown in figure 1.

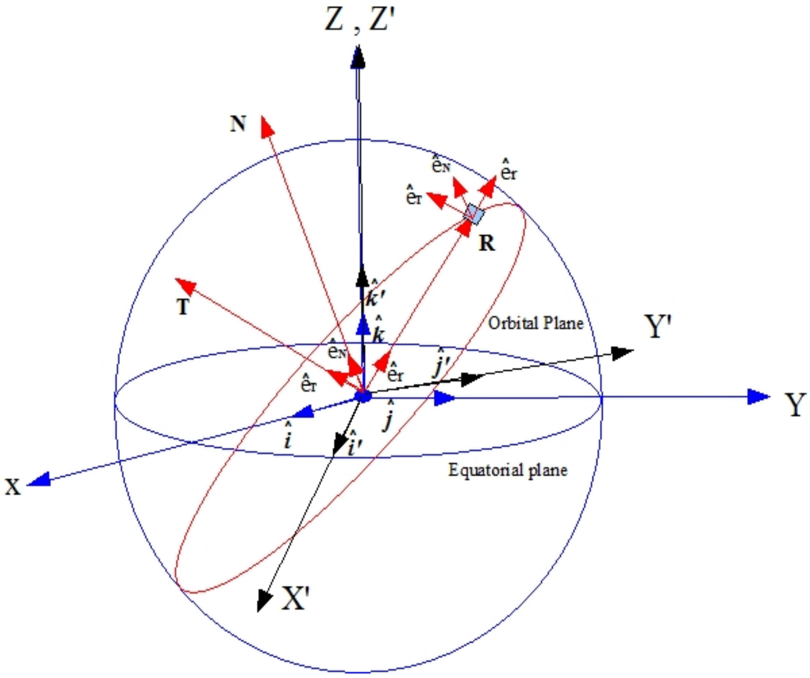

Figure 1. Coordinate systems.

(i) The Earth Centered Inertial ECI coordinate system $OXYZ$. In this system the X-axis is directed toward the vernal Equinox, the Y-axis is in the equatorial plane and normal to the X-axis, and the Z-axis is directed along the rotation axis of the Earth (i.e. normal to the equatorial plane). The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are taken in the directions of the X-axis, Y-axis and Z-axis respectively. (ii) The Earth Centered Earth Fixed $ECEF$ coordinate system $OXYZ$. In this system the X-axis is directed to- ward Greenwich, the Y-axis is in the equatorial plane and normal to the X-axis, and the Z-axis is directed along the rotation axis of the Earth. The unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are taken in the directions of the X-axis, Y-axis and Z-axis respectively. (iii) The Orbital Coordinate $ORTN$ coordinate system . In this system the R -axis is directed along the radius vector of the satellite, the T -axis is in the local orbital plane and normal to the R-axis, and the N-axis is normal to the orbital plane. The unit vectors $\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_N$ are taken in the directions of the R-axis, T-axis and N-axis respectively. (iv) The Topocentric Horizon (SEZ) coordinate system. In this system the fundamental plane is the observer's horizon plane, the positive x-axis is directed in the south direction, the y-axis is directed toward the East and z-axis is directed toward the observer's zenith.

3. Earth's oblateness

Earth is an oblate spheroid. A truncated gravitational potential up J_3 is given by:

$$U_g = -\frac{\mu R_\oplus^2}{r^3}\left[J_2\left(1-\frac{3}{2}\sin^2\phi\right)+J_3\frac{R_\oplus}{r}\left(5\sin^3\phi-3\sin\phi\right)\right], \tag{3.1}$$

where μ is the gravitational constant and ϕ is the angle between the Earth's spin axis and satellite radius.

The gradient of this potential gives the perturbing gravitational force in ECI (see (Schaub & Junkins, 2009))

$$\begin{aligned}\mathbf{F}_g \;=\; & -\frac{3}{2}J_2\left(\frac{\mu}{r^3}\right)\left(\frac{R_\oplus}{r}\right)\begin{Bmatrix}(1-5\sin^2\phi)x\\(1-5\sin^2\phi)y\\(3-5\sin^2\phi)z\end{Bmatrix}+\\ & -\frac{1}{2}J_3\left(\frac{\mu}{r^3}\right)\left(\frac{R_\oplus}{r}\right)^3\begin{Bmatrix}5(7\sin^3\phi-3\sin\phi)x\\5(7\sin^3\phi-3\sin\phi)y\\(-105\sin^4\phi+30\sin^2\phi-3)z\end{Bmatrix}\end{aligned} \tag{3.2}$$

and this force is expressed in the orbital frame as

$$\mathbf{F}_g = F_R\,\hat{\mathbf{e}}_R + F_T\,\hat{\mathbf{e}}_T + F_N\,\hat{\mathbf{e}}_N, \tag{3.3}$$

where, by setting $S_\bullet = \sin(\bullet)$, and $C_\bullet = \cos(\bullet)$, and $\theta = \omega + f$, the expressions of F_R, F_T, and F_N are

$$\begin{aligned}F_R &= -3\mu\frac{R_\oplus^2}{r^4}\left[\frac{J_2}{2}(1-3S_i^2S_\theta^2)+J_3\frac{R_\oplus}{r}(-15S_iS_\theta-3S_i^2S_\theta^2+40S_i^3S_\theta^3+30S_i^4S_\theta^4-70S_i^5S_\theta^5)\right]\\ F_T &= -3\mu\frac{R_\oplus^2}{r^4}\left\{J_2S_i^2S_\theta C_\theta+J_3\frac{R_\oplus}{r}\left[5S_i(C_i^2+S_i^2S_\theta^2)(-3+7S_i^2S_\theta^2)-S_i^2C_\theta^2(3-30S_i^2S_\theta^2+35S_i^4S_\theta^4)\right]\right\}\\ F_N &= -3\mu\frac{R_\oplus^2}{r^4}\left\{J_2S_iS_\theta C_i+J_3\frac{R_\oplus}{r}\left[S_i^2(-15S_i^3S_\theta^3)+C_i^2(-3+30S_i^2S_\theta^2-35S_i^4S_\theta^4)\right]\right\},\end{aligned}$$

where i is the orbit inclination, ω the argument of perigee, f the true anomaly, and $\hat{\mathbf{e}}_R$, $\hat{\mathbf{e}}_T$, and $\hat{\mathbf{e}}_N$ are the unit-vectors of the orbital reference frame axes.

4. Solar radiation pressure

A simplified expression for SRP acceleration vector was given in (Schaub & Junkins, 2009) by

$$\mathbf{a} = -C_R\,P_\odot\,S\,m\,\frac{\mathbf{r}_{s\odot}}{r_{s\odot}^3} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{i}} + a_z\,\hat{\mathbf{k}},$$

where $P_\odot \approx 4.56\cdot 10^{-6}$ Nm^{-2} is the solar radiation pressure coefficient, S is the surface area, and m the satellite mass, $\mathbf{r}_{s\odot} = \mathbf{r}_\odot - \mathbf{r}$ is the position vector of the Sun with respect to the satellite, and C_R is the radiation pressure coefficient, which is a function of the reflectivity coefficient, ϵ. The reflectivity coefficient becomes $\epsilon = 0$ when the satellite surface absorbs all the solar radiation while it becomes $\epsilon = 1$ when it reflects all the solar radiation.

Using a pseudo potential function, The acceleration components of the SRP can be expressed in the ECI frame as

$$\begin{Bmatrix}a_x\\a_y\\a_z\end{Bmatrix} = -\frac{C_RP_\odot S}{m\,r_{s\odot}}\begin{Bmatrix}x_\odot-x\\y_\odot-y\\z_\odot-z\end{Bmatrix}\qquad\text{where}\qquad\begin{Bmatrix}x_\odot\\y_\odot\\z_\odot\end{Bmatrix} = r_{s\odot}\begin{Bmatrix}\cos\lambda_\odot\\\sin\lambda_\odot\,\cos\varepsilon\\\sin\lambda_\odot\,\sin\varepsilon\end{Bmatrix}, \tag{4.1}$$

where $\lambda_\odot$ is the sun ecliptic longitude and ε is the obliquity of the ecliptic.

Equations (4.1) are transformed to orbital coordinate system using the transformation

$$\begin{bmatrix} \hat{\mathbf{e}}_R \\ \hat{\mathbf{e}}_T \\ \hat{\mathbf{e}}_N \end{bmatrix}^{\mathbf{T}} = R_{313} \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix}^{\mathbf{T}},$$

where R_{313} is the transformation matrix that can be expressed by the "3-1-3" Euler sequence

$$R_{313} = R_3(\omega + f)\, R_1(i)\, R_2(\Omega).$$

So that the SRP force can be expressed as three components in the directions of $(\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_N)$ coordinate system as a_R, a_T, a_N.

5. Perturbed motion

In case of unperturbed motion, the angles ω, Ω, and i are constant. These angles are used in the transformation equations between coordinate systems and also can be used to determine the position and velocity of the satellite at any given time. The orbit of the satellite undergoes perturbations from several environmental forces resulting in changes in the elements of the orbits.

The rates of change of the orbital elements $(a, e, i, \omega, \Omega, M)$ due to a perturbing force

$$\mathbf{F} = F_R\, \hat{\mathbf{e}}_R + F_T\, \hat{\mathbf{e}}_T + F_N\, \hat{\mathbf{e}}_N \tag{5.1}$$

are given in (Guochang, 2008) and called Gaussian planetary equations. These equations are:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}[e\cos f\, F_R + (1 + e\cos f)\, F_T)] \tag{5.2}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}[\sin f\, F_R + (\cos E + \cos f)\, F_T] \tag{5.3}$$

$$\frac{di}{dt} = \frac{(1 - e\cos E)\cos(\omega + f)}{na\sqrt{1-e^2}}\, F_N \tag{5.4}$$

$$\frac{d\Omega}{dt} = \frac{(1 - e\cos E)\sin(\omega + f)}{na\sqrt{1-e^2}\sin i}\, F_N \tag{5.5}$$

and

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left(-\cos f\, F_R + \sin f \frac{2 + e\cos f}{1 + e\cos f} F_T\right) - \cos i \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{nae}\left[-\left(\cos f - \frac{2e}{1 + e\cos f}\right) F_R + \sin f \frac{2 + e\cos f}{1 + e\cos f} F_T\right],$$

where a is the semi-major axis, e is the eccentricity of the orbit, n is the mean motion, E is the eccentric anomaly, and M is the mean anomaly. We solve this system of differential equations to get the elements $(a, e, i, \Omega, \omega, M)$ as functions of time. Having these elements one can find the position and velocity at any time. The angles (i, Ω, ω) are needed for the transformations between coordinate systems. We need to compute the radius vector $\mathbf{r}$ in the ECI reference frame.

6. Position vector of the ground station

6.1. Position of the ground station in ECI frame

Assuming the Earth is an oblate spheroid, the position vector of the station in the *ECI* frame has the components :

$$\begin{aligned} R_i &= (N+H)\cos\lambda_E\,\cos\theta, \\ R_j &= (N+H)\cos\lambda_E\,\sin\theta, \\ R_k &= (N(1-e_E^2)+H)\sin\lambda_E, \end{aligned}$$

where $N = \dfrac{a_E}{\sqrt{1-e_E^2\sin^2\lambda_E}}$, is the Earth's mean radius, λ_E is the geodetic longitude of the station and H is the height of the station.

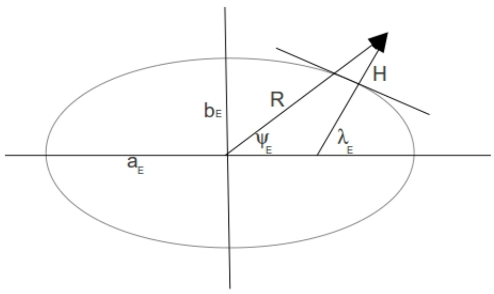

Figure 2. Ground Station Geodetic Coodinates.

Earth rotates around the $\hat{\mathbf{k}}$-axis with angular velocity $\omega_\oplus = 7.2921158553 \cdot 10^{-5}$ rad/s. The angle θ between the $\hat{\mathbf{i}}$-axis and the $\hat{\mathbf{i}}'$-axis is a function of time and is related to $\omega_\oplus$ by

$$\alpha(t) = \alpha_0 + \omega_\oplus(t - t_0).$$

The angle α, called *Greenwich hour angle*, is the right ascension of the Greenwich meridian.

6.2. Satellite range

The range of the satellite is given by

$$\boldsymbol{\rho} = \mathbf{r}_{sat} - \mathbf{R}_{station}.$$

We have described both $\mathbf{r}_{sat}$ and $\mathbf{R}_{station}$ in the *ECI* frame. Now we need to have an expression of this range as seen in the observer's Topocentric Horizon coordinate system (local, on the Earth surface). In this reference rame the fundamental plane is the observer's horizon plane, the positive $\hat{\mathbf{x}}$-axis is taken in the South direction, the $\hat{\mathbf{y}}$-axis is pointing toward the East, and $\hat{\mathbf{z}}$-axis pointing toward the observer's Zenith. The frame is referred to as SEZ frame.

The transformation of the range vector from the *ECI* frame to the SEZ frame is done using the transformation equation

$$\boldsymbol{\rho}_{SEZ} = A_{tp}\,\boldsymbol{\rho}_{ECI}$$

where the transformation matrix is given as

$$A_{tp} = \begin{bmatrix} \sin\psi_E \cos\theta & \sin\psi_E \sin\theta & -\cos\psi_E \\ -\sin\theta & \cos\theta & 0 \\ \cos\psi_E \cos\theta & \cos\psi_E \sin\theta & \sin\psi_E \end{bmatrix},$$

where ψ_E is the angle between the radius vector of the station and the semi-major axis of the spheroidal Earth. The magnitude of the range is given by

$$\rho = \sqrt{\rho_S^2 + \rho_E^2 + \rho_Z^2}.$$

The time derivative of the range gives the relative velocity magnitude of the satellite with respect to the station in the observer's local frame (SEZ frame). We have from the above equation

$$v_R = \dot{\rho} = \frac{1}{\rho}(\rho_S\,\dot{\rho}_S + \rho_E\,\dot{\rho}_E + \rho_Z\,\dot{\rho}_Z).$$

7. Numerical example

Considering a GPS satellite with initial values $a = 26,550$ km, $e = 0.02$, $i = 55°$, $s/m = 0.02$ m^2/kg, $\Omega = 0°$, $\omega = 0°$, and $M = 0°$. The period of this GPS Satellite is 12 hours and data are computed for 4 days. Figures 3 through 8 show the perturbation in the elements of the orbit. Figure 9 shows the change in range as seen from the ECI coordinate system and Figure 10 shows the change in range as seen from the topo-centric coordinate system.

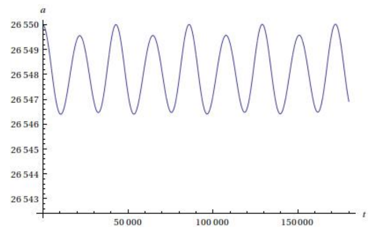

Figure 3. Perturbation in the semi major axis of a GPS satellite.

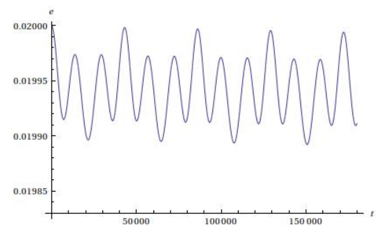

Figure 4. Perturbation in the eccentricity of a GPS satellite.

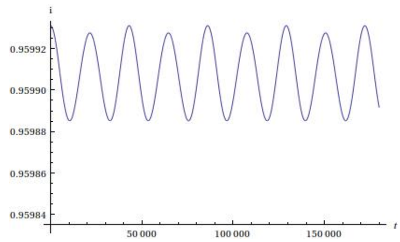

Figure 5. Perturbation in the inclination of a GPS satellite.

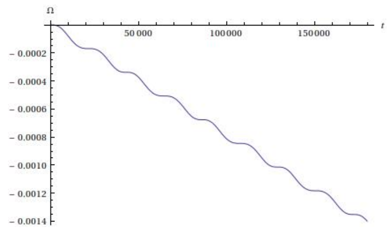

Figure 6. Perturbation in the longitude of ascending node of a GPS satellite.

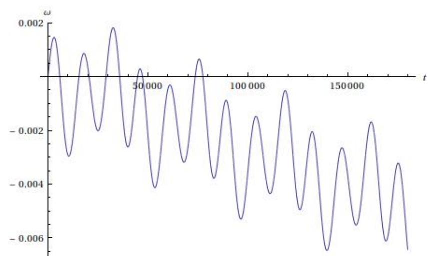

Figure 7. Perturbation in the argument of perigee of a GPS satellite.

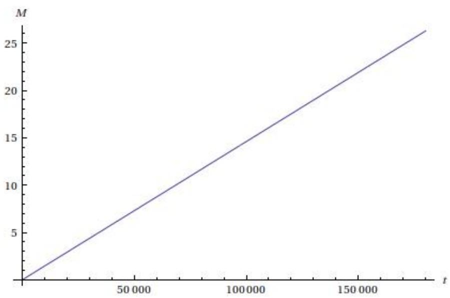

Figure 8. Perturbation in the mean anomaly of a GPS satellite.

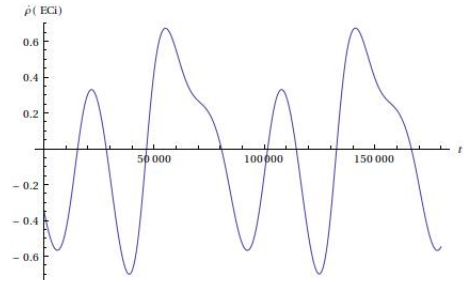

Figure 9. Change in the range as it is seen from the ECI coordinate system.

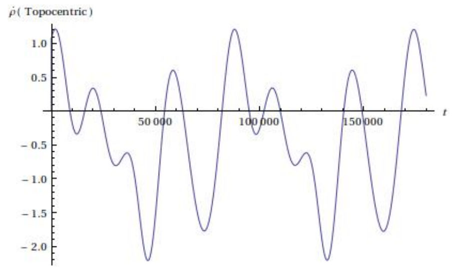

Figure 10. Change in the range as it is seen from the topocentric coordinate system.

8. Conclusion

In this paper we have computed the range and the change in range (relative velocity) of a GPS satellite as seen by an observer in the ground station. The relative motion of the satellite with respect to the ground station is affected by the rotation of the Earth and by the perturbation of the satellite. The GPS satellite's motion was under the effect of the perturbation of the oblateness of the Earth up to J_3 and the perturbation of the Solar Radiation Pressure force.

9. Acknowledgment

This project was supported financially by the Science and Technology Development Fund (STDF), Egypt, Grant No 1834.

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Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 2 (1) (2012) 61–71

A_σ-Double Sequence Spaces and Double Statistical Convergence in 2-Normed Spaces Defined by Orlicz Functions

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Abstract

The main aim of this paper is to introduce a new class of sequence spaces which arise from the notion of invariant means, de la Valee-Pousin means and double lacunary sequence with respect to an Orlicz function in 2-normed space. Some properties of the resulting sequence space were also examined. Further we study the concept of uniformly $(\bar{\lambda}, \sigma)$-statistical convergence and establish natural characterization for the underline sequence spaces.

Keywords: Double sequence spaces, 2-normed space, Double statistical convergence, Orlicz function.
2000 MSC: 46E30, 46E40, 46B20.

1. Introduction

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences $x = (x_i)$, with complex terms respectively, normed by $\|x\|_\infty = \sup_i |x_i|$, where $i \in \mathbb{N}$. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits that is to say, if and only if, for all $i = 0, j = 0, \sigma^j(i) \neq i$ and T be the operator defined on l_∞ by $(T(x_i)_{i=1}^\infty) = (x_{\sigma(i)})_{i=1}^\infty$.

A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_i)$ has $x_i \geq 0$ for all i,
2. $\phi(e) = 1$, where $e = \{1, 1, 1, ...\}$ and
3. $\phi(x_{\sigma(i)}) = \phi(x)$ for all $x \in l_\infty$.

The space $[V_\sigma]$ of strongly $\sigma-$convergent sequence was introduced by Mursaleen in (Mursaleen, 1983). A sequence $x = (x_k)$ is said to be strongly $\sigma-$convergent if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k} |x_{\sigma^j}(m) - L| \to \infty$$

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as $k \to \infty$ uniformly in m. If we take $\sigma(m) = m + 1$ then $[V_\sigma] = [\hat{c}]$, which was defined by Maddox in (Maddox, 1967).

If $x = (x_i)$ write $Tx = (Tx_i) = (x_{\sigma(i)})$. It can be shown that

$$V_\sigma = \Big\{x = (x_i) : \sum_{m=1}^{\infty} t_{m,i}(x) = L \text{ uniformly in i, } L = \sigma - \lim x\Big\} \tag{1.1}$$

where $m \geq o$, $i > 0$.

$$t_{m,i}(x) = \frac{x_i + x_{\sigma(i)} + ... + x_{\sigma^m(i)}}{m+1} \text{ and } t_{-1,i} = 0, \tag{1.2}$$

where, $\sigma^m(i)$ denote the mth iterate of $\sigma(i)$ at i. In the case σ is the translation mapping, $\sigma(i) = i + 1$ is often called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequence(see (Móricz & Rhoades, 1988)). Subsequently invariant means have been studied by Ahmad and Mursaleen in (Ahmad & Mursaleen, 1988), (Raimi, 1963) and many others.

The concept of 2-normed spaces was initially introduced by (Gähler, 1963) in the mid of 1960's. Since then, many researchers have studied this concept and obtained various results, see for instance (Gähler, 1965, 1964; Gunawan & Mashadi, 2001).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to R$ which satisfies the following four conditions (Khan & Tabassum, 2011*b*, 2010):

(i) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent;
(ii) $\|x_1, x_2\| = \|x_2, x_1\|$;
(iii) $\|\alpha x_1, x_2\| = \alpha\|x_1, x_2\|$, for any $\alpha \in R$;
(iv) $\|x + x', x_2\| \leq \|x, x_2\| + \|x', x_2\|$.

The pair $(X, \|.,.\|)$ is then called a 2-normed space.

Exemple 1.1. A standard example of a 2-normed space is R^2 equipped with the following 2-norm: $\|x, y\| :=$ the area of the triangle having vertices $0, x, y$.

Exemple 1.2. Let Y be a space of all bounded real-valued functions on R. For f, g in Y, define $\|f, g\| = 0$ if f, g are linearly dependent, $\|f, g\| = \sup_{t \in R} |f(t).g(t)|$, if f, g are linearly independent. Then $\|.,.\|$ is a 2-norm on Y.

An *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M satisfies the $\Delta_2 - condition$ ($M \in \Delta_2$ for short) if there exist constant $K \geq 2$ and $u_0 > 0$ such that $M(2u) \leq KM(u)$ whenever $|u| \leq u_0$.

An Orlicz function M can always be represented in the integral form $M(x) = \int_0^x q(t)dt$, where q known as the kernel of M, is right differentiable for $t \geq 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1,$$

since M is convex and $M(0) = 0$.

Lindesstrauss and Tzafriri in (Lindenstrauss & Tzafriri, 1971) used the idea of Orlicz sequence space;

$$l_M := \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\},$$

which is Banach space with the norm

$$\|x\|_M = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

Throughout $x = (x_{jk})$ is a double sequence that is a double infinite array of elements x_{jk}, for $j, k \in \mathbb{N}$.

Double sequence have been studied by Vakeel A. Khan and S. Tabassum in (Khan, 2010; Khan & Tabassum, 2012, 2011*b*,*a*, 2010) and many others.

The following inequality will be used throughout

$$|x_{jk} + y_{jk}|^{p_{jk}} \le D(|x_{jk}|^{p_{jk}} + |y_{jk}|^{p_{jk}}), \tag{1.3}$$

where x_{jk} and y_{jk} are complex numbers, $D = \max(1, 2^{H-1})$ and $H = \sup_{j,k} p_{jk} < \infty$.

Definition 1.1. A double sequence $x = (x_{jk})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \epsilon$ whenever $j, k > N$. We shall describe such an x more briefly as $P - convergent$.

Definition 1.2. (Savaş & Patterson, 2007) The four dimensional matrix $A = (a_{m,n,j,k})$ is said to be RH-regular if it maps every bounded P-convergent sequences into a P-convergent sequence with the same P-limit.

Theorem 1.3. *(Savaş & Patterson, 2007) The four dimensional matrix $A = (a_{m,n,j,k})$ is said to be RH-regular if and only if*

- **(i)** $P - \lim_{m,n} a_{m,n,j,k} = 0$ *for each* j, k;
- **(ii)** $P - \lim_{m,n} \sum_{j,k=1}^{\infty} a_{m,n,j,k} = 1$;
- **(iii)** $P - \lim_{m,n} \sum_{j=1}^{\infty} |a_{m,n,j,k}| = 0$; *for each* k;
- **(iv)** $P - \lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,j,k}| = 0$; *for each* j;
- **(v)** $\sum_{j,k=1}^{\infty} |a_{m,n,j,k}|$ *is P-convergent and*
- **(vi)** *there exist positive numbers A and B such that* $\sum_{j,k>B} |a_{m,n,j,k}| < A$.

2. Main Results

Let M be an Orlicz function $P = (p_{jk})$ be any factorable double sequence of strictly positive real numbers. Let $A = (a_{m,n,j,k})$ be a non negative RH-regular summabilty matrix method, $(X, \|.,.\|)$ be 2-norm space, σ be an injection of the set of positive integers $\mathbb{N}$ into itself and $p,\ q \in \mathbb{N}$. We define the following double sequence spaces:

$${}_2W_o(A_\sigma, M, p, \|.,.\|) = \Big\{x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0 \text{ and } z\in X\Big\}$$

$$_2W(A_\sigma, M, p, \|.,.\|) = \Big\{x=(x_{jk}) : P-\lim_{m,n}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}-L,z\|}{\rho}\Big)\Big]^{p_{jk}}=0,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0, L>0 \text{ and } z\in X\Big\}$$

$$_2W_\infty(A_\sigma, M, p, \|.,.\|) = \Big\{x=(x_{jk}) : \sup_{m,n,j,k}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)},z\|}{\rho}\Big)\Big]^{p_{jk}}<\infty,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0 \text{ and } z\in X\Big\}$$

Let us consider a few special cases of above definitions:
(**i**) In particular, when $\sigma(p,q)=(p+1,q+1)$, we have

$$_2W_o(A, M, p, \|.,.\|) = \Big\{x=(x_{jk}) : P-\lim_{m,n}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q},z\|}{\rho}\Big)\Big]^{p_{jk}}=0,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0 \text{ and } z\in X\Big\}$$

$$_2W(A, M, p, \|.,.\|) = \Big\{x=(x_{jk}) : P-\lim_{m,n}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q}-L,z\|}{\rho}\Big)\Big]^{p_{jk}}=0,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0, L>0 \text{ and } z\in X\Big\}$$

$$_2W_\infty(A, M, p, \|.,.\|) = \Big\{x=(x_{jk}) : \sup_{m,n,j,k}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\Big[M\Big(\frac{\|x_{j+p,k+q},z\|}{\rho}\Big)\Big]^{p_{jk}}<\infty,$$

$$\text{uniformly in } p,q,\ \text{ for some } \rho>0 \text{ and } z\in X\Big\}$$

(**ii**) If $M(x)=x$ then we have

$$_2W_o(A_\sigma, p, \|.,.\|) = \Big\{x=(x_{jk}) : P-\lim_{m,n}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)},z\|^{p_{jk}}=0,$$

$$\text{uniformly in } p,q,\ \text{ and } z\in X\Big\}$$

$$_2W(A_\sigma, p, \|.,.\|) = \Big\{x=(x_{jk}) : P-\lim_{m,n}\sum_{j,k=0}^{\infty} a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)}-L,z\|^{p_{jk}}=0,$$

$$\text{uniformly in } p,q \text{ and } L>0, z\in X\Big\}$$

$$ {}_2W_\infty(A_\sigma, p, \|.,.\|) = \Big\{ x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \|x_{\sigma^j(p),\sigma^k(q)}, z\|^{p_{jk}} < \infty, $$
$$ \text{uniformly in } p, q, \text{ and } z \in X \Big\} $$

(iii) If $p_{jk} = 1$ for all (j,k), we have

$$ {}_2W_o(A_\sigma, M, \|.,.\|) = \Big\{ x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho} \Big) \Big] = 0, $$
$$ \text{uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X \Big\} $$

$$ {}_2W(A_\sigma, M, \|.,.\|) = \Big\{ x = (x_{jk}) : P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho} \Big) \Big] 0, $$
$$ \text{uniformly in } p, q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X \Big\} $$

$$ {}_2W_\infty(A_\sigma, M, \|.,.\|) = \Big\{ x = (x_{jk}) : \sup_{m,n,j,k} \sum_{j,k=0}^{\infty} a_{m,n,j,k} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho} \Big) \Big] < \infty, $$
$$ \text{uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X \Big\}. $$

Definition 2.1. (Savaş & Patterson, 2008) A bounded double sequence $x = (x_{jk})$ of real number is said to be $(\bar{\lambda}, \sigma)$-convergent to L provided that

$$ P - \lim_{r,s} T^{p,q}_{r,s} = L \text{ uniformly in } (p,q), $$

where

$$ T^{r,s}_{p,q} = \frac{1}{\bar{\lambda}_{r,s}} \sum_{(j,k)\in \bar{I}_{r,s}} x_{\sigma^j(p),\sigma^k(q)}. $$

In this case we write $(\bar{\lambda}, \sigma) - \lim x = L$.

One can see that in contrast to the case for single sequences, a P-convergent sequences need not be $(\bar{\lambda}, \sigma)$−convergent. But it is easy to see that every bounded P-convergent double sequence is $(\bar{\lambda}, \sigma)$−convergent. In addition, if we let $\sigma(p) = p+1, \sigma(q) = q+1$, and $\bar{\lambda}_{r,s} = rs$ in the above definition then $(\bar{\lambda}, \sigma)$−convergence reduces to almost P-convergence which was defined by Moricz and Rhoades in (Móricz & Rhoades, 1988).

Definition 2.2. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non decreasing sequences of positive real numbers both of which tends to ∞ as r, s approach ∞, respectively. Also let $\lambda_{r+1} \le \lambda_r + 1, \lambda_1 = 0$ and $\mu_{s+1} \le \mu_s + 1, \mu_1 = 0$. We write the generalized double de la Valee-Pousin mean by

$$ t_{r,s}(x) = \frac{1}{\lambda_r \mu_s} \sum_{j\in I_r} \sum_{k \in I_s} x_{j,k}, $$

where $I_r = [r - \lambda_r + 1, r]$ and $I_s = [s - \mu_s + 1, s]$.

We shall denote $\lambda_r\mu_s$ by $\bar{\lambda}rs$ and $(j \in I_r,\ k \in I_s)$ by $(j,k) \in \bar{I}_{r,s}$. Let M be an Orlicz function, x_{jk} be double sequence space and $p = (p_{jk})$ be any factorable double sequence of strictly positive real numbers. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be the same as defined above and $(X, \|.,.\|)$ be 2-norm space. If we take

$$a_{r,s,j,k} = \begin{cases} \frac{1}{\bar{\lambda}rs} \text{ if } & (j,k) \in \bar{I}_{r,s}, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in\bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$
$$\text{uniformly in } p,q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|] = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in\bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$
$$\text{uniformly in } p,q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_\infty = \Big\{x = (x_{jk}) : \sup_{r,s,p,q} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in\bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} < \infty,$$
$$\text{for some } \rho > 0 \text{ and } z \in X\Big\}.$$

Definition 2.3. The double lacunary sequence was defined by E. Savaş and R. F. Patterson (Savaş & Patterson, 1994) as follows:
The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations : $k_{r,s} = k_r l_s,\ \ h_{r,s} = h_r \bar{h}_s$.
The following intervals are determined by θ :

$$I_r = \{(k_r) : k_{r-1} < k < k_r\}, I_s = \{(l) : l_{s-1} < l < l_s\},$$

$$I_{r,s} = \{(k,l) : k_{r-1} < k < k_r \text{ and } l_{s-1} < l < l_s\},$$

$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r\bar{q}_s$. We will denote the set of all double lacunary sequences by $N_{\theta_{r,s}}$. The space of double lacunary strongly convergent sequence is defined as follows

$$N_{\theta_{r,s}} = \Big\{x = (x_{k,l}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l} - L| = 0 \text{ for some } L\Big\}$$

see (Savaş & Patterson, 1994).
If we take

$$a_{r,s,j,k} = \begin{cases} \frac{1}{\bar{h}_{rs}} & \text{if } (j,k) \in I_{r,s}, \\ 0 & \text{otherwise .} \end{cases}$$

We have

$$[{}_2W_\sigma, \theta, M, p, \|.,.\|]_o = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p, q, \text{ for some } \rho > 0 \text{ and } z \in X\Big\}$$

$$[{}_2W_\sigma, \theta, M, p, \|.,.\|] = \Big\{x = (x_{jk}) : P - \lim_{r,s} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\Big)\Big]^{p_{jk}} = 0,$$

$$\text{uniformly in } p, q, \text{ for some } \rho > 0, L > 0 \text{ and } z \in X\Big\}$$

$$[{}_2W_\sigma, M, \theta, p, \|.,.\|]_o = \Big\{x = (x_{jk}) : \sup_{r,s,p,q} \frac{1}{\bar{h}rs} \sum_{(j,k)\in I_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho}\Big)\Big]^{p_{jk}} < \infty,$$

$$\text{for some } \rho > 0 \text{ and } z \in X\Big\}.$$

Theorem 2.1. *Let $P = p_{jk}$ be bounded. Then ${}_2W(A_\sigma, M, p, \|.,.\|)$, ${}_2W_o(A_\sigma, M, p, \|.,.\|)$ and ${}_2W_\infty(A_\sigma, M, p, \|.,.\|)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.2. *Let $P = p_{jk}$ be bounded. Then $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o$, $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]$ and $[{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_{jk})$ and $y = (y_{jk}) \in [{}_2V_\sigma, \bar{\lambda}, M, p, \|.,.\|]_o$ and $\alpha, \beta \in \mathbb{C}$ then there exist two positive numbers ρ_1, ρ_2 such that

$$P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1}\Big)\Big]^{p_{jk}} = 0,$$

$$P - \lim_{r,s} \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2}\Big)\Big]^{p_{jk}} = 0,$$

uniformly in (p, q). Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex, we have

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|\alpha x_{\sigma^j(p),\sigma^k(q)} + \beta y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3}\Big)\Big]^{p_{jk}} = \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|\alpha x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3} + \frac{|\beta y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_3}\Big)\Big]^{p_{jk}}$$

$$\le \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \frac{1}{2^{p_{jk}}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1}\Big) + M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2}\Big)\Big]^{p_{jk}}$$

$$\le \frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in \bar{I}_{r,s}} \Big[M\Big(\frac{\|x_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_1}\Big) + M\Big(\frac{\|y_{\sigma^j(p),\sigma^k(q)}, z\|}{\rho_2}\Big)\Big]^{p_{jk}}$$

$$\leq D\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)},z\|}{\rho_1}\right)\right]^{p_{jk}} + D\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in \bar{I}_{r,s}}\left[M\left(\frac{\|y_{\sigma^j(p),\sigma^k(q)},z\|}{\rho_2}\right)\right]^{p_{jk}}.$$

(From equation (1.1).)

Now since the last inequality tends to zero as (r,s) approaches in Pringsheim sense, uniformly in (p,q), $[{}_2V_\sigma,\bar{\lambda},M,p,\|.,.\|]_o$ is linear. The proof of others follow in similar manner. □

Theorem 2.3. *Let $P = p_{jk}$ be bounded. Then $[{}_2W_\sigma,\theta,M,p,\|.,.\|]_o$, $[{}_2W_\sigma,\theta,M,p,\|.,.\|]$ and $[{}_2W_\sigma,\theta,M,p,\|.,.\|]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.4. *Let A be non negative RH regular summability matrix method and M be an Orlicz function which satisfies $\triangle_2$ condition. Then ${}_2W_o(A_\sigma,p,\|.,.\|)\subset {}_2W_o(A_\sigma,M,p,\|.,.\|)$, ${}_2W(A_\sigma,p,\|.,.\|)\subset {}_2W(A_\sigma,M,p,\|.,.\|)$ and ${}_2W(A_\sigma,p,\|.,.\|)_\infty\subset {}_2W(A_\sigma,M,p,\|.,.\|)_\infty$.*

Proof. Let $x=(x_{jk})\in {}_2W(A_\sigma,p,\|.,.\|)$, then

$$P-\lim_{m,n}\sum_{j,k=0}^{\infty}a_{m,n,j,k}\|x_{\sigma^j(p),\sigma^k(q)},z\|^{p_{jk}}\to 0, \tag{2.1}$$

as $m,n\to\infty$ uniformly in (p,q). Let $\epsilon>0$ and choose $0<\delta<1$ such that $M(t)<\frac{\epsilon}{2}$ for $0\leq t\leq\delta$. Write $y_{jk}=\|x_{\sigma^j(p),\sigma^k(q)},z\|$ and consider

$$\sum_{j,k=0}^{\infty}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}=\sum_{1}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}+\sum_{2}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}.$$

Where the first summation is over $y_{jk}\leq\delta$ and the second summation is over $y_{jk}>\delta$. Since M is continuous, we have

$$\sum_{1}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}\leq\epsilon^H\sum_{j,k=0}^{\infty}a_{m,n,j,k}.$$

For $y_{jk}>\delta$, we use the fact that

$$y_{jk}<\frac{y_{jk}}{\delta}\leq 1+\left(\frac{y_{jk}}{\delta}\right).$$

Since M is non decreasing and convex, it follows that

$$M(y_{jk})<M(1+\delta^{-1}y_{jk})=M\left(\frac{2}{2}+\frac{2}{2}\delta^{-1}y_{jk}\right)<\frac{1}{2}M(2)+\frac{1}{2}M(2\delta^{-1}y_{jk}).$$

Since M satisfies Δ_2-condition, there is a constant $K>2$ such that

$$M(2\delta^{-1}y_{jk})\leq\frac{1}{2}K\delta^{-1}y_{jk}M(2).$$

Hence

$$\sum_{2}[M(y_{jk})]^{p_{jk}}<\max(1,(K\delta^{-1}M(2)))\sum_{j,k=0}^{\infty}[M(y_{jk})]^{p_{jk}}.$$

Thus we have

$$\sum_{(j,k=0)}^{\infty}[M(y_{jk})]^{p_{jk}}<\max(1,\epsilon^H)\sum_{j,k=0}^{\infty}a_{m,n,j,k}+\max(1,(K\delta^{-1}M(2)))\sum_{j,k=0}^{\infty}a_{m,n,j,k}[M(y_{jk})]^{p_{jk}}.$$

Thus (2.1) and R-H Regularity of A grants us ${}_2W(A_\sigma,p,\|.,.\|)\subset {}_2W(A_\sigma,M,p,\|.,.\|)$. Similarly we can prove the other two inclusion relations. □

3. Double Statistical Convergence

The concept of statistical convergence was first introduced by Fast in (Fast, 1951) and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Šalát (Tibor, 1980), Fridy in (Fridy, 1985) and many others.

Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is called statistically convergent to L if

$$\lim_n \frac{1}{n}|k : |x_k - L| \geq \epsilon, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$.

The following definition was presented by Mursaleen in (Mursaleen, 2000). A sequence x is said to be $\lambda-$statistical convergent to L, if for $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n}|k \in I_n : |x_k - L| \geq \epsilon, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set and $I_n = [n - \lambda_n + 1, n]$. In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$.

Savaş (Savaş, 2000) presented and studied the concepts of uniformly $\lambda-$statistical convergence as follows: A sequence x is said to uniformly $\lambda-$statistical convergent to L , if for $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \max_m |k \in I_n : |x_{k+m} - L| \geq \epsilon| = 0.$$

In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(_\lambda)$.

A double sequence (x_{jk}) is called statistically convergent to L if

$$\lim_{m,n\to\infty} \frac{1}{mn}|(j,k) : |x_{jk} - L| \geq \epsilon, j \leq m, k \leq n| = 0,$$

where the vertical bars indicate the number of elements in the set.(see[10])

Definition 3.1. (Savaş & Patterson, 2008) A double sequence $x = (x_{jk})$ is said to be uniformly $(\bar{\lambda}, \sigma)$-statistical convergent to L, provided that for every $\epsilon > o$

$$P - \lim_{r,s} \frac{1}{\lambda_{rs}} \max_{p,q} |\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L| \geq \epsilon\}| = 0.$$

In this case we write $_2S_{(\bar{\lambda},\sigma)} - \lim x = L$ or $x_{jk} \to L(_2S_{(\bar{\lambda},\sigma)})$.

Theorem 3.1. *Let M be an Orlicz Function and $0 < h = \inf p_{jk} \leq p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$ then $[_2V_\sigma, \bar{\lambda}, M, p, ||., .||] \subset {_2S_{(\bar{\lambda},\sigma)}}$.*

Proof. Let $x = (x_{jk}) \in [_2V_\sigma, \bar{\lambda}, M, p, ||., .||]$. Then there exists $\rho > 0$ such that

$$\frac{1}{\bar{\lambda}rs} \sum_{(j,k)\in\bar{I}_{r,s}} \left[M\Big(\frac{||x_{\sigma^j(p),\sigma^k(q)} - L, z||}{\rho}\Big)\right]^{p_{jk}} = 0,$$

as $r, s \to \infty$ in the Pringsheim sense uniformly in (p, q).

If $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{\rho}$, then we obtain the following:

$$\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in\bar{I}_{r,s}}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}} = \frac{1}{\bar{\lambda}rs}\sum_{1}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}}$$

$$+\frac{1}{\bar{\lambda}rs}\sum_{2}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}}$$

Where the first summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| \geq \epsilon$ and the second summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| < \epsilon$

$$\geq \frac{1}{\bar{\lambda}rs}\sum_{1}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}} \geq \frac{1}{\bar{\lambda}rs}\sum_{1}[M(\epsilon_1)]^{p_{jk}} \geq \frac{1}{\bar{\lambda}rs}\sum_{1}\min\{[M(\epsilon_1)]^{p_{jk}}, [M(\epsilon_1)]^{H}\}$$

$$\geq \frac{1}{\bar{\lambda}rs}|\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L, z| \geq \epsilon\}| \min\{[M(\epsilon_1)]^{h}, [M(\epsilon_1)]^{H}\}.$$

This implies that $x \in {}_2S_{(\bar{\lambda},\sigma)}$. □

Theorem 3.2. *Let M be a bounded Orlicz function and* $0 < h = \inf p_{jk} \leq p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$ *then* ${}_2S_{(\bar{\lambda},\sigma)} \subset [{}_2V_\sigma, \bar{\lambda}, M, p, \|., .\|]$

Proof. Since M is bounded there exists an integer K such that $M(x) < K$ for $x > 0$. Thus

$$\frac{1}{\bar{\lambda}rs}\sum_{(j,k)\in\bar{I}_{r,s}}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}} = \frac{1}{\bar{\lambda}rs}\sum_{1}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}}$$

$$+\frac{1}{\bar{\lambda}rs}\sum_{2}\left[M\left(\frac{\|x_{\sigma^j(p),\sigma^k(q)} - L, z\|}{\rho}\right)\right]^{p_{jk}}.$$

Where the first summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| \geq \epsilon$ and the second summation is over $\|x_{\sigma^j(p),\sigma^k(q)} - L, z\| < \epsilon \leq \frac{1}{\bar{\lambda}rs}\sum_{1}\max\{K^h, K^H\} + \frac{1}{\bar{\lambda}rs}\sum_{2}\left[M\left(\frac{\epsilon}{\rho}\right)\right]^{p_{jk}}$

$$\leq \max\{K^h, K^H\}\frac{1}{\bar{\lambda}rs}|\{(j,k) \in \bar{I}_{r,s} : |x_{\sigma^j(p),\sigma^k(q)} - L, z| \geq \epsilon\}| + \max\left\{\left[M\left(\frac{\epsilon}{\rho}\right)\right]^{h}, \left[M\left(\frac{\epsilon}{\rho}\right)\right]^{H}\right\}.$$

Hence $x \in [{}_2V_\sigma, \bar{\lambda}, M, p, \|., .\|]$. □

Acknowledgments. The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

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