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On Some Generalized I-Convergent Sequence Spaces Defined by a Modulus Function

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Abstract

In this article we introduce the sequence spaces $c_0^I(f,p)$, $c^I(f,p)$ and $l_\infty^I(f,p)$ for a modulus function f, $p = (p_k)$ is a sequence of positive reals and study some of the properties of these spaces.

Keywords: Ideal, filter, paranorm, modulus function, I-convergent sequence spaces, Lipschitz function, I-convergence field.
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1. Introduction

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H. Fast (Fast, 1951). Later on it was studied by J. A. Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilczyński (P. Kostyrko & Wilczyński, 2000). Later on it was studied by Šalát, Tripathy and Ziman (T. Šalát & Ziman, 2004, 2005), Esi and Ozdemir (Esi & Ozdemir, 2012), Hazarika and Esi (Esi & Hazarika, 2012) and Demirci (Demirci, 2001).

Here we give some preliminaries about the notion of I-convergence.

Let N be a non empty set. Then a family of sets I$\subseteq 2^N$(power set of N) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

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A non-empty family of sets $£(I) \subseteq 2^N$ is said to be filter on N if and only if $\Phi \notin £(I)$,for $A, B \in £(I)$ we have $A \cap B \in £(I)$ and for each $A \in £(I)$ and $A \subseteq B$ implies $B \in £$(I).

An Ideal I$\subseteq 2^N$ is called non-trivial if I$\neq 2^N$.

A non-trivial ideal I$\subseteq 2^N$ is called admissible if $\{x : \{x\} \in N\} \subseteq$I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J≠I containing I as a subset. For each ideal I, there is a filter £(I) corresponding to I, i.e £(I) = $\{K \subseteq N : K^c \in I\}$,where K^c = N-K.

Definition 1.1. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write I-lim $x_k = L$.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq\} \in I,\ for\ some\ L \in \mathbb{C}\}.$$

Definition 1.2. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 .In this case we write I-lim $x_k = 0$.

Definition 1.3. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.4. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in \mathbb{N} : |x_k| > M\}$.

Definition 1.5. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for *almost all k relative to I (a.a.k.r.I)*, if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$

Definition 1.6. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \text{ for all } x \in E\}$$ (see (Simons, 1965)).

Definition 1.7. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$. (Tripathy & Hazarika, 2011).

Definition 1.8. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (See (T. Šalát & Ziman, 2005)).

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function (c.f (Dems, 2005; T. Šalát & Ziman, 2004; Gurdal, 2004; Tripathy & Hazarika, 2009, 2011; Khan, 2005; Khan & Ebadullah, 2011*b*,*a*, 2012; Vakeel. A. Khan & Ahmad, 2012).

Definition 1.9. The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value.
Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,
(PI) $g(x) = 0 \; if \; x = \theta$,
(P2) $g(-x) = g(x)$,
(P3) $g(x + y) \leq g(x) + g(y)$,
(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \; (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \;\; (n \to \infty)$, in the sense that $g(x_n - a) \to 0 \;\; (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \;\; (n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space. See (Maddox, 1969).
The idea of modulus was structured in 1953 by Nakano. See (Nakano, 1953).

A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if:

(1) f(t) = 0 if and only if t = 0,
(2) f(t+u)≤ f(t)+ f(u) for all t, u≥0,
(3) f is increasing, and
(4) f is continuous from the right at zero.

Ruckle [17-19] used the idea of a modulus function f to construct the sequence space:

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space,and Ruckle[19 - 21] proved that that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[19- 21] proved that, for any modulus f,

$$X(f) \subset l_1 \text{ and } X(f)^{\alpha} = l_{\infty}$$

where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty. \text{ (See[17-19])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch in (Gramsch, n.d.). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H

Garling (Garling, 1966, 1968),G. Köthe (Köthe, 1970) and W. H. Ruckle ((Ruckle, 1968), (Ruckle, 1967), (Ruckle, 1973)).

The following subspaces of ω were first introduced and discussed by Maddox ((Maddox, 1986), (Maddox, 1969)):

$$l(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\},$$

$$l_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\},$$

$$c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0,\ for\ some\ l \in \mathbb{C}\},$$

$$c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0\},$$

where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides ((Lascarides, 1971, 1983)) defined the following sequence spaces:

$$l_\infty\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\},$$

$$c_0\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0\ \},$$

$$l\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\},$$

where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

We need the following lemmas in order to establish some results of this article.

Lemma 1.1. *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following conditions are equivalent.(See (Lascarides, 1983)).*
(a) $H < \infty$ and $h > 0$,
(b) $c_0(p) = c_0$ or $l_\infty(p) = l_\infty$,
(c) $l_\infty\{p\} = l_\infty(p)$,
(d) $c_0\{p\} = c_0(p)$,
(e) $l\{p\} = l(p)$.

Lemma 1.2. *Let $K \in$ £(I) and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$. (See (T. Šalát & Ziman, 2004), (Tripathy & Hazarika, 2011)).*

Lemma 1.3. *If $I \subset 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$. (See (T. Šalát & Ziman, 2004), (Tripathy & Hazarika, 2011)).*

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded, I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$c^I(f,p) = \{(x_k) \in \omega : f(|x_k - L|^{p_k}) \geq \epsilon \text{ for some L}\} \in I,$$

$$c_0^I(f,p) = \{(x_k) \in \omega : f(|x_k|^{p_k}) \geq \epsilon\} \in I,$$

$$l_\infty^I(f,p) = \{(x_k) \in \omega : \sup_k f(|x_k|^{p_k}) < \infty\} \in I.$$

Also we write

$$m^I(f,p) = c^I(f,p) \cap l_\infty(f,p)$$

and

$$m_0^I(f,p) = c_0^I(f,p) \cap l_\infty(f,p).$$

2. Main Results

Theorem 2.1. *Let $(p_k) \in l_\infty$. Then $c^I(f,p), c_0^I(f,p), m^I(f,p)$ and $m_0^I(f,p)$ are linear spaces.*

Proof. Let $(x_k),(y_k) \in c^I(f,p)$ and α,β be two scalars. Then for a given $\epsilon > 0$ we have:

$$\left\{k \in \mathbb{N} : f(|x_k - L_1|^{p_k}) \geq \frac{\epsilon}{2M_1}, for\ some\ L_1 \in \mathbb{C}\right\} \in I$$

$$\left\{k \in \mathbb{N} : f(|y_k - L_2|^{p_k}) \geq \frac{\epsilon}{2M_2}, for\ some\ L_2 \in \mathbb{C}\right\} \in I$$

where

$$M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$$

$$M_2 = D.max\{1, \sup_k |\beta|^{p_k}\},$$

and

$$D = max\{1, 2^{H-1}\} \text{ where } H = \sup_k p_k \geq 0.$$

Let

$$A_1 = \left\{k \in \mathbb{N} : f(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}, for\ some\ L_1 \in \mathbb{C}\right\} \in I,$$

$$A_2 = \left\{k \in \mathbb{N} : f(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}, for\ some\ L_2 \in \mathbb{C}\right\} \in I$$

be such that $A_1^c, A_2^c \in I$.

Then

$$A_3 = \{k \in \mathbb{N} : f(|(\alpha x_k + \beta y_k) - f(\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\}$$
$$\supseteq \{k \in \mathbb{N} : |\alpha|^{p_k} f(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}|\alpha|^{p_k}.D\}$$
$$\cap\{k \in \mathbb{N} : |\beta|^{p_k} f(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}|\beta|^{p_k}.D\}.$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in c^I(f, p)$. Therefore $c^I(f, p)$ is a linear space. The rest of the result follows simililarly. □

Theorem 2.2. *Let $(p_k) \in l_\infty$. Then $m^I(f, p)$ and $m_0^I(f, p)$ are paranormed spaces, paranormaed by $g(x_k) = \sup_k f(|x_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$*

Proof. Let $x = (x_k), y = (y_k) \in m^I(f, p)$.
(1) Clearly, $g(x) = 0$ if and only if $x = 0$.
(2) $g(x) = g(-x)$ is obvious.
(3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality and the definition of f we have:

$$\sup_k f(|x_k + y_k|^{\frac{p_k}{M}}) \leq \sup_k f(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|y_k|^{\frac{p_k}{M}}).$$

(4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.
Let $x_k \in m^I(f, p)$ such that $f(|x_k - L|^{p_k}) \geq \epsilon$.
Therefore, $g(x_k - L) = \sup_k f(|x_k - L|^{\frac{p_k}{M}}) \leq \sup_k f(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|L|^{\frac{p_k}{M}})$.
Hence $g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x_k) + \lambda g(L)$ as $(k \to \infty)$.
Hence $m^I(f, p)$ is a paranormed space.
The rest of the result follows similarly. □

Theorem 2.3. *A sequence $x = (x_k) \in m^I(f, p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p). \tag{2.1}$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L|^{p_k} < \frac{\epsilon}{2}\} \in m^I(f, p). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_k|^{p_k} \leq |x_{N_\epsilon} - L|^{p_k} + |L - x_k|^{p_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$.

Conversely, suppose that $\{k \in \mathbb{N} : f(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(f, p)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(f, p)$ for all $\epsilon > 0$.

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(f,p)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(f,p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(f,p)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(f,p)$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.

In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2} diamI_{k-1}$ for (k = 2, 3, 4, ...) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(f,p)$ for (k=1,2,3,4,......).

Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$.

□

Theorem 2.4. *Let $H = \sup_k p_k < \infty$ and I an admissible ideal. Then the following are equivalent.*
(a) $(x_k) \in c^I(f,p)$;
(b) there exists$(y_k) \in c(f,p)$ such that $x_k = y_k$, for a.a.k.r.I;
(c) there exists$(y_k) \in c(f,p)$ and $(x_k) \in c_0^I(f,p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f(|y_k - L|^{p_k}) \geq \epsilon\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in £(I)$ and $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.

Proof. (a) implies (b). Let $(x_k) \in c^I(f,p)$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

Define a sequence(y_k) as

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x_k - L|^{p_k} < t^{-1}, \\ & L, otherwise. \end{cases}$$

Then $(y_k) \in c(f,p)$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : f(|x_k - L|^{p_k}) \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.
(b) implies (c).For $(x_k) \in c^I(f, p)$. Then there exists $(y_k) \in c(f, p)$ such that $x_k = y_k$, for a.a.k.r.I.
Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $k \in I$.
Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, otherwise. \end{cases}$$

Then $z_k \in c_0^I(f, p)$ and $y_k \in c(f, p)$.
(c) implies (d).Let $P_1 = \{k \in \mathbb{N} : f(|x_k|^{p_k}) \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.
(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim_{n\to\infty} f(|x_{k_n} - L|^{p_{k_n}}) = 0$.
Then for any $\epsilon > 0$, and Lemma 1.10, we have

$$\{k \in \mathbb{N} : f(|x_k - L|^{p_k}) \geq \epsilon\} \subseteq K^c \cup \{k \in K : f(|x_k - L|^{p_k}) \geq \epsilon\}.$$

Thus $(x_k) \in c^I(f, p)$. □

Theorem 2.5. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, p) \supseteq m_0^I(f, q)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. Let $\liminf_{k\in K} \frac{p_k}{q_k} > 0$. and $(x_k) \in m_0^I(f, q)$. Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m_0^I(f, q)$,for a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : f(|x_k|^{q_k}) \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : f(|x_k|^{p_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : f(|x_k|^{\beta q_k}) \geq \epsilon\} \in I.$$

Therefore $(x_k) \in m_0^I(f, p)$. □

Theorem 2.6. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, q) \supseteq m_0^I(f, p)$ if and only if $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. The proof follows similarly as the proof of Theorem 2.5. □

Theorem 2.7. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(f, q) = m_0^I(f, p)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, and $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

Proof. On combining Theorem 2.5 and 2.6 we get the required result.

□

Theorem 2.8. *Let* $h = \inf_k p_k$ *and* $H = \sup_k p_k$*. Then the following results are equivalent.*
(a) $H < \infty$ *and* $h > 0$*.*
(b) $c_0^I(f, p) = c_0^I$*.*

Proof. Suppose that $H < \infty$ and $h > 0$, then the inequalities $min\{1, s^h\} \leq s^{p_k} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalent of (a) and (b) is obvious. □

Theorem 2.9. *Let* f *be a modulus function. Then* $c_0^I(f, p) \subset c^I(f, p) \subset l_\infty^I(f, p)$ *and the inclusions are proper.*

Proof. Let $(x_k) \in c^I(f, p)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f(|x_k - L|^{p_k}) = 0$. We have $f(|x_k|^{p_k}) \leq \frac{1}{2} f(|x_k - L|^{p_k}) + \frac{1}{2} f(|L|^{p_k})$. Taking supremum over k both sides we get $(x_k) \in l_\infty^I(f, p)$ and the inclusion $c_0^I(f, p) \subset c^I(f, p)$ is obvious. Hence $c_0^I(f, p) \subset c^I(f, p) \subset l_\infty^I(f, p)$ and the inclusions are proper. □

Theorem 2.10. *If* $H = \sup_k p_k < \infty$*, then for any modulus* f*, we have* $l_\infty^I \subset M(m^I(f, p))$*, where the inclusion may be proper.*

Proof. Let $a \in l_\infty^I$.This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all k. Therefore $x \in m^I(f, p)$ implies $\sup_k f(|a_k x_k|^{p_k}) \leq (1 + K)^H \sup_k f(|x_k|^{p_k}) < \infty$. which gives $l_\infty^I \subset M(m^I(f, p))$.

To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all k. Take $a_k = k$ for all k. Therefore $x \in m^I(f, p)$ implies $\sup_k f(|a_k x_k|^{p_k}) \leq \sup_k f(|k|^{\frac{1}{k}}) \sup_k f(|x_k|^{p_k}) < \infty$. Thus in this case $a = (a_k) \in M(m^I(f, p))$ while $a \notin l_\infty^I$. □

Theorem 2.11. *The function* $\hbar : m^I(f, p) \to \mathbb{R}$ *is the Lipschitz function,where* $m^I(f, p) = c^I(f, p) \cap l_\infty(f, p)$*, and hence uniformly continuous.*

Proof. Let $x, y \in m^I(f, p)$,$x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} \geq ||x - y||\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} \geq ||x - y||\} \in I.$$

Here $||x - y|| = \sup_k f(|x_k - y_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$
Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} < ||x - y||\} \in m^I(f, p),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} < ||x - y||\} \in m^I(f, p).$$

Hence also $B = B_x \cap B_y \in m^I(f, p)$,so that $B \neq \phi$.
Now taking k in B,

$$|\hbar(x) - \hbar(y)|^{p_k} \leq |\hbar(x) - x_k|^{p_k} + |x_k - y_k|^{p_k} + |y_k - \hbar(y)|^{p_k} \leq 3||x - y||.$$

Thus $\hbar$ is a Lipschitz function. For $m_0^I(f, p)$ the result can be proved similarly. □

Theorem 2.12. *If $x, y \in m^I(f,p)$,then $(x.y) \in m^I(f,p)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)|^{p_k} < \epsilon\} \in m^I(f,p),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)|^{p_k} < \epsilon\} \in m^I(f,p).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)|^{p_k} &= |x_k y_k - x_k\hbar(y) + x_k\hbar(y) - \hbar(x)\hbar(y)|^{p_k} \\ &\leq |x_k|^{p_k}|y_k - \hbar(y)|^{p_k} + |\hbar(y)|^{p_k}|x_k - \hbar(x)|^{p_k}. \end{aligned} \tag{2.2}$$

As $m^I(f,p) \subseteq l_\infty(f,p)$,there exists an $M \in \mathbb{R}$ such that $|x_k|^{p_k} < M$ and $|\hbar(y)|^{p_k} < M$.
Using (2.2) we get

$$|x_k y_k - \hbar(x)\hbar(y)|^{p_k} \leq M\epsilon + M\epsilon = 2M\epsilon,$$

for all $k \in B_x \cap B_y \in m^I(f.p)$. Hence $(x.y) \in m^I(f,p)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For $m_0^I(f,p)$ the result can be proved similarly. □

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A Variant of Classical Von Kármán Flow for a Second Grade Fluid due to a Rotating Disk

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Abstract

An attempt is made to examine the classical Von Kármán flow problem for a second grade fluid by using a generalized non-similarity transformation. This approach is different from that of Von Kármán's evolution of the flow in such a way that the physical quantities are allowed to develop non-axisymmetrically. The three-dimensional equations of motion for the second grade fluid are treated analytically yielding the derivation of the exact solutions for the velocity components. The physical interpretation of the velocity components, vorticity components, shear stresses and boundary layer thickness are also presented.

Keywords: Non-axisymmetric flow, rotating disk, second grade fluid.
2010 MSC: 76-XX.

1. Introduction

The theoretical study of the flow near a rotating disk of infinite extent can be traced back to Von Kármán's similarity analysis. That is why the flow is widely known as Von Kármán flow. He assumed that the flow possessed axial symmetry, and introduced a similarity transformation which reduced the Navier-Stokes equation into a system of coupled nonlinear ordinary differential equations. These equations have been used as a test problem for numerical methods, and in the study of matched asymptotic expansions. This problem has received considerable attention over the years and different extensions of Von Kármánn's swirling flow problem have been made to address various applications, see for instance (Benton, 1966; Kuiken, 1971; Riley, 1964; Sahoo, 2009; Ariel, 2003). However, the possibility of an exact solution for the flow due to a rotating

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disk in a fluid which is at infinity and is rotating rigidly has been implied by Berker (Berker, 1982). Parter and Rajagopal (Parter & Rajagopal, 1984) have established the existence of solutions which do not possess axial symmetry, to the Navier-Stokes equations for the problem governing the flow of two infinite disks rotating about a common axis. Based on that work, Huilgol and Rajagopal(Huilgol & Rajagopal, 1987) have shown that in the case of certain non-Newtonian fluid models, solutions that lack axisymmetry are possible. Recently, Turkyilmazoglu (Turkyilmazoglu, 2009) has obtained exact solutions to the Navier-Stokes equations for the swirling flow problem in such a way that the physical quantities are allowed to develop non-axisymmetrically over a rotating disk.

It is a well-known fact that the Navier-Stokes equations seem to be a weak model for a class of real fluids, called non-Newtonian fluids. During the last few decades, considerable efforts have been devoted to the study of flow of non-Newtonian fluids because of their technological applications. A vast amount of literature is now available for the flow problems associated with non-Newtonian fluids in a variety of situations. One important and simple model of non-Newtonian fluids for which one can reasonably hope to obtain analytical solutions is the second grade fluid. Keeping this in mind, the aim of this work is to extend the analysis of (Turkyilmazoglu, 2009) for a second grade fluid. Undoubtedly, the equations of motion for a second grade fluid are more complicated with highly non-linear terms which make the question of well-posedness extremely difficult to address. Here, it is shown that by using a generalized transformation, the governing equations for the second grade fluid are transformed into a well posed second order system of ODEs whose exact solution is straightforward. In solving this problem we have relaxed the axisymmetric condition of the traditional Von Kármán flow. This analysis is important, not only from a mathematical point of view, but mainly as an essential test for the underlying physical model. The practical applications that can be envisaged for this problem are in the design of thrust bearings, radial diffusers etc., used in the defence industry for instance. We note that a similar problem of a Jeffrey Fluid, has been addressed by (Siddiqui *et al.*, 2013).

The following structure is pursued in the rest of the paper. In section two mathematical formulation of the problem is given. Section three concerns with the flow analysis and section four contains some concluding remarks.

2. Formulation of the problem

Consider the three dimensional flow of an incompressible second grade fluid due to an infinite disk which rotates in the plane $z = 0$ about its axis of rotation z with a constant angular velocity Ω. In cylindrical coordinates (r, θ, z) which rotates with the disk, the governing equations of motion of the second grade fluid are the laws of conservation of mass and momentum which are

$$\nabla.\mathbf{V} = 0, \tag{2.1}$$

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla P + \nabla.\sigma, \tag{2.2}$$

where $\mathbf{V} = (u, v, w)$ is the velocity vector, $\frac{d}{dt}$ is the material time derivative, ρ is the fluid density and P is the pressure. For the second grade fluid the extra stress tensor σ is given by (Ariel, 1997)

$$\sigma = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.3}$$

in which μ is the dynamic viscosity, $\alpha_i (i = 1, 2)$ are material constants satisfying $\alpha_1 \geq 0,$ and $\alpha_1 + \alpha_2 = 0$, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T, \quad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 (\nabla \mathbf{V}) + (\nabla \mathbf{V})^{\mathrm{T}} \mathbf{A}_1, \tag{2.4}$$

where the superscript T is the transpose of the matrix. In the present analysis the flow is assumed to take place in the semi-infinite space $z \geq 0$. Boundary conditions accompanying (2.1)-(2.2) are such that the fluid adheres to the wall at $z = 0$ with a given axial velocity and the velocities are bounded at far distances from the disk.

The flow in this analysis is such that the physical quantities are allowed to develop non-axisymmetrically and we assume that there is no flow along the normal, thus the velocity field can be taken in the form

$$\mathbf{V} = [u(r, \theta, z), v(r, \theta, z), 0] \tag{2.5}$$

We introduce the following dimensionless variables:

$$r^* = \frac{r}{L}, z^* = \frac{z}{L}, u^* = \frac{u}{U}, v^* = \frac{v}{U}, P^* = \frac{P}{\rho U^2}, \sigma_{ij} = \frac{\sigma_{ij}}{\frac{\mu U}{L}}, i, j = r, \theta, z, \tag{2.6}$$

where L is the length scale and $U = L\Omega$. Hence, the dimensionless form of the continuity and the equations of motion, after dropping the *s are given by

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = 0, \tag{2.7}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial P}{\partial r} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right] \tag{2.8}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{\mathbf{2}\sigma_{\theta\theta}}{r}\right] \tag{2.9}$$

$$0 = -\frac{\partial P}{\partial z} + \frac{1}{\mathrm{Re}}\left[\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right] \tag{2.10}$$

In equations (2.7)-(2.10), σ_{rr}, $\sigma_{r\theta}$, σ_{rz}, $\sigma_{\theta z}$, $\sigma_{\theta\theta}$ and σ_{zz}, are the components of the stress tensor σ in (2.3), and are given by

$$c\sigma_{rr} = 2\frac{\partial u}{\partial r} + 2\lambda_1 \left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\frac{\partial u}{\partial r} + \left(\frac{v}{r} - \frac{\partial v}{\partial r}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + 2\left(\frac{\partial u}{\partial r}\right)^2\right\}$$
$$+\lambda_2 \left\{4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial \mathrm{u}}{\partial z}\right)^2\right\}, \tag{2.11}$$

$$c\sigma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \lambda_1 \left\{\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + 2\frac{\partial u}{\partial r}\left(\frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r}\right)\right\}$$
$$+\lambda_2 \left\{\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right)\right\}, \tag{2.12}$$

$$c\sigma_{rz} = \frac{\partial u}{\partial z} + \lambda_1 \left\{ \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + 3\frac{\partial u}{\partial r} \right) \frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} \left(2\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{2v}{r} \right) \right\} + \lambda_2 \left\{ 2\left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) \right\}, \tag{2.13}$$

$$c\sigma_{\theta z} = \frac{\partial v}{\partial z} + \lambda_1 \left\{ \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} \right) \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z} \left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \right) - 3\frac{\partial u}{\partial r}\frac{\partial v}{\partial z} \right\} + \lambda_2 \left\{ \frac{\partial u}{\partial z} \left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \right) - 2\frac{\partial u}{\partial r}\frac{\partial v}{\partial z} \right\}, \tag{2.14}$$

$$c\sigma_{\theta\theta} = -2\left(\frac{\partial u}{\partial r}\right) + \lambda_1 \left\{ 4\left(\frac{\partial u}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial u}{\partial \theta}\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) - 2\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta}\right)\left(\frac{\partial u}{\partial r}\right) \right\} + \lambda_2 \left\{ 4\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right\}, \tag{2.15}$$

$$\sigma_{zz} = (2\lambda_1 + \lambda_2)\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right), \tag{2.16}$$

where $\lambda_1 = \frac{\alpha_{1U_c}}{\mu L}$ and $\lambda_2 = \frac{\alpha_{2U_c}}{\mu L}$ are the material parameters of the second grade fluid.

3. Flow analysis

In this section we restrict ourselves to the stationary mean flow relative to the rotating disk. Within this view, via a coordinate transformation $\zeta = \sqrt{\frac{\mathrm{Re}}{2}}z$, we assume a solution of the form (Turkyilmazoglu, 2009)

$$u = aF(\theta,\zeta),\ v = r + aW(\theta,\zeta), w = 0, P = \frac{r^2}{2} - ra\cos(\theta - \sigma) + a^2 p(\zeta), \tag{3.1}$$

such that, non-axisymmetric and periodic solutions with respect to θ of F and W are determined here, subjected to the pressure field given by (3.1). The parameters a and σ correspond to the polar representation of a fixed point on the disk surface and $p(\zeta)$ is some function of ζ.

The transformations (2.7)-(2.10) along with (2.11)-(2.16), satisfy the continuity equation directly, and for the momentum equations, the periodicity assumption of F and W with respect to θ, gives the set of ordinary differential equations

$$F_{\zeta\zeta} + \lambda_1 W_{\zeta\zeta} + 2W = -2\cos(\theta - \sigma), \tag{3.2}$$

$$W_{\zeta\zeta} - \lambda_1 F_{\zeta\zeta} - 2F = 2\sin(\theta - \sigma), \tag{3.3}$$

$$p(\zeta) = \frac{(2\lambda_1 + \lambda_2)}{2}(F_\zeta^2 + W_\zeta^2) + K, \tag{3.4}$$

where the constant K is determined from the pressure prescribed at the disk surface. The boundary conditions for the problem reduce to

$$F = 0, \quad W = 0 \text{ at } \zeta = 0, \quad F, W \text{ bounded, as } \zeta \to \infty. \tag{3.5}$$

Introducing a new function of the form $V = F + iW$, transforms the pair of equations (3.2)-(3.3) into a single complex differential equation with real variables

$$(1 - i\lambda_1)V_{\zeta\zeta} - 2iV = -2((\cos(\theta - \sigma) - i\sin(\theta - \sigma)), \tag{3.6}$$

whose solution is bounded with respect to ζ and can be immediately expressed as

$$V = Ce^{m\zeta} - i\left(\cos((\theta - \sigma) - i\sin(\theta - \sigma)\right), \tag{3.7}$$

where, C is a complex integration constant depending on θ and is determined by using the no-slip condition on the wall and the constant $m = -\sqrt{\frac{2}{1+\lambda_1^2}(i - \lambda_1)}$. Equating real and imaginary parts of the solution given in (3.7), F and W are found to be

$$F(\zeta, \theta) = f(\zeta)\cos(\theta - \sigma) + g(\zeta)\sin(\theta - \sigma), \tag{3.8}$$

$$W(\zeta, \theta) = -f(\zeta)\sin(\theta - \sigma) + g(\zeta)\cos(\theta - \sigma), \tag{3.9}$$

where

$$f(\zeta) = \sin(d_2\zeta)e^{-d_1\zeta}, \quad g(\zeta) = -1 + \cos(d_2\zeta)e^{-d_1\zeta}, \tag{3.10}$$

and where

$$d_1 = \sqrt{\frac{\left(-\lambda_1 + \sqrt{(1+\lambda_1^2)}\right)}{(1+\lambda_1^2)}}, d_2 = \sqrt{\frac{\left(\lambda_1 + \sqrt{(1+\lambda_1^2)}\right)}{(1+\lambda_1^2)}}. \tag{3.11}$$

As $\zeta \to \infty$ we note from (3.10), that the velocities far away from the disk turn out to be $u = -a\sin(\theta - \sigma)$, $v = r - a\cos(\theta - \sigma)$ different from the no-slip velocities. In order to see the effects of the material parameter of the second grade fluid on the flow, graphs of f and $-g$ are plotted for various values of λ_1 in Figure 1. These graphs clearly indicate that the flow exhibits a boundary layer like behavior near the disk. It is also seen that when λ_1 increases, the oscillatory behavior of the flow becomes more prominent and can be seen up to a considerable distance from the disk. It should be noted that when $\lambda_1 = 0$, our results are in agreement with those of (Turkyilmazoglu, 2009) without suction and injection. Moreover, (3.10) shows that the velocity distribution is in the form of an Ekman spiral representing the flow over a disk in a rotating system similar to (Siddiqui *et al.*, 2013).

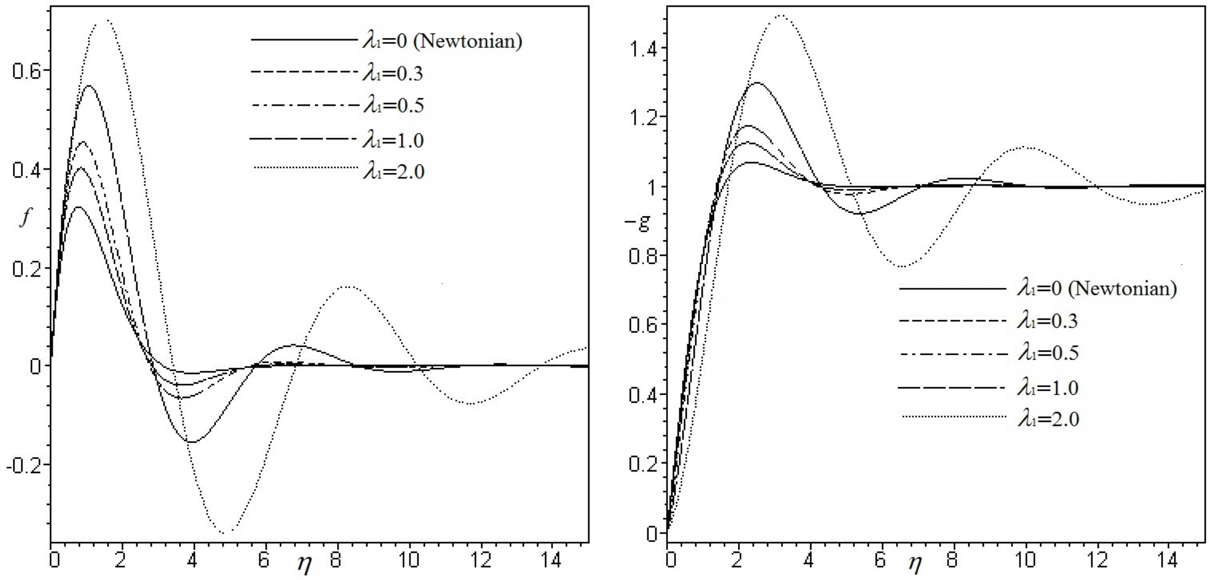

Figure 1. Variation of f and $-g$ with η for different values of λ_1.

The effects of viscosity in the fluid adjacent to the disk tends to develop some tangential shear stress which opposes the rotation of the disk. There is also a surface shear stress in the radial direction. The dimensionless expressions for the tangential and radial stresses are given as

$$\sigma_{\theta z} = a\left[W_\zeta - \lambda_1 F_\zeta\right]_{\zeta=0} = a\sqrt{\frac{\text{Re}}{2}}\left[(-d_2+\lambda_1 d_1)\cos(\theta-\sigma) - (d_1+\lambda_1 d_2)\sin(\theta-\sigma)\right], \quad (3.12)$$

$$\sigma_{rz} = a\left[F_\zeta + \lambda_1 W_\zeta\right]_{\zeta=0} = -a\sqrt{\frac{\text{Re}}{2}}\left[(-d_2+\lambda_1 d_1)\sin(\theta-\sigma) + (d_1+\lambda_1 d_2)\cos(\theta-\sigma)\right]. \quad (3.13)$$

In the particular case when $\theta = \sigma$, we obtain $\sigma_{\theta z} = a\sqrt{\frac{\text{Re}}{2}}(-d_2+\lambda_1 d_1)$ and $\sigma_{rz} = -a\sqrt{\frac{\text{Re}}{2}}(d_1 + \lambda_1 d_2)$. Moreover, when $\sigma = 0$, the results obtained point out the fact that maximum resistance due to viscosity of the fluid will take place at the locations $\theta = \tan^{-1}\left(\frac{-d_2+\lambda_1 d_1}{d_1+\lambda_1 d_2}\right)$ and $\theta = \tan^{-1}\left(\frac{-d_2+\lambda_1 d_1}{d_1+\lambda_1 d_2}\right)+\pi$ for the tangential stress and at the locations $\theta = \tan^{-1}\left(\frac{d_1+\lambda_1 d_2}{d_2-\lambda_1 d_1}\right)$ and $\theta = \tan^{-1}\left(\frac{d_1+\lambda_1 d_2}{d_2-\lambda_1 d_1}\right)+\pi$ for the radial stress. From the above equations one can easily find out the locations at which the minimum and maximum skin friction occurs against the flow.

The fluid dynamic thickness in radial and tangential directions are evaluated as

$$\delta_r = \int_0^\infty f(\zeta)d\zeta = \frac{d_2}{d_1^2+d_2^2}, \;\; \delta_\theta = \int_0^\infty (1+g(\zeta))d\zeta = \frac{d_1}{d_1^2+d_2^2}. \quad (3.14)$$

Hence, an increase in λ_1 results in an increase in the boundary layer thickness, this is clearly because as λ_1 increases d_1 and d_2 decrease and tend to zero.

The vorticity components $(\omega_r, \omega_\theta, \omega_z) = \nabla \times \mathbf{V}$ that exists within the fluid can be found out exactly with the help of equations (3.8)-(3.10), which are respectively

$$\omega_r = -\frac{\partial v}{\partial z} = -a\sqrt{\frac{\text{Re}}{2}}W_\zeta, \;\; \omega_\theta = \frac{\partial u}{\partial z} = a\sqrt{\frac{\text{Re}}{2}}F_\zeta, \omega_z = 2 \quad (3.15)$$

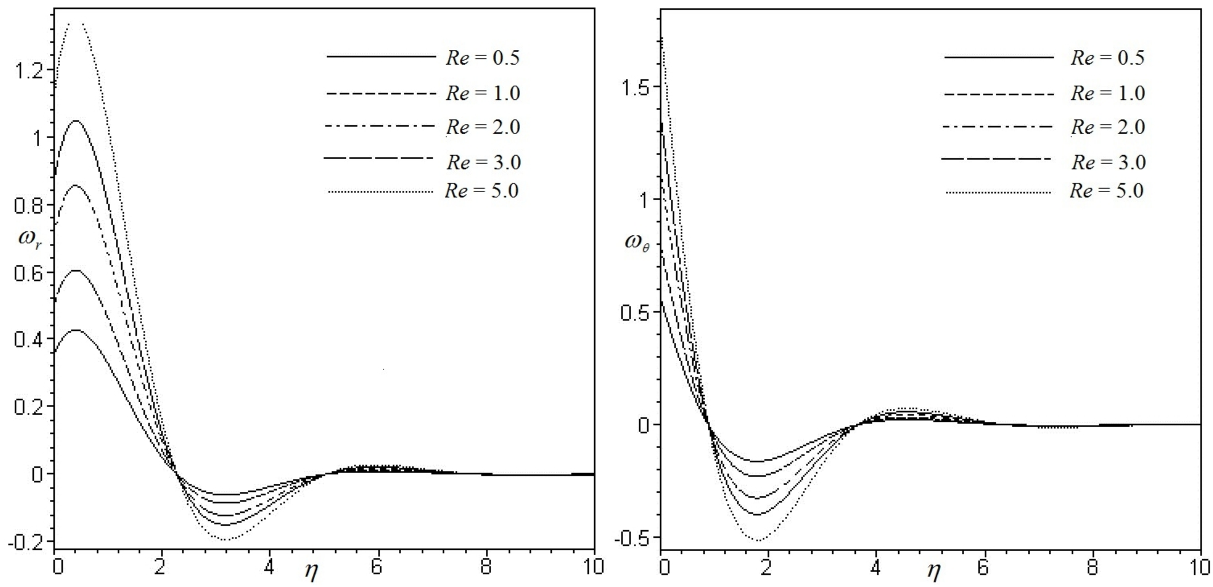

Figure 2. Variation of ω_r and ω_θ with η for different values of *Re* keeping $\lambda_1 = 0.5$.

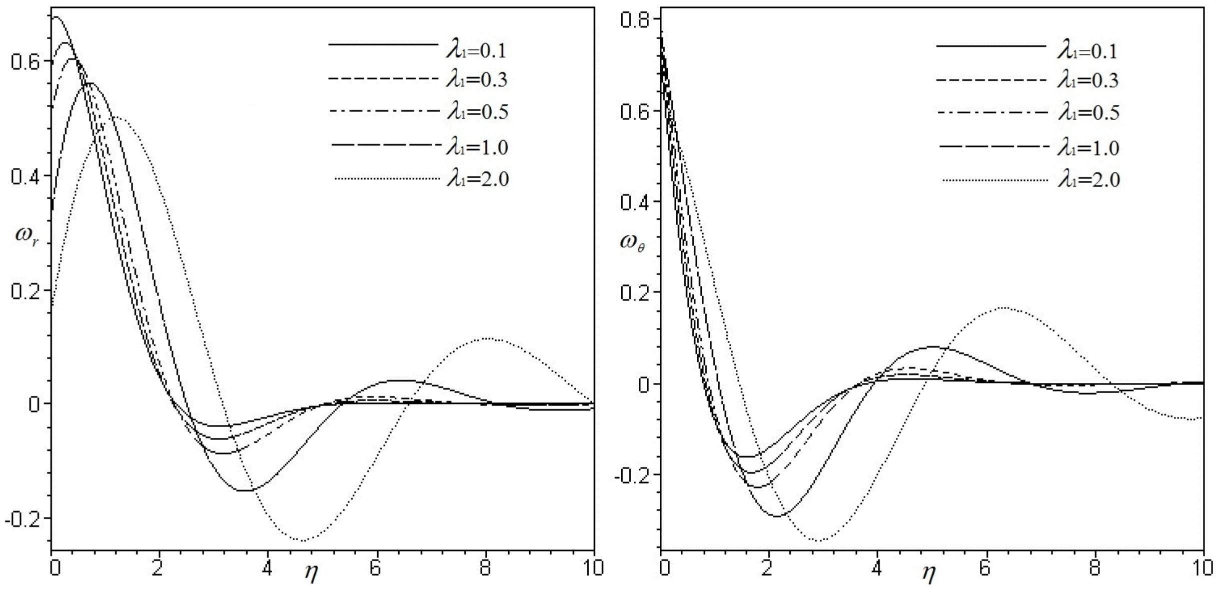

Figure 3. Variation of ω_r and ω_θ with η for different values of λ_1 when $Re = 1$.

In order to get the nature of the vorticity near the disk the expressions for ω_r and ω_θ are plotted for different values of Re and λ_1 when $\sigma = \theta$. It is observed from Figure 2 that both the components increase near the disk with increasing values of Re and show oscillatory behavior before approaching the asymptotic limits. Figure 3 is to demonstrate the effects of λ_1 on ω_r and ω_θ. It is noted that ω_r decreases whereas ω_θ increases near the disk with increasing values of λ_1. However, a large gradient is observed for ω_r near the wall. Actually, these vorticity components are responsible for driving the motion of fluid flow considered in the current study.

4. Concluding remarks

In this article, an exact solution for three-dimensional equations governing the incompressible second grade fluid flow over a single rotating disk has been obtained in such a way that the physical quantities are allowed to develop non-axisymmetrically within a no-normal flow assumption. We have worked through cylindrical coordinates which rotate with the disk, whose polar representation is (a, σ). The particular case $a = 0$ is associated with the rigid body rotation. The non-zero choice of a has enabled us to achieve the solutions bounded away from the disk. These results point out that a boundary layer structure develops near the surface of the disk whose far away behavior is distinct from the near wall solutions. It is observed that increases in λ_1 cause an increase in the boundary layer thickness. There is no effect of the material parameter λ_2 on the velocity field since both the disk and the fluid rotate with the same speed. We also note that this technique can also be applied to other non-Newtonian fluid flow problems successfully.

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A Simplified Architecture of Type-2 TSK Fuzzy Logic Controller for Fuzzy Model of Double Inverted Pendulums

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Abstract

This paper proposes a novel inference mechanism for an interval type-2 Takagi-Sugeno-Kang fuzzy logic control system (IT2 TSK FLCS). This paper focuses on control applications for case both plant and controller use A2-C0 TSK models. The defuzzified output of the T2FLS is then obtained by averaging the defuzzified outputs of the resultant four embedded T1FLSs in order to reduce the computational burden of T2 TSK FS. A simplified T2 TSK FS based on a hybrid structure of four type-1 fuzzy systems (T1 TSK FS). A simulation example is presented to show the effectiveness of this method.

Keywords: Fuzzy control systems, simplified type-2 fuzzy logic system, double inverted pendulums.

1. Introduction

Fuzzy systems of Takagi-Sugeno (T-S) models (Takagi & Sugeno, 1985) have become an effective method to represent nonlinear system by fuzzy sets and fuzzy reasoning. In (Echanobe *et al.*, 2005) presented some important aspects concerning the analysis and implementation of a piecewise linear (PWL) fuzzy model with universal approximation capability. Reference (Sadighi & jong Kim, 2010) presented a combination of a Sugeno fuzzy model and neural networks. In (Guechi *et al.*, 2010) presented a new technique for tracking-error model-based Parallel Distributed Compensation (PDC) control and stabilizing controller by solved by LMI conditions for the tracking-error model.

A new stability analysis method for nonlinear processes with T-S fuzzy logic controllers (FLCs) without process linearization and without using the quadratic Lyapunov functions in the derivation and proof of the stability conditions was designed in (Tomescu *et al.*, 2007). In (Precup *et al.*, 2009) studied a new framework for the design of generic two-degree-of-freedom (2-DOF), linear and fuzzy, controllers dedicated to a class of integral processes specific to servo systems.

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Fuzzy systems first introduced by Zadeh. The membership degree of Type-1 fuzzy set is crisp value but it isT1FS in the Type-2 fuzzy sets (Mendel, 2001; Wu & Mendel, 2001; Mendel, 2007). Researchers have shown that T1FLS have difficulty in modeling and minimizing the effect of uncertainties (Zadeh, 1975).

In (Biglarbegian *et al.*, 2010), the WuMendel uncertainty bounds (WM UBs) to design stable interval type-2 TSK fuzzy logic control systems (IT2 TSK FLCS). Proposed Inference Methods for IT2 TSK FLCSs in (Mohammad, 2010). In (Ren *et al.*, 2011) showed IT2 TSK FLSs analyzes the sensibility of the outputs of a type-2 TSK fuzzy system, and discusses the approximation capacities of type-2 TSK FLS and its type-1 counterpart. In (Wu & Tan, 2004) the study is conducted by utilizing a type-2 FLC, evolved by a genetic algorithm(GA), to control a liquid-level process.The proposed algorithm of interval type-2 TSK FLS has been used in fuzzy modeling and uncertainty prediction in high precision manufacturing (Ren *et al.*, 2009).

In this paper, Proposed the new inference mechanisms. we reduced the computational burden of T2 TSK FS.A simplified T2 TSK FS have a hybrid structure of four type-1 fuzzy systems (T1 TSK FS).The final output of the T2 TSK FLS is then obtained by averaging the defuzzified outputs of each T1 TSK FLC. The rest of the chapter is organized as follows: Section II, we present an overview of dynamic Takagi Sugeno systems. In this section, deals with analytical design of Type-2 TSK fuzzy control and introduces the proposed simplified implementation of T2 TSK FLS using four embedded T1FSs. Some simulations are executed to verify the validity of the proposed approach in Section III. Section IV concludes the paper.

2. Takagi-sugeno fuzzy model

A dynamic T-S fuzzy model is described by a set of fuzzy IF THEN rules with fuzzy sets in the antecedents and dynamic linear time-invariant systems in the consequents. A generic T-S plant rule can be written as follows (Dorato *et al.*, 1995; Khaber *et al.*, 2006):

$$i^{th} Plant\ Rule:\ IF x_1(t)\ is\ M_{i1} \ldots, x_n(t)\ is\ M_{in}\ THEN\ \dot{x} = A_i x + B_i u,$$

where $x_{n\times1}$ is the state vector, r is the number of rules, M_{ij} are input fuzzy sets, $u_{m\times1}$ is the input and $A_{n\times n}$, $B_{n\times m}$ are state matrix and input matrix respectively. Using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as:

$$\dot{x} = \frac{\sum_{i=1}^{r} \omega_i(x)(A_i x + B_i u)}{\sum_{i=1}^{r} \omega_i(x)}, \tag{2.1}$$

with the term ω_i is defined by:

$$\omega_i(x) = \prod_{j=1}^{n} \mu_{ij(x_j)}, \tag{2.2}$$

where μ_{ij} is the membership function of the jth fuzzy set in the ith rule. Defining the coefficients α_i as:

$$\alpha_i = \frac{\omega_i}{\sum_{i=1}^{r} \omega_i} \tag{2.3}$$

we can write (2.1) as:

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) i = 1, \ldots, r, \tag{2.4}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r} \alpha_i (x) = 1$.

Using the same method for generating T-S fuzzy rules for the controller, we have:

$$i^{th} controllerRule :$$

$$IF x_1 (t) \ is\ M_1^i \ and \ \ldots x_n(t) \ is\ M_n^i \ then\ u(t) = -K_i x(t), \ i = 1, \ldots, r,$$

The over all controllers would be

$$u = -\sum_{i=1}^{r} \alpha_i (x) K_i x. \tag{2.5}$$

Replacing (2.5) in (2.4), we obtain the following equation for the closed loop system:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i (x) \alpha_j (x) (A_i x + B_i u) x. \tag{2.6}$$

3. IT2 TSK FLSs

This chapter first presents the design of IT2 TSK FLSs for modeling and control applications. Second, WM UBs are introduced and third, a new inference engines for IT2 TSK FLSs are introduced. The general structure of an interval A2-C0 TSK model for a system is as follows (Mohammad, 2010):

$$If\ x_1\ is\ \tilde{F}_1^i\ and\ x_2\ is\ \tilde{F}_2^i\ and\ \ldots x_n\ is\ \tilde{F}_n^i,\ Then\ y_i = a_0^i x_1 + a_0^i x_2 + \ldots + a_n^i x_n \tag{3.1}$$

where $\tilde{F}_j^i$, $i = 1, \ldots, M$ represents the IT2 FS of input state j in rule i, $x_1, \ldots, x_n$ are states, $a_0^i, \ldots, a_n^i$ are the coefficients of the output function for rule i (and hence are crisp numbers, i.e., type-0 FSs), y_i is the output of the i^{th} rule, and M is the number of rules. The above rules allow us to model the uncertainties encountered in the antecedents.

In an IT2 TSK A2-C0 model, $\bar{f}^i(x)$ and $\underline{f}^i(x)$, lower and upper firing strengths of the i^{th} rule, respectively, are given by

$$\bar{f}^i(x^*) = \bar{\mu}_{\tilde{F}_1^i}(x_1) \star \ldots \star \bar{\mu}_{\tilde{F}_n^i}(x_n), \tag{3.2}$$

$$\underline{f}^i(x^*) = _{\tilde{F}_1^i}(x_1) \star \ldots \star _{\tilde{F}_n^i}(x_n), \tag{3.3}$$

where $\underline{\mu}_{\tilde{F}_j^i}$ and $\bar{\mu}_{\tilde{F}_j^i}$ represent the j^{th} ($j = 1 \ldots M$) lower and upper MFs of rule i, and "$\star$" is a t-norm operator. State vector is defined as

$$x = [x_1, x_2, \ldots, x_n]^T \tag{3.4}$$

The final output of the IT2 TSK A2-C0 is given as:

$$Y_{TSK/A2-C0} = [y_l, y_r] = \int_{f^1 \in [\underline{f}^1, \bar{f}^1]} \ldots \int_{f^M \in [\underline{f}^M, \bar{f}^M]} 1 \Big/ \frac{\sum_{k=1}^{M} f^i(x) y_i}{\sum_{k=1}^{M} f^i(x)}, \tag{3.5}$$

where y_i is given by the consequent part of (3.1). $Y_{TSK/A2-C0}$ is an interval T1 set and only depends on its left and right end-points y_l, y_r, which can be computed using the iterative KM algorithms. Therefore, the final output is given as The final output of the IT2 TSK A2-C0 is given as:

$$Y_{output}(x) = \frac{y_r(x) + y_l(x)}{2}. \tag{3.6}$$

KM Algorithm (Mohammad, 2010):

The KM algorithm presents iterative procedures to compute y_l, y_r in as follows:

Set $y^i = y_l^i$ (or y_r^i) for $i = 1, \ldots, N$;

Arrange y^i in ascending order;

Set $f^i = \frac{\underline{f}^i + \bar{f}^i}{2}$ for $i = 1, \ldots, N$;

$y' = \frac{\sum_{i=1}^{N} y^i f^i}{\sum_{i=1}^{N} f^i}$;

Do

$y'' = y'$;

Find $k \in [1, N-1]$ such that $y^k \leq y' \leq y^{k+1}$;

Set $f^i = \bar{f}^i$ (or ^{i}) for $i \leq k$;

Set $f^i = \underline{f}^i$ (or $\bar{f}^i$) for $i \geq k+1$;

$y' = \frac{\sum_{i=1}^{N} y^i f^i}{\sum_{i=1}^{N} f^i}$;

While $y' \neq y''$

y_l (or y_r) $= y'$.

It has been proven that this iterative procedure can converge in at most N iterations (Mohammad, 2010).

4. A simplified implementation of T2 TSK FS

As shown in the Figure 1, each T2MF can represents by two T1MFs, upper MF and lower MF. Therefore, each one of two neighbor T2MFs intersects each other in four points and object to get four MFs, upper MF, lower MF, left MF and right MF showing in Figure 2 (Hameed *et al.*, 2011). Thus four T1 TSK Fuzzy controller supplanted are used discretely. The MFs in each controller supplanted by upper MF, lower MF, left MF and right MF, and will create upper fuzzy controller (UFC), lower fuzzy controller (LFC), left fuzzy controller (LEFTFC) and right fuzzy controller (RFC) respectively.

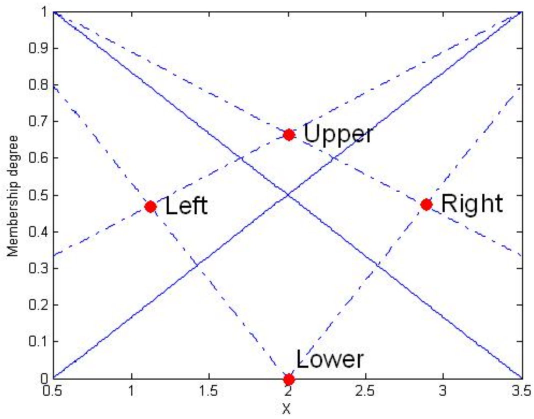

Figure 1. Illustration of decomposing T2MFs into 4 T1MFs.

The defuzzified output of the T2FLS is then obtained by averaging the defuzzified outputs of the resultant four embedded T1FLSs, as shown in Figure 3.

$$Y(x) = \frac{1}{4}y_{upper}(x) + \frac{1}{4}y_{lower}(x) + \frac{1}{4}y_{left}(x) + \frac{1}{4}y_{right}(x). \tag{4.1}$$

5. Simulation

A two-inverted pendulum system is shown in Figure 4. It consists of two cart-pole inverted pendulums. The inverted pendulums are linked by a spring in the middle. The carts will move to and from during the operation. The control objective is to balance the inverted pendulums vertically despite the movings of the spring and carts by applying forces to the tips of the pendulums. Referring to Figure 4, M and m are the masses of the carts and the pendulums, respectively, m=10 kg and M =100 kg. L =1 m is the length of the pendulums. The spring has a stiffness constant $k = 1N/m$. $y_1(t) = sin(2t)$ and $y_2(t) = L + sin(3t)$ are the trajectories of the moving carts. $u_1(t)$ and $u_2(t)$ are the forces applied to the pendulums. $\theta_1(t)$ and $\theta_2(t)$ are the angular displacements of the pendulums measured from the vertical. The dynamic equation of the two-inverted pendulum system can be written as follows (Lam *et al.*, 2000):

$$\dot{X} = A(x(t))\,x(t) + Bu(t) \tag{5.1}$$

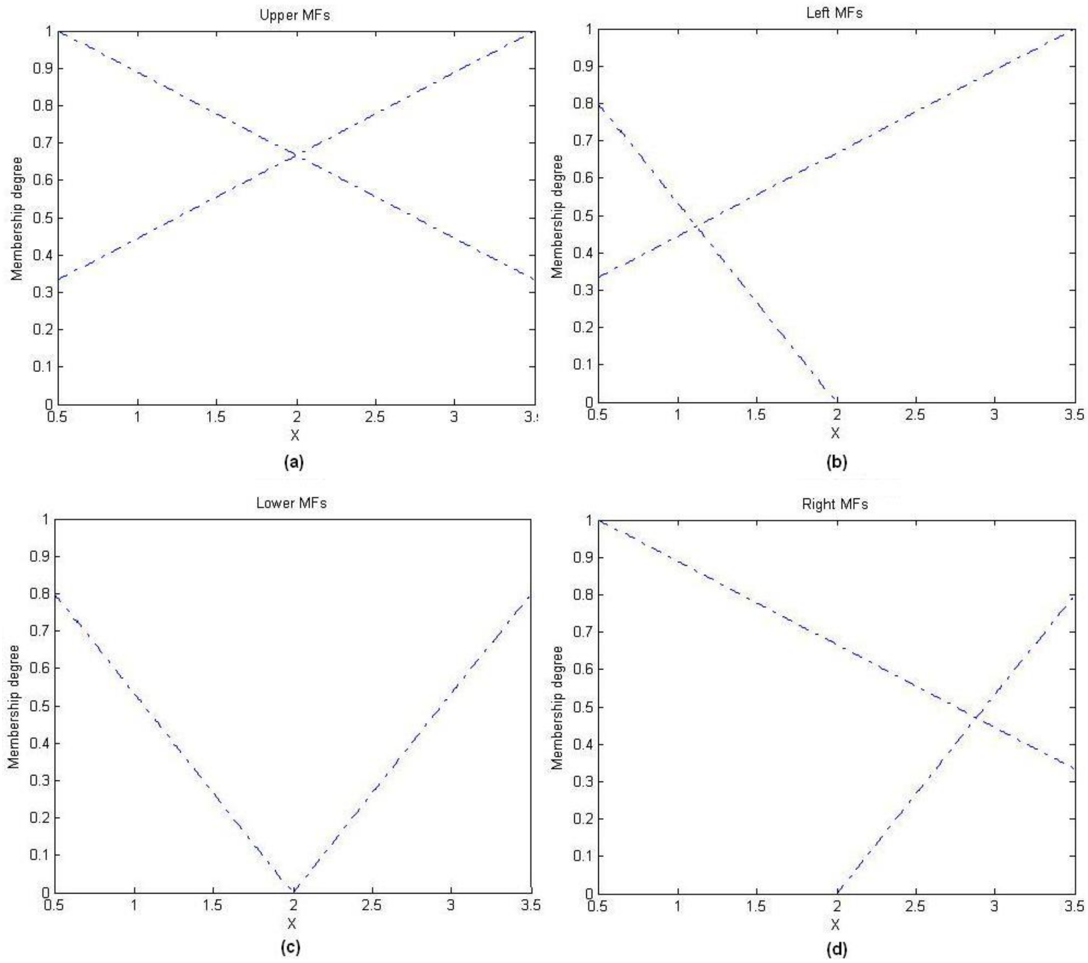

Figure 2: (a) Membership functions of upper intersection points. (b) Membership functions of left intersection points. (c) Membership functions of lower intersection points, and (d) Membership functions of right intersection points.

Where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \theta_1(t) \\ \dot{\theta}_1(t) \\ \theta_2(t) \\ \dot{\theta}_2(t) \end{bmatrix}, x_1 \in [x_{1min} x_{1max}] = \left[-\tfrac{\pi}{2} \tfrac{\pi}{2}\right], x_3 \in [x_{3min} x_{3max}] = \left[-\tfrac{\pi}{2} \tfrac{\pi}{2}\right],$$

$$A(x(t)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_1(x_1(t)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_2(x_3(t)) & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \\ 0 & 0 \\ 0 & \lambda \end{bmatrix} \text{and } u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

$$f_1(t) = \tfrac{2}{L} - \tfrac{m}{M}\sin(x_1(t))\, x_1(t), \; f_2(t) = \tfrac{2}{L} - \tfrac{m}{M}\sin(x_3(t))\, x_3(t) \text{ and } \lambda = \tfrac{2}{mL^2}.$$

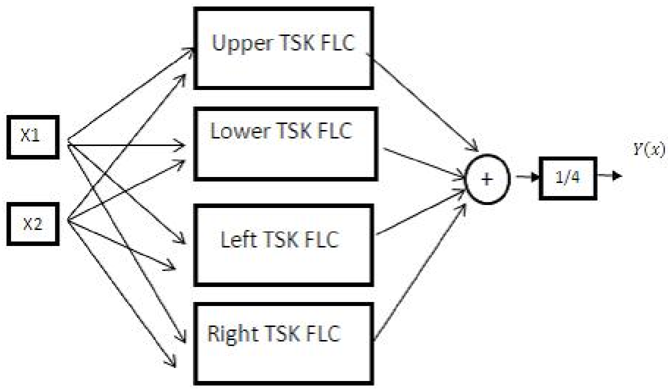

Figure 3. Simplified type-2 TSK fuzzy Logic Controller.

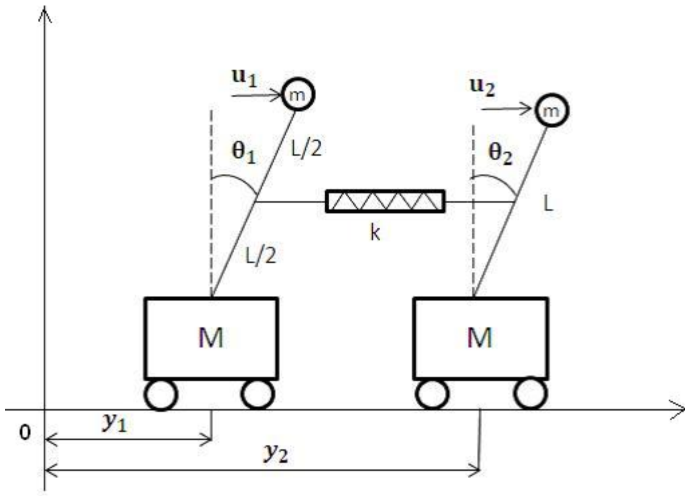

Figure 4. Two-inverted pendulum system.

A four-rule TS-fuzzy plant model is used to represent the two inverted pendulum system. The *i-th* rule of the TS-fuzzy plant model is given by

$$Rule\ i = IF f_1\left(x_1\left(t\right)\right)\ is\ M_{i1}\ and\ f_2\left(x_3\left(t\right)\right)\ is\ M_{i2}\ then\ \dot{X} = A_i x\left(t\right) + Bu\left(t\right),\ i = 1,2,3,4 \quad (5.2)$$

where M_i is a fuzzy term of rule i, $i = 1,2,3,4$. Then, the system dynamics is described by

$$\dot{X} = \sum_{i=1}^{4} w_i \left[A_i x\left(t\right) + Bu\left(t\right)\right], \quad (5.3)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1min} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2min} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1min} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2max} & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1max} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2min} & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ f_{1max} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f_{2max} & 0 \end{bmatrix},$$

$$w_i = \frac{\mu_{M_1^i}\left(f_1\left(x_1\left(t\right)\right)\right) \times \mu_{M_2^i}\left(f_2\left(x_3\left(t\right)\right)\right)}{\sum_{i=1}^{4}\left(\mu_{M_1^i}\left(f_1\left(x_1\left(t\right)\right)\right) \times \mu_{M_2^i}\left(f_2\left(x_3\left(t\right)\right)\right)\right)},$$

$\mu_{M_1^\beta}\left(f_1\left(x_1\left(t\right)\right)\right) = \frac{-f_1(x_1(t))+f_{1max}}{f_{1max}-f_{1min}}$ for $\beta = 1, 2$ and $\mu_{M_1^\delta}\left(f_1\left(x_1\left(t\right)\right)\right) = 1-\mu_{M_1^1}\left(f_1\left(x_1\left(t\right)\right)\right)$ for $\delta = 3, 4$
$\mu_{M_1^\varepsilon}\left(f_2\left(x_3\left(t\right)\right)\right) = \frac{-f_2(x_3(t))+f_{2max}}{f_{2max}-f_{2min}}$ for $\varepsilon = 1, 3$ and $\mu_{M_1^\phi}\left(f_2\left(x_3\left(t\right)\right)\right) = 1-\mu_{M_2^1}\left(f_2\left(x_3\left(t\right)\right)\right)$ for $\delta = 2, 4$
$f_{1max} = \frac{2}{L} + x_{1max}$ and $f_{1min} = \frac{2}{L} + x_{1min}$, $f_{2max} = f_{1max}$ and $f_{2min} = f_{1min}$.

Figure 5 shows a controller in which the inputs are the states *x(k)* and the output is *u(k)*. For this system, the general *i-th* rule has the following form:

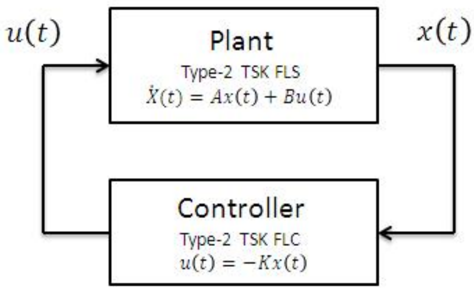

Figure 5. Closed-loop T2 TSK fuzzy control system.

To compare the performance of the IT2 TS FLC with the T1 controller, the model of the plant is kept as a T1 TS and only the controller is replaced with an IT2 TS model. To make a fair comparison, the parameters of the plants and controllers are kept unchanged for four control systems, and only the MFs for the IT2 controller are designed. MFs for this Example showed Figures 1 and 2. In this paper, the simplified Type 2 TSK Fuzzy controller of scaling factors are tuned by trial-and-error approach. A four-rule fuzzy controller is designed as following equation (Lam *et al.*, 2000).

$$Rule\ i = If\ x_1\left(t\right)\ is\ \tilde{M}_1^i\ and\ x_3\left(t\right)\ is\ \tilde{M}_2^i\ then\ u\left(t\right) = G_j x\left(t\right)\ j = 1, 2, 3, 4.$$

The feedback gains for each fuzzy controller are then chosen as:

$$G_1 = \begin{bmatrix} -116.6410 & -119.7827 & -95.0589 & -39.6463 \\ -79.0293 & -40.024 & -260.5216 & -180.2173 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -116.6410 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.024 & -260.5216 & -180.2173 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} -179.4729 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.024 & -260.5216 & -180.2173 \end{bmatrix}$$

and

$$G_4 = \begin{bmatrix} -179.4729 & -119.7827 & -95.0589 & -39.6463 \\ -97.0293 & -40.024 & -323.3534 & -180.2173 \end{bmatrix}$$

The zero-input responses of the system under the initial conditions:

$$x(0)(rad) = [\frac{88\pi}{180}0 - \frac{88\pi}{180}0].$$

The responses for T1 TSK Fuzzy and simplified T2 TSK Fuzzy controllers are shown in Figures 6-7 comparison between the two types of TSK FLCs have done. The reciprocal of the Root squared error (RMSE) of the response showed in Table I.

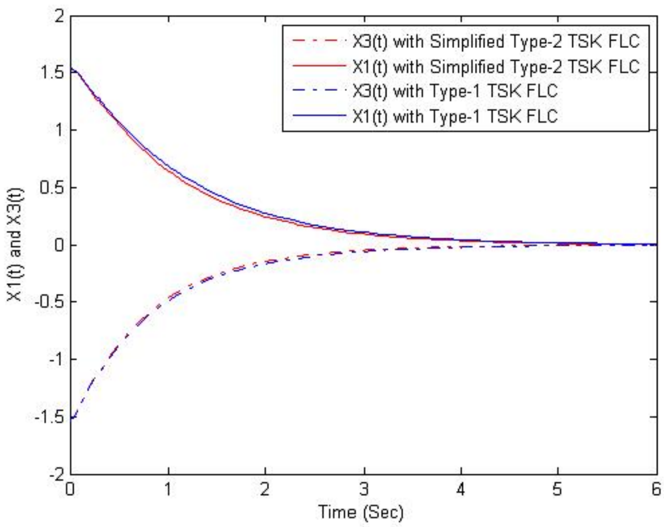

Figure 6: Responses of (solid line) and (dotted line) of the two-inverted pendulum system under T1 TSK FLC and Simplified T2 TSK FLC with M = 100 kg.

Table 1
RMSE of the responses

RMSE	X1(t)	X2(t)	X3(t)	X4(t)
T1 TSK FLC	6.8299	3.8769	6.3953	5.1933
T2 TSK FLC	6.7483	4.0474	6.3542	5.3271

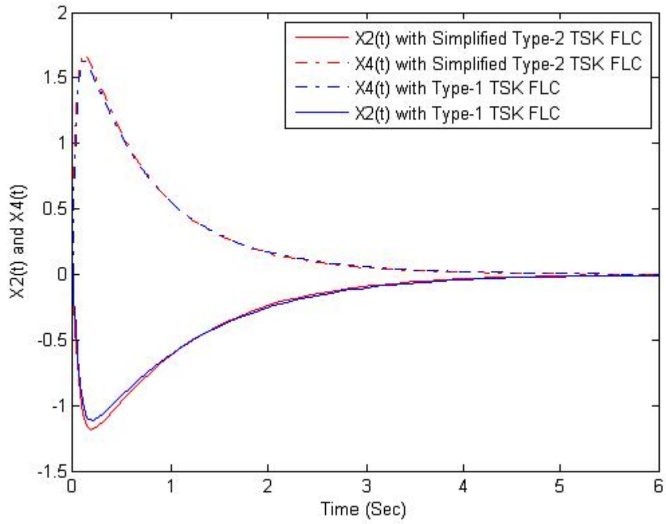

Figure 7: Responses of (solid line) and (dotted line) of the two-inverted pendulum system under T1 TSK FLC and Simplified T2 TSK FLC with M = 100 kg.

6. Conclusion

The nonlinear, T1, and IT2 controllers are capable of stabilizing the system. With attention to table. 1 in before section, value RMSE reduced in the X1(t) and X3(t) in the T2 TSK FLC with respect to T1 TSK FLC. Therefore output system is robustness. In this case study, it is shown that the proposed IT2 TSK FLC is capable of stabilizing the coupled two inverted pendulum while achieving a better performance compared to its T1 TSK FLC

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Second Order $(\Phi, \Psi, \rho, \eta, \theta)-$Invexity Frameworks and $\epsilon-$Efficiency Conditions for Multiobjective Fractional Programming

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Abstract

A generalized framework for a class of second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities is developed, and then some parametric sufficient efficiency conditions for multiobjective fractional programming problems are established. The obtained results generalize and unify a wider range of investigations in the literature on applications to other results on multiobjective fractional programming.

Keywords: Generalized invexity, multiobjective fractional programming, $\epsilon-$ efficient solutions, parametric sufficient $\epsilon-$ efficiency conditions.
2010 MSC: 90C30, 90C32, 90C34.

1. Introduction

Zalmai and Zhang (see (Zalmai & Zhang, 2007*a*)) have established a set of necessary efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for semi-infinite discrete minimax fractional programming problems. Recently, Verma (see (Verma, 2013)) developed a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\epsilon-$efficient solutions. On the other hand, the work of Kim, Kim and Lee (see (Kim *et al.*, 2011)) extends the results of Kim and Lee (see (Kim & Lee, 2013)) on $\epsilon-$optimality theorems for a convex multiobjective optimization problem to a multiobjective fractional optimization problem, while this has been followed by other research advances. They also applied the generalized Abadie constraint qualification to the context of the optimal solvability of a semi-infinite discrete minimax fractional programming problems.

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Based on the recent advances in the study of $\epsilon-$optimality and weak $\epsilon-$optimality conditions for multiobjective fractional programming problems, we first generalize the $(\rho, \eta, \theta)-$invexities to second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities, and we introduce some parametric sufficient efficiency conditions for multiobjective fractional programming to achieve $\epsilon-$efficient solutions to multiobjective fractional programming problems. The results established in this communication, not only generalize the results on weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems, but also generalize the second order invexity results in general setting. The notion of the second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities encompass most of the existing notions of the generalized invexities (see (Ben-Israel & Mond, 1986), (Caiping & Xinmin, 2009), (Hanson, 1981) (Jeyakumar, 1985), (Liu, 1999), (Mangasarian, 1975), (Mishra, 1997), (Mishra, 2000), (Mishra & Rueda, 2000), (Mishra & Rueda, 2006), (Mond & Weir, 1981-1983), (Mond & Zhang, 1995), (Mond & Zhang, 1998), (Patel, 1997), (Srivastava & Bhatia, 2006), (Srivastava & Govil, 2000), (Suneja *et al.*, 2003), (Vartak & Gupta, 1987), (Yang, 1995), (Yang, 2009),(Yang & Hou, 2001), (Yang *et al.*, 2004*a*), (Yang *et al.*, 2003), (Yang *et al.*, 2005), (Yang *et al.*, 2008), (Yang *et al.*, 2004*b*), (Yokoyama, 1996), (Zalmai, 2007), (Zalmai, 2007), (Zhang & Mond, 1996), (Zhang & Mond, 1997)). There exists a vast literature on higher order generalized invexity and duality models in mathematical programming. For more details, we refer the reader (see (Verma, 2012), (Verma, 2013), (Zalmai, 2012), (Zalmai & Zhang, 2007*b*), (Zeidler, 1985)).

We consider under the general framework of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities of functions, the following multiobjective fractional programming problem:

(P)

$$Minimize\Big(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$,

where X is an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:

(Pλ)

$$Minimize\Big(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x)\Big)$$

subject to $x \in Q$ with

$$\lambda = \Big(\lambda_1, \lambda_2, \cdots, \lambda_p\Big) = \Big(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\Big),$$

where x^* is an efficient solution to (P).

General Mathematical programming problems serve a significant useful purpose, especially in terms of applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, and others.

2. Generalized second order invexities

In this section, we develop some concepts and notations for the problem on hand. Let X be an open convex subset of $\mathfrak{R}^n$ (n-dimensional Euclidean space). Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $\eta : X \times X \to \mathfrak{R}^n$ be a function. Suppose that f is a real-valued twice continuously differentiable function defined on X, and that $\triangledown f(y)$ and $\nabla^2 f(y)$ denote, respectively, the gradient and hessian of f at y.

Definition 2.1. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$invex at x^* of second order if there exist a superlinear function $\Phi : \mathfrak{R}^n \to \mathfrak{R}$, a sublinear function $\Psi : \mathfrak{R}^n \to \mathfrak{R}$ and a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Phi\Big(f(x) - f(x^*)\Big) \geq \langle \triangledown f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2.$$

Definition 2.2. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exist a superlinear function $\Phi : \mathfrak{R}^n \to \mathfrak{R}$, a sublinear function $\Psi : \mathfrak{R}^n \to \mathfrak{R}$ and a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2 \geq 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.3. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be strictly $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2 \geq 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) > 0.$$

Definition 2.4. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be prestrictly $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\langle \triangledown f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2 > 0 \Rightarrow \Phi\Big(f(x) - f(x^*)\Big) \geq 0.$$

Definition 2.5. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Psi(f(x) - f(x^*)) \leq 0 \Rightarrow \langle \triangledown f(x^*), \eta(x, x^*) \rangle + \frac{1}{2} \langle z, \nabla^2 f(x^*) z \rangle + \rho(x, x^*) \|\theta(x, x^*)\|^2 \leq 0.$$

Definition 2.6. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be strictly $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Psi(f(x) - f(x^*)) \leq 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^r < 0.$$

Definition 2.7. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be prestrictly $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$,

$$\Psi(f(x) - f(x^*)) < 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^r \leq 0.$$

We observe that the second order generalized $(\Phi, \Psi, \rho, \eta, \theta)-$invexities can be specialized to second order $(\rho, \eta, \theta)-$invexities.

Definition 2.8. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exist a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \Rightarrow f(x) - f(x^*) \geq 0.$$

Definition 2.9. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be strictly $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \Rightarrow f(x) - f(x^*) > 0.$$

Definition 2.10. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be prestrictly $(\rho, \eta, \theta)-$pseudo-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$\langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 > 0 \Rightarrow f(x) - f(x^*) \geq 0.$$

Definition 2.11. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be $(\rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$f(x) - f(x^*) \leq 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

Definition 2.12. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be strictly $(\rho, \eta, \theta)-$quasi-invex at x^* of second if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$f(x) - f(x^*) \leq 0 \Rightarrow \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0.$$

Definition 2.13. A twice differentiable function $f : X \to \mathfrak{R}$ is said to be prestrictly $(\rho, \eta, \theta)-$quasi-invex at x^* of second order if there exists a function $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that for each $x \in X$, $\rho : X \times X \to \mathfrak{R}$, $\theta : X \times X \to \mathfrak{R}^n$ and $z \in \mathfrak{R}^n$

$$f(x) - f(x^*) < 0 \Rightarrow \langle \triangledown f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

3. The $\epsilon-$Solvability Conditions

Now we consider the $\epsilon-$solvability conditions for (P) and (Pλ) problems motivated by the publications (see (Kim *et al.*, 2011)), where they have investigated the $\epsilon-$efficiency as well as the weak $\epsilon-$efficiency conditions for multiobjective fractional programming problems under constraint qualifications. Based on these developments in the literature, first we introduce a second order generalization of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities to the existing notion of $(\rho, \eta, \theta)-$invexities, and then using the parametric approach, we develop some parametric sufficient $\epsilon-$efficiency conditions for multiobjective fractional programming problem (P) under this framework. We need to recall some auxiliary results crucial to the problem on hand.

Definition 3.1. A point $x^* \in Q$ is an $\epsilon-$efficient solution to (P) if there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i \,\forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j, \; some\; j \in \{1, \cdots, p\},$$

where ϵ_i=$(\epsilon_1, \cdots, \epsilon_p)$ is with $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, Definition 3.1 reduces to the case that $x^* \in Q$ is an efficient solution to (P).

Definition 3.2. A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \,\forall\, i = 1, \cdots, p.$$

Next to this context, we have the following auxiliary problem:
(P$\bar{\lambda}$)

$$minimize_{x \in Q}(f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,

where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, $\epsilon_i^* = \epsilon_i g_i(x^*)$ and $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$.

Next, we introduce the ϵ^*-solvability conditions for (P$\bar{\lambda}$) problem.

Definition 3.3. A point $x^* \in Q$ is an ϵ^*-efficient solution to (P$\bar{\lambda}$) if there does not exist an $x \in Q$ such that

$$f_i(x) - \bar{\lambda}_i g_i(x) \leq f_i(x^*) - \bar{\lambda}_i g_i(x^*) - \epsilon_i^* \,\forall\, i = 1, \cdots, p,$$

$$f_j(x) - \bar{\lambda}_j g_j(x) < f_j(x^*) - \bar{\lambda}_j g_j(x^*) - \epsilon_j^*, \; some\; j \in \{1, \cdots, p\},$$

where $\bar{\lambda}_i = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$, and $\epsilon_i^* = \epsilon_i g_i(x^*)$ with $\epsilon = (\epsilon_1, \cdots, \epsilon_p)$, $\epsilon_i \geq 0$ for $i = 1, \cdots, p$.

For $\epsilon = 0$, it reduces to the case that x^* is an efficient solution to (P) if there exists no $x \in Q$ such that

$$\Big(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\Big) \le \Big(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}\Big).$$

Lemma 3.1. *(Kim et al., 2011) Let $x^* \in Q$. Suppose that $f_i(x^*) \ge \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an ϵ−efficient solution to (P).

(ii) x^* is an ϵ^*−efficient solution to $(P\bar{\lambda})$, where

$$\bar{\lambda} = \Big(\frac{f_1(x^*)}{g_1(x^*)} - \epsilon_1, \cdots, \frac{f_p(x^*)}{g_p(x^*)} - \epsilon_p\Big)$$

and $\epsilon^* = (\epsilon_1 g_1(x^*), \cdots, \epsilon_p g_p(x^*))$.

Lemma 3.2. *(Kim et al., 2011) Let $x^* \in Q$. Suppose that $f_i(x^*) \ge \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an ϵ−efficient solution to (P).

(ii) There exists $c = (c_1, \cdots, c_p) \in \mathfrak{R}^p_+ \setminus \{0\}$ such that

$$\Sigma_{i=1}^p c_i\Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)\Big] \ge 0 = \Sigma_{i=1}^p c_i\Big[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x^*)\Big] - \Sigma_{i=1}^p c_i\epsilon_i g_i(x^*),$$

for any $x \in Q$.

Lemma 3.3. *Let $x^* \in Q$. Suppose that $f_i(x^*) \ge \epsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) x^* is an ϵ^*−efficient solution to $(P\bar{\lambda})$.

(ii) There exists $c = (c_1, \cdots, c_p) \in \mathfrak{R}^p_+ \setminus \{0\}$ such that

$$\Sigma_{i=1}^p c_i\Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)\Big] \ge 0 = \Sigma_{i=1}^p c_i\Big[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x^*)\Big] - \Sigma_{i=1}^p c_i\epsilon_i g_i(x^*),$$

for any $x \in Q$.

4. Auxiliary results on Parametric sufficiency conditions

This section deals with some auxiliary parametric sufficient ϵ− efficiency conditions for problem (P) under the generalized frameworks for generalized invexity. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i\Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big)g_i(x)\Big],\ i \in \{1, \cdots, p\},$$

and

$$B_j(.,v) = v_j H_j(x), \;\; j = 1, \cdots, m.$$

Verma (see (Verma, 2013)) recently established the following result based on parametric sufficient weak $\epsilon-$ efficiency conditions for problem (P) under the generalized (ρ, η, θ) frameworks for generalized invexities. These results are significant to developing our main results on hand.

Theorem 4.1. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)] + \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*) \rangle \geq 0 \,, \tag{4.1}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{4.2}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$) :

(i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are $(\rho, \eta, \theta)-$quasi-invex at x^*.

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly $(\rho, \eta, \theta)-$pseudo-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly $(\rho, \eta, \theta)-$quasi-invex at x^*.

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are prestrictly $(\rho, \eta, \theta)-$quasi-invex at x^*, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are strictly $(\rho, \eta, \theta)-$pseudo-invex at x^*.

(iv) For each $i \in \{1, \cdots, p\}$, f_i is $(\rho_1, \eta, \theta)-$invex and $-g_i$ is $(\rho_2, \eta, \theta)-$invex at x^*. $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is $(\rho_3, \eta, \theta)-$quasi-invex at x^*, and $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \Sigma_{i=1}^p u_i^* (\rho_1 + \phi(x^*)\rho_2)$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$.

Then x^* is a weakly $\epsilon-$efficient solution to (P).

Next, we recall the following result (see (Verma & Zalmai, 2012)) that is crucial to developing the results for the next section based on second Order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities.

Theorem 4.2. *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \rightarrow G_j(z, \overline{t})$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \rightarrow H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\begin{aligned} &\{\Big(\nabla G_j(x^*, t), \langle y, \nabla^2 G_j(x^*, t) y \rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \\ + \;\; & span\{\Big(\nabla H_k(x^*, s), \langle y, \nabla^2 H_k(x^*, s) y \rangle\Big) : s \in S_k, k \in \underline{r}\}, \\ & where \;\; \hat{T}_j(x^*) \equiv \{t \in T_j \;:\; G_j(x^*, t) = 0\}, \end{aligned}$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^{p} u_i = 1\}$ *and integers* ν_0^* *and* ν^**, with* $0 \leq \nu_0^* \leq \nu^* \leq n+1$*, such that there exist* ν_0^* *indices* j_m*, with* $1 \leq j_m \leq q$*, together with* ν_0^* *points* $t^m \in \hat{T}_{j_m}(x^*)$*,* $m \in \underline{\nu_0^*}$*,* $\nu^* - \nu_0^*$ *indices* k_m*, with* $1 \leq k_m \leq r$*, together with* $\nu^* - \nu_0^*$ *points* $s^m \in S_{k_m}$ *for* $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$*, and* ν^* *real numbers* v_m^**, with* $v_m^* > 0$ *for* $m \in \underline{\nu_0^*}$*, with the property that*

$$\sum_{i=1}^{p} u_i^* [\nabla f_i(x^*) - \lambda^* (\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* [\nabla G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*, s^m) = 0, \tag{4.3}$$

$$\langle y, \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*, t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*, s^m) \Big] y \rangle \geq 0, \tag{4.4}$$

where $\hat{T}_{j_m}(x^*) = \{t \in T_{j_m} : G_{j_m}(x^*, t) = 0\}$, $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^{p} u_i = 1\}$*, and* $\underline{\nu^*} \backslash \underline{\nu_0^*}$ *is the complement of the set* $\underline{\nu_0^*}$ *relative to the set* $\underline{\nu^*}$*.*

5. Second Order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities

This section deals with some parametric sufficient $\epsilon-$ efficiency conditions for problem (P) under the generalized frameworks of $(\Phi, \Psi, \rho, \eta, \theta)-$invexities for generalized invex functions. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i [f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i\Big) g_i(x)],\ i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x),\ \ j = 1, \cdots, m.$$

Theorem 5.1. *Let* $x^* \in Q$*. Let* f_i, g_i *for* $i \in \{1, \cdots, p\}$ *with* $f_i(x^*) \geq \epsilon_i g_i(x^*)$*,* $g_i(x^*) > 0$ *and* H_j *for* $j \in \{1, \cdots, m\}$ *be twice continuously differentiable at* $x^* \in Q$*, and let there exist* $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$*,* $v^* \in \Re_+^m$ *and* $z \in \Re^n$ *such that*

$$\Sigma_{i=1}^{p} u_i^* [\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)] + \Sigma_{j=1}^{m} v_j^* \triangledown H_j(x^*) = 0, \tag{5.1}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z \Big\rangle \geq 0, \tag{5.2}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{5.3}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$*):*

(i) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0$.

(ii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are prestrictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0.$

(iii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\Phi, \Psi, \rho, \eta, \theta)-$*quasi-invex at* x^* *for* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0.$

(iv) For each $i \in \{1, \cdots, p\}$*,* f_i *is* $(\Phi, \Psi, \rho_1, \eta)-$*invex and* $-g_i$ *is* $(\Phi, \Psi, \rho_2, \eta)-$*invex at* x^**.* $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ *is* $(\Phi, \Psi, \rho_3, \eta)-$*quasi-invex at* x^**,* $\Phi(a) \geq 0 \Rightarrow a \geq 0$ *and* $b \leq 0 \Rightarrow \Psi(b) \leq 0,$ *and* $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i.$

Then x^* *is an* $\epsilon-$*efficient solution to (P).*

Proof. If (i) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0\ \forall\, x \in Q, \quad (5.4)$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0. \quad (5.5)$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and in light of the $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, and assumptions on Ψ, we find

$$\Psi\Big(\Sigma_{j=1}^m v_j^* H_j(x) - \Sigma_{j=1}^m v_j^* H_j(x^*)\Big) \leq 0,$$

which results in

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \quad (5.6)$$

It follows from (5.3), (5.4), (5.5) and (5.6) that

$$\langle \Sigma_{i=1}^p u_i^*[\nabla f_i(x^*) - (\tfrac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \tfrac{1}{2}\Big\langle z, \Sigma_{i=1}^p u_i^*[\nabla^2 f_i(x^*) z - (\tfrac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*) z]\Big\rangle \geq \rho(x, x^*)\|\theta(x, x^*)\|^2. \quad (5.7)$$

As a result, since $\rho(x, x^*) \geq 0$, applying the $(\Phi, \Psi, \rho, \eta, \theta)-$ pseudo-invexity at x^* to (5.7) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^p u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \epsilon_i) g_i(x)] - \Sigma_{i=1}^p u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)]\Big) \geq 0,$$

which implies

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)]$$

$$\geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] - \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) = 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq 0. \tag{5.8}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ \textit{some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

Next, if (ii) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*)\rangle + \langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0 \ \forall\, x \in Q, \tag{5.9}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{5.10}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and in light of the strict $(\Phi, \Psi, \rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, and assumptions on Ψ, we find

$$\Psi\Big(\Sigma_{j=1}^{m} v_j^* H_j(x) - \Sigma_{j=1}^{m} v_j^* H_j(x^*)\Big) \leq 0,$$

which results in

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{5.11}$$

It follows from (5.3), (5.9), (5.10) and (5.11) that

$$\begin{aligned} & \langle \Sigma_{i=1}^{p} u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*) \rangle \\ + & \frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla^2 g_i(x^*) z]\Big\rangle \\ > & \rho(x, x^*) \|\theta(x, x^*)\|^2. \end{aligned} \tag{5.12}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrict $(\Phi, \Psi, \rho, \eta, \theta)-$pseudo-invexity at x^* to (5.12) and assumptions on Φ, we have

$$\Phi\Big(\Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \epsilon_i) g_i(x)] - \Sigma_{i=1}^{p} u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)]\Big) \geq 0,$$

which implies

$$\begin{aligned} & \Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq \Sigma_{i=1}^{p} u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] \\ \geq & \Sigma_{i=1}^{p} u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] - \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq 0. \tag{5.13}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \ \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \ some\ j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

The proofs applying (iii) is similar to that of (ii), so we just need to include the proof using (iv) as follows: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, which implies $\Psi\Big(H_j(x) - H_j(x^*)\Big) \leq 0$.

Then applying the $(\Phi, \Psi, \rho_3, \eta)-$quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \eta(x, x^*) \rangle + \frac{1}{2}\Big\langle z, \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z\Big\rangle \leq -\Sigma_{j=1}^{m} v_j^* \rho_3 \|\theta(x, x^*)\|^2.$$

Since $u^* \geq 0$ and $f_i(x^*) \geq \epsilon_i g_i(x^*)$, it follows from $(\Phi, \Psi, \rho_3, \eta)-$invexity assumptions that

$$\begin{aligned}
& \Phi\Big(\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)]\Big) \\
= \; & \Phi\Big(\Sigma_{i=1}^{p} u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)[g_i(x) - g_i(x^*)] + \epsilon_i g_i(x^*)\}\Big) \\
\geq \; & \Sigma_{i=1}^{p} u_i^*\{\langle \nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*), \eta(x, x^*)\rangle\} \\
+ \; & \frac{1}{2}\langle z, \Sigma_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z\rangle] \\
+ \; & \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) \\
\geq \; & -[\langle \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\Big\langle z, \Sigma_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)z\Big\rangle] \\
+ \; & \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) \\
\geq \; & (\Sigma_{j=1}^{m} v_j^*\rho_3 + \Sigma_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2])\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) \\
= \; & (\Sigma_{j=1}^{m} v_j^*\rho_3 + \rho^*)\|\theta(x, x^*)\|^2 + \Sigma_{i=1}^{p} u_i^*\epsilon_i g_i(x^*) \\
\geq \; & (\Sigma_{j=1}^{m} v_j^*\rho_3 + \rho^*)\|\theta(x, x^*)\|^2,
\end{aligned}$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$ and $\rho^* = \Sigma_{i=1}^{p} u_i^*(\rho_1 + \phi(x^*)\rho_2)$. □

We note that Theorem 5.1 can be specialized to the context of second order $(\rho, \eta, \theta)-$ invexities as follows:

Theorem 5.2. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1, \cdots, p\}$ with $f_i(x^*) \geq \epsilon_i g_i(x^*)$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \Re^p : u > 0, \Sigma_{i=1}^{p} u_i = 1\}$ and $v^* \in \Re_+^m$ such that*

$$\Sigma_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)] + \Sigma_{j=1}^{m} v_j^* \nabla H_j(x^*) = 0 \tag{5.14}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0, \tag{5.15}$$

and

$$v_j^* H_j(x^*) = 0, \; j \in \{1, \cdots, m\}. \tag{5.16}$$

Suppose, in addition, that any one of the following assumptions holds (for $\rho(x, x^*) \geq 0$):

(i) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(ii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are prestrictly* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(iii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are strictly* $(\rho, \eta, \theta)-$*pseudo-invex at* x^**, and* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are strictly* $(\rho, \eta, \theta)-$*quasi-invex at* x^**.*

(iv) *For each* $i \in \{1, \cdots, p\}$*,* f_i *is* $(\rho_1, \eta, \theta)-$*invex and* $-g_i$ *is* $(\rho_2, \eta, \theta)-$*invex at* x^**.* $H_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *is* $(\rho_3, \eta, \theta)-$*quasi-invex at* x^**, and* $\Sigma_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \Sigma_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i$*.*

Then x^* *is an* $\epsilon-$*efficient solution to (P).*

Proof. If (i) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*\rangle + \langle \Sigma_{j=1}^m v_j^* \nabla H_j(x^*), \eta(x, x^*)\rangle = 0\ \forall\, x \in Q, \tag{5.17}$$

$$\Big\langle z, \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{5.18}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(x^*),$$

and in light of the $(\rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we have

$$\langle \nabla H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \tag{5.19}$$

It follows from (5.19) that

$$\begin{aligned} & \langle \Sigma_{i=1}^p u_i^* [\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \nabla g_i(x^*)], \eta(x, x^*\rangle \\ + \ & \frac{1}{2}\Big\langle z, \sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*) z]\Big\rangle \\ \geq \ & \rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{5.20}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the $(\rho, \eta, \theta)-$ pseudo-invexity at x^* to (5.20), we have

$$\begin{aligned} & \Sigma_{i=1}^p u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x)] \geq \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] \\ \geq \ & \Sigma_{i=1}^p u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) g_i(x^*)] - \Sigma_{i=1}^p u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{5.21}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \;\forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \; some\; j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P).

Next, if (ii) holds, and if $x \in Q$, then it follows from (5.1) and (5.2) that

$$\begin{aligned} &\langle \Sigma_{i=1}^{p} u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x, x^*)\rangle \\ + \; &\langle \Sigma_{j=1}^{m} v_j^* \triangledown H_j(x^*), \eta(x, x^*)\rangle = 0 \;\forall\, x \in Q, \end{aligned} \tag{5.22}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0. \tag{5.23}$$

Since $v^* \geq 0$, $x \in Q$ and (5.3) holds, we have

$$\Sigma_{j=1}^{m} v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^{m} v_j^* H_j(x^*),$$

and in light of the strict $(\rho, \eta, \theta)-$quasi-invexity of $B_j(., v^*)$ at x^*, we find

$$\langle \triangledown H_j(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0. \tag{5.24}$$

It follows from (5.23) and (5.24) that

$$\begin{aligned} &\langle \Sigma_{i=1}^{p} u_i^*[\triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \triangledown g_i(x^*)], \eta(x, x^*)\rangle \\ + \; &\frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle > \rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned} \tag{5.25}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the prestrict $(\rho, \eta, \theta)-$pseudo-invexity at x^* to (5.25), we have

$$\begin{aligned} &\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq \Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] \\ \geq \; &\Sigma_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x^*)] - \Sigma_{i=1}^{p} u_i^* \epsilon_i g_i(x^*) = 0. \end{aligned}$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i)g_i(x)] \geq 0. \tag{5.26}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - (\frac{f_i(x^*)}{g_i(x^*)} - \epsilon_i) \leq 0 \; \forall\, i = 1, \cdots, p,$$

and

$$\frac{f_j(x)}{g_j(x)} - (\frac{f_j(x^*)}{g_j(x^*)} - \epsilon_j) < 0, \; some\; j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\epsilon-$efficient solution to (P). □

6. Concluding Remarks

We observe that the obtained results in this communication can be generalized to the case of multiobjective fractional subset programming with generalized invex functions, for instance based on the work of Mishra et al. (see (Mishra *et al.*, 2010)) and Verma (see (Verma, 2013))) to the case of the $\epsilon-$ efficiency and weak $\epsilon-$efficiency conditions to the context of minimax fractional programming problems involving n-set functions.

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Refined Estimates for the Equivalence Between Ditzian-Totik Moduli of Smoothness and K-Functionals

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Abstract

The aim of this note is to study the magnitude of the constants in the equivalence between the first and second order Ditzian-Totik moduli of smoothness and related K-functionals. Applications to some classic approximation operators are given.

Keywords: Ditzian-Totik moduli of smoothness, K-functional, smooth functions, constants in the equivalence.
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1. Introduction and main results

Ditizian-Totik moduli of smoothness have become a standard tool in approximation theory. This is true in particular for second order moduli which play a crucial role in approximation by positive linear operators. For their properties and many applications see (Z. Ditzian, 1987). For $f \in C[0,1]$ the second order Ditzian-Totik modulus is defined by

$$\omega_2^{\varphi}(f,h) = \sup\{|\Delta^2_{\rho\varphi(x)} f(x)|,\ x \pm \rho\varphi(x) \in [0,1],\ 0 < \rho \leq h.\} \tag{1.1}$$

Here $\varphi(x) = \sqrt{x(1-x)}$, and $\Delta^2_{\eta} f(y) = f(y-\eta) - 2f(y) + f(y+\eta)$ if $\eta > 0$ and $y \pm \eta \in [0,1]$ and as 0 otherwise. In the sequel we will use the following notation:

$$AC_{loc}[0,1] := \{h : h \text{ is absolutely continuous in } [a,b] \text{ for every } 0 < a < b < 1\};$$

$$W^{\varphi}_{2,\infty}[0,1] := \{g : g' \in AC_{loc}[0,1] \text{ and } \|\varphi^2 g''\|_\infty < \infty\}.$$

The related K-functional $K_2^{\varphi}(f,h^2)$ is given by

$$K_2^{\varphi}(f,h^2) := \inf_g\{\|f-g\|_\infty + h^2\|\varphi^2 g''\|_\infty\}. \tag{1.2}$$

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Here the infimum is taken over all $f \in W^{\varphi}_{2,\infty}[0,1]$. The definitions of the second order Ditzian-Totik modulus of smoothness and related K-functional can be generalized in a natural way for all $r \geq 1$. The equivalence between these two constructive characteristics is well-known (see Theorem 6.2 in (DeVore & Lorentz, 1993)). We cite it here as

Theorem A *There are constants $c_1, c_2 > 0$, which depend only on r, such that for all $f \in C[0,1]$*

$$c_1\omega_r^{\varphi}(f,h) \leq K_r^{\varphi}(f,h^r) \leq c_2\omega_r^{\varphi}(f,h),\ 0 < h \leq (2r)^{-1}. \tag{1.3}$$

In many problems in approximation theory it is a difficult task to prove direct or inverse estimates directly in terms of the Ditzian-Totik moduli of smoothness. Instead of this, the K-functionals have become a powerful tool to establish such statements. However, the main disadvantage of the latter, is the fact, that practically it is impossible to calculate the value of the K-functional for a given function f. Therefore, the usual way is first to prove direct or inverse estimates in terms of the K-functional, and after this using Theorem A to reformulate the results in terms of the moduli of smoothness. This explains how important is to have a good information about the magnitude of the constants c_1, c_2. To the best of our knowledge the problem to find the best possible values of c_2 and c_1 (in the sense c_2-minimal and c_1-maximal in (1.3)) is still not solved. Hardly anything appears to be known about the explicit description of the size of c_1 and c_2 for $r = 1, 2$ and about their asymptotic dependence on r for $r > 2$. The first attempt in this direction is Theorem 3.5 in (Gonska & Tachev, 2003) which we cite here as

Theorem B *For $m \geq 2$, $h \in \left[\frac{\sqrt{2}}{md(m)}, \frac{\sqrt{2}}{(m-1)d(m-1)}\right]$ the following inequalities hold for any $f \in C[0,1]$:*

$$\frac{1}{16}\omega_2^{\varphi}(f,h) \leq K_2^{\varphi}(f,h^2) \leq c_2(m)\omega_2^{\varphi}(f,h),$$

where

$$c_2(m) := 1 + \left(\frac{m}{m-1}\right)^2 \frac{48}{d^2(m-1)},$$

and the sequence $d(m)$ is defined as

$$d(m) = \frac{\sqrt{m^4+m^2+1}-1}{\sqrt{m^4+m^2+1}+m^2},\ d(m) \to \frac{1}{2},\ m \to \infty.$$

It is clear that $\lim_{m\to\infty} c_2(m) = 193$. If we restrict our attention to values $h \leq 1$, as a corollary from Theorem B we get

$$\frac{1}{16}\omega_2^{\varphi}(f,h) \leq K_2^{\varphi}(f,h^2) \leq 404 \cdot \omega_2^{\varphi}(f,h). \tag{1.4}$$

The difficulties in the proof of Theorem B are connected with the construction of an appropriate auxiliary function g in the definition of K_2^{φ}. Actually we apply a "smoothing" technique to the linear interpolant on certain places near the points of interpolation to obtain an appropriate quadratic C^1-spline based upon the knot sequence. This method was developed in (Gonska & Kovacheva, 1994; Gonska & Tachev, 2003; H. Gonska, 2002) and further refined in (Gavrea, 2002). In this note we essentially improve the value of the constant 404 in (1.4). Our main result states the following:

Theorem 1 *The following inequalities hold for any $f \in C[0,1]$, $h \in (0,1]$:*

$$\frac{1}{16}\omega_2^{\varphi}(f,h) \le K_2^{\varphi}(f,h^2) \le (5+2\sqrt{2})\cdot\omega_2^{\varphi}(f,h). \tag{1.5}$$

In Section 2 we give the proof of Theorem 1. In Section 3 we apply Theorem 1 to obtain quantitative estimates in terms of second order Ditzian-Totik modulus of smoothness for approximation by genuine Bernstein-Durrmeyer operator, considered in (P. E. Parvanov, 1994) and also for pointwise estimates, established in (Felten, 1998). In the last section we consider the case $r = 1$, which is closely related to piecewise linear interpolation at specific knot sequence.

2. Proof of Theorem 1

To obtain as small as possible value of the constant c_2 in (1.3) we need an appropriate auxiliary function g in the definition of the K-functional. We use the construction, developed by Gavrea in (Gavrea, 2002) and based on the ideas from (Gonska & Kovacheva, 1994; Gonska & Tachev, 2003). Let m be fixed natural number, $m \ge 1$. The patition Δ_m of the interval $[0,1]$ is given by

$$\Delta_m: \ 0 = x_0 < x_1 < \cdots < x_{2m+2} = 1,$$

where

$$x_k = \sin^2\frac{k\pi}{4(m+1)},\ k = 0,1,\ldots,2m+2. \tag{2.1}$$

We denote by $S_m(f)$ a piecewise linear interpolant with interpolation knots-the points x_k, $k = 0,1,\ldots,2m+2$. Each point $(x_k, S_m(f,x_k))$, $k = 1,2,\ldots,2m+1$ we associate with two other points $(a_k, S_m(f,a_k)), (b_k, S_m(f,b_k))$ such that

$$a_1 = \frac{x_1}{2},\ b_1 - x_1 = x_1 - a_1,$$

and

$$a_k = \frac{x_k + x_{k-1}}{2},\ b_k - x_k = x_k - a_k,\ k = 1,2,\ldots,2m+1.$$

The function g is defined as follows:

For $x \in [0,a_1] \cup [b_{2m+1},1]$ we set $g(x) = S_m(f,x)$.

For $x \in [a_k,b_k]$, $k = 1,\ldots,2m+1$, $g(x)$ is the 2nd degree Bernstein polynomial over the interval $[a_k,b_k]$, determined by the ordinates $S_m(f,a_k), f(x_k), S_m(f,b_k)$.

For $x \in [b_k,a_{k+1}]$, $k = 1,2,\ldots,2m$ we set $g(x) = S_m(f,x)$. Thus $g(x)$ is uniquely determined by the interpolation conditions and is C^1-continuous. For this function the following two crucial estimates are proved in Theorem 6 in (Gavrea, 2002):

$$\|f-g\|_\infty \le \omega_2^{\varphi}\left(f, \sin\frac{\pi}{2(m+1)}\right), \tag{2.2}$$

$$\|\varphi^2 g''\|_\infty \le \frac{1}{\sin^2 \frac{\pi}{4(m+1)}} \cdot \omega_2^\varphi\left(f, \sin \frac{\pi}{2(m+1)}\right). \tag{2.3}$$

For any positive number $h \in (0, 1]$ there exists a natural number $m \ge 1$, such that

$$h \in \left[\sin \frac{\pi}{2(m+1)}, \sin \frac{\pi}{2m}\right].$$

Hence (2.2) and (2.3) imply

$$\|f - g\|_\infty \le \omega_2^\varphi(f, h), \tag{2.4}$$

$$h^2\|\varphi^2 g''\|_\infty \le \frac{\sin^2 \frac{\pi}{2m}}{\sin^2 \frac{\pi}{4(m+1)}} \cdot \omega_2^\varphi(f, h). \tag{2.5}$$

It is easy to verify that the sequence

$$c(m) := \frac{\sin^2 \frac{\pi}{2m}}{\sin^2 \frac{\pi}{4(m+1)}}$$

is monotone decreasing, i.e. $c(m) \le c(1) = 4 + 2\sqrt{2}$. Consequently the right-hand side of (1.5) is proved. Lastly we point out that the constant $\frac{1}{16}$ in (1.5) could be derived from Theorem 6.1 in (DeVore & Lorentz, 1993). Thus the proof of Theorem 1 is completed.

3. Applications

1. The genuine Bernstein-Durrmeyer operator. As first application of Theorem 1 let us consider the so-called genuine Bernstein-Durrmeyer operator, introduced by Goodman and Sharma in (Goodman & Sharma, 1991) and given by

$$U_n(f, x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t)f(t)dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}, k = 0, \ldots, n$, are the fundamental Bernstein polynomials. Parvanov and Popov proved in (P. E. Parvanov, 1994) in an elementary and very elegant manner a direct and a strong converse inequality of type A, thus completely characterizing the approximation speed of the operators. The main result in (P. E. Parvanov, 1994) states the following:

For any $f \in C[0, 1]$ we have

$$\frac{1}{2}\|U_n f - f\|_\infty \le K_2^\varphi(f, \frac{1}{2n}) \le (4 + \sqrt{2})\|U_n f - f\|_\infty. \tag{3.1}$$

As a corollary from Theorem 1 and (3.1) we obtain

$$\frac{1}{2(5 + 2\sqrt{2})}\|U_n f - f\|_\infty \le \omega_2^\varphi(f, \frac{1}{\sqrt{2n}}) \le 16(4 + \sqrt{2})\|U_n f - f\|_\infty. \tag{3.2}$$

2. The Bernstein operator The classical Bernstein operator $B_n(f,x)$ for a given function $f \in C[0,1]$ is defined by

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) p_{n,k}(x).$$

Let $\Phi : [0,1] \to R$, $\Phi \neq 0$ be a function such that Φ^2 is concave. Then the pointwise approximation

$$|B_n(f,x) - f(x)| \leq 2K_2^{\varphi}\left(f, n^{-1}\frac{\varphi^2(x)}{\Phi^2(x)}\right), \ x \in [0,1], \tag{3.3}$$

holds true for all $f \in C[0,1], n \in N$. This result was proved by Felten in (Felten, 1998). As a straightforward corollary from Theorem 1 we get

$$|B_n(f,x) - f(x)| \leq 2(5 + 2\sqrt{2})\omega_2^{\varphi}\left(f, n^{-\frac{1}{2}}\frac{\varphi(x)}{\Phi(x)}\right), \ x \in [0,1]. \tag{3.4}$$

4. The case $r = 1$

In this section we consider the interval $[-1,1]$ instead of $[0,1]$. After a linear transformation it is clear that each estimate in one of these two cases can be obtained from the other. The weight function over $[-1,1]$ is now $\varphi(x) = \sqrt{1-x^2}$. Let $\Delta_n : -1 = x_0 < x_1 < \cdots < x_n = 1$ be a partition of the interval $[-1,1]$ such that the inequalities

$$c_3(x_{k+1} - x_k) \leq \frac{\varphi(x)}{n} \leq c_4(x_{k+1} - x_k) \tag{4.1}$$

are satisfied for $k = 1, 2, \ldots, n-2$, $x \in [x_k, x_{k+1}]$, and also

$$c_3(x+1) \leq \frac{\varphi(x)}{n} \leq c_4(x+1), \ x \in [x_0, x_1],$$

$$c_3(1-x) \leq \frac{\varphi(x)}{n} \leq c_4(1-x), \ x \in [x_{n-1}, x_n],$$

where $c_i, i = 3,4$ are absolute positive constants independent of n. The function g in the definition of K_1^{φ} we define as the linear interpolant of $f \in C[-1,1]$ with knots $\{x_k\}$. For $x \in [x_k, x_{k+1}]$, $k = 1, \ldots, n-2$, from the properties of linear interpolation it follows that

$$|g(x) - f(x)| \leq \omega_1(f, x_{k+1} - x_k) = \sup\{|f(x + \frac{h}{2}) - f(x - \frac{h}{2})|, \ x, x \pm \frac{h}{2} \in [x_k, x_{k+1}]\} \leq \omega_1^{\varphi}(f, \frac{1}{c_3 n}). \tag{4.2}$$

Let $x \in [-1, x_1]$. The case $x \in [x_{n-1}, 1]$ is analogous. Obviously

$$|g(x) - f(x)| \leq \sup\left\{|f(x + \frac{h}{2}) - f(x - \frac{h}{2})|, \ x, x \pm \frac{h}{2} \in [-1, x_1]\right\}.$$

The inequality $x - \frac{h}{2} \geq -1$ yields $h \leq 2(x+1) \leq \frac{2}{c_3}\frac{\varphi(x)}{n}$, which follows from (4.1). To summarize we proved

$$\|f - g\|_\infty \leq \omega_1^{\varphi}(f, \frac{2}{c_3 n}). \tag{4.3}$$

Next we evaluate the second term in the definition of the K-functional.

For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \dots, n-2$, it is easy to verify that

$$\frac{1}{n}|\varphi(x)g'(x)| = \frac{\varphi(x)}{n(x_{k+1} - x_k)}|f(x_k) - f(x_{k+1})|.$$

Using (4.1) and (4.2) we get

$$\frac{1}{n}\|\varphi g'\|_{L_\infty[x_1, x_{n-1}]} \le c_4 \omega_1^{\varphi}(f, \frac{1}{c_3 n}).$$

It remains to consider $x \in [-1, x_1]$. In this case we observe that

$$\frac{\varphi(x)}{n} \le c_4(x+1).$$

Therefore for $x \in [-1, x_1]$ we have

$$\frac{1}{n}|\varphi(x)g'(x)| \le c_4 \omega_1^{\varphi}(f, \frac{2}{c_3 n}).$$

Finally we arrive at

$$\frac{1}{n}\|\varphi g'\| \le c_4 \omega_1^{\varphi}(f, \frac{2}{c_3 n}). \tag{4.4}$$

For every $0 < t < 1$ there exists $n \ge 2$ such that

$$\frac{2}{c_3 n} < t < \frac{2}{c_3(n-1)}.$$

Combining (4.3) and (4.4) we get

$$K_1^{\varphi}(f, t) \le \left[1 + \frac{2c_4}{c_3}\left(\frac{n}{n-1}\right)\right]\omega_1^{\varphi}(f, t). \tag{4.5}$$

It is clear that the condition number $\frac{c_4}{c_3}$ of our system of knots determines the value of the constant in front of the modulus. By the previous considerations we have shown the validity of

Theorem 2. *For $f \in C[-1, 1]$, $n \ge 2$, $t \le \frac{1}{2}$ we have*

$$\frac{1}{8}\omega_1^{\varphi}(f, t) \le K_1^{\varphi}(f, t) \le c_2(n)\omega_1^{\varphi}(f, t), \tag{4.6}$$

where

$$c_2(n) = \left[1 + \frac{2c_4}{c_3}\left(\frac{n}{n-1}\right)\right].$$

Remark 1. The constant $\frac{1}{8}$ in the left side of (4.6) follows easily if we verify the computations made in Theorem 6.1 in Chapter 6 in (DeVore & Lorentz, 1993).

Remark 2. If we strictly follow the construction in the proof of Theorem 2, it is possible to improve the value of $c_2(n)$, i.e. to obtain the value of the latter as small as possible. In order to do this, we would find an optimal set of knots, satisfying (4.1) with a condition number as small as possible. In this case we formulate the following

Open problem. Find the optimal set $\{x_k\}$ satisfying (4.1) in the sense that the condition number $\frac{c_4}{c_3}$ is minimal.

Here we give two examples of knots.

Example A. In this example we choose $\{x_k\}$ to be the well-known zeros of the Chebyshev polynomial of the first kind

$$x_k = \cos\theta_k,\ \theta_k := \frac{(2k-1)\pi}{2n},\ k = 1,\dots,n,\ x_{n+1} := -1, x_0 := 1.$$

Following (7.7-7.8) in Chapter 8 in [1] we get $c_3 = \frac{1}{3\pi}$, $c_4 = 3\pi$. The condition number is $9\pi^2$.

Example B. Here we choose the extremal points of the Chebyshev polynomial of the first kind

$$x_k = \cos(\frac{k\pi}{n}),\ k = 0,\dots,n.$$

This is the same set of interpolation knots , considered in Section 2 for the interval $[0,1]$. In this case we compute $c_3 = \frac{1}{2\pi}$, $c_4 = 2\pi$. The condition number is $4\pi^2$-better as in Example A.

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Guidelines for Improvement Information Processes in Commerce by Implementing the Link Between a Web Application and Cash Registers

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Abstract

The main task of the article is clarifying the content of different types of costs (which are associated with providing information about business processes) and on this basis an identification of opportunities for their reduction may be found. In this regard, this paper examines business processes in retail in rendering account of sales. A special place in the article is devoted to the analysis of existing models of information technology systems in commercial enterprises. The important question for the adaptation of web applications in commercial enterprises has been developed. Factors contributing adaptation of Web applications in business practices are taken in account. Factors hindering their adaptation (such as the relationship of a web application with cash registers a problem that is not widely discussed) are also reviewed. A discussion of various options for improving both the technological model of the information system in enterprises and specific guidelines for the quantification of the proposed approaches to reduce costs is made.

Keywords: Web application, cash register, fiscal printer, commerce, Delphi, Intraweb, Bulgaria.
CCS: D2.
JEL: I23.

1. Introduction

Reducing the cost of providing information for business processes can be made after a thorough analysis of the performed information processing. The study of information processes is performed in order to reduce the cost of hardware and software. An especially acute problem is the problem for measuring the cost of hardware and software (for purchase, maintenance and power supply). Under current conditions, operating costs of buying and maintaining hardware and software are distinct from an accounting point of view, but they are not subject of extensive study by a managerial perspective. This feature prevents their full, thorough and objective study to find

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specific ways for their reduction.

Rationalization of IT processes as an indirect effect gives rise to conditions for reducing managerial costs. Even though the cost of computer hardware and software declines steadily, companies do not report cost reductions. The article gives specific suggestions for reducing the cost of buying and maintaining hardware and software systems in commercial enterprises. Despite the fact that the costs of buying and maintaining hardware and software systems in commercial enterprises have relatively small share of total expenditures, these costs are subject to monitoring by managers. The article discusses concrete opportunities to reduce these costs in implementing the managerial processes. Several approaches for cost reduction are discussed. Options for rationalization of communication processes are considered.

At this stage, information systems used for sales in stores (known as POS systems) are generally associated with a cash register or a fiscal printer. The Bulgarian market offers a wide variety of both: (1) cash registers and fiscal printers and (2) software to track sales. It should be noted that some software products to track sales technologically implement the connection with a cash register. In the rest of software products for tracking sold goods the sales registering process is done twice at a computer and at a cash register. It is obvious that connecting an electronic cash register (ECR) with fiscal memory (or a fiscal printer) to a computer is not an easy task. There are examples of software companies where programmers applying for jobs are not approved because they do not know how to connect software to record sales with an ECR or a fiscal printer.

The *purpose* of this article is to improve the existing technology model of an information system for recording sales in retail outlets in order to reduce the cost of hardware and software. To achieve the objective we have to solve the following *tasks*: (1) to examine the current technological models, (2) to make a proposal for improvement, (3) to demonstrate the need to connect a web application with cash registers (4) to explore existing ICT and (5) to develop a specific program to connect a web application with cash registers. The *subject* of this study is information technology for development of web applications. The *objects* of this study are communication technologies (both low and high level) for communication between software (both desktop application and web application) and cash registers.

The paper is organized as follows. Section 2 expresses an analysis of existing technologies for providing information for retail business. Two existing technology models of information systems in retail are described. Section 3 presents several guidelines for improving the model of information support of commercial processes. Subsection 3.1 contains a new (enhanced) model the second technology model is further developed. Thin clients are used instead of desktop computers. The adoption of the enhanced model and the link between a desktop application and cash registers are described in subsection 3.2. Software aspects for the implementation the link between a web application and fiscal printers are illustrates in subsection 3.3. The conclusions are outlined in Section 4.

2. Analysis of existing technologies for providing information for retail business

Commercial processes are widely known both at home and abroad. The need for rapid recording of sales leads to adaptation of sales software systems. In the course of time many experts have studied the business processes. They have offered a variety of improvements. The use of barcodes for automatic identification of goods and materials is a nice example. Sales are registered in a database. Nowadays analogues of barcodes and databases are not found.

In the early 90-ies of XX century, some software companies connect their automated information system (AIS) with a cash register. The technological achievement is significant and it is appreciated by retailers. For software developers (Application Software Providers ASP) the implementation of such a system leads to realization a significant revenue for a short period of time (having in mind that the software market does not offer similar products). Normally other software vendors also try to enter the market as an attempt to connect their AIS with a fiscal printer (FP). The available technical documentation offered 20 years ago by producers of fiscal printers (and ECRs) in Bulgaria is clear that a FP is connected to COM port via RS-232. Most modern laptops do not have a COM port.

Unfortunately manufacturers of FPs for a long period of time do not offer the communication protocol extremely necessary to developers of application software to connect their AIS with FPs. Users of specialized software fall into a situation where they have to enter one and the same data in two places in their AIS and on their ECR. Such duplicate data entry takes considerable time and it is a prerequisite for admission of technical errors. There are two possibilities to users of POS systems: (1) to ask the software vendor to enhance his software system so that it connects with a FP or (2) to change the POS system with another POS system offered by another software provider (ASP). Examples from both directions may be given.

There are two recognized process models of software systems for recording sales in retail outlets. It is typical for *the first technology model* of retail that sales are recorded in a cash register and in a local database. A cash receipt is printed. At the end of the day records from local databases are merged on the server database (Figure 1).

The mentioned approach guarantees quick registering of sales. *The first technology model of a software system for recording sales* in cash desks zone is adopted in many enterprises in Bulgaria. We have to highlight that each POS terminal consists of a computer with an installed operating system, a database management system (DMBS), a local database (for registering sales at each cash desk) and a software module which transfers sales transactions from local databases to the server database offline. The first technology model is characterized with high performance and reliability of the software system for registering sales transactions. The high speed of the software is due to the fact the software system is undependable from the local area network (LAN). A significant drawback is the operability of the transmission of data and up to date information on the server. Because data is transferred offline (not online), operational managers do not have updated information about sales. If a manager wants to obtain information online, he cannot obtain it. Such organization of work is suitable for small and medium-sized shops.

To overcome the shortcomings of the first technology model a number of companies apply a different technological model for recording sales that we would conditionally call *second technology model* (Figure 2).

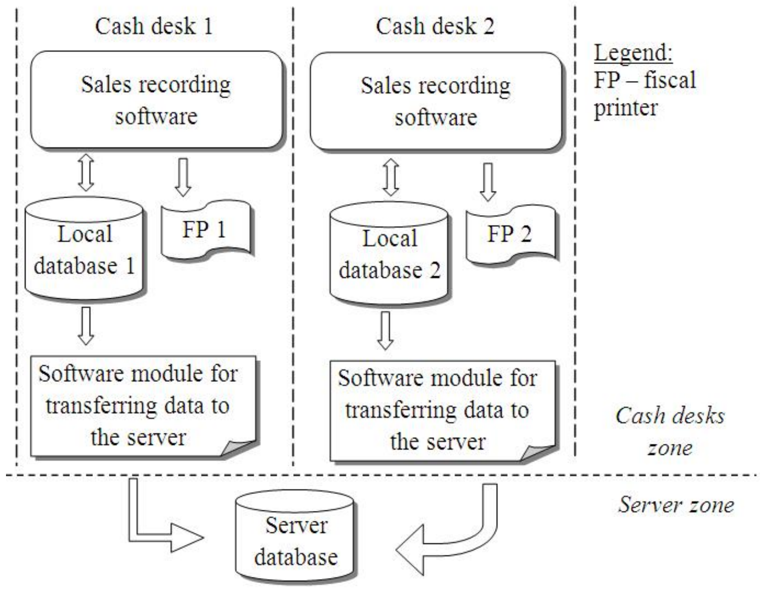

Figure 1. First technology model of a software system for recording sales in cash desks zone.

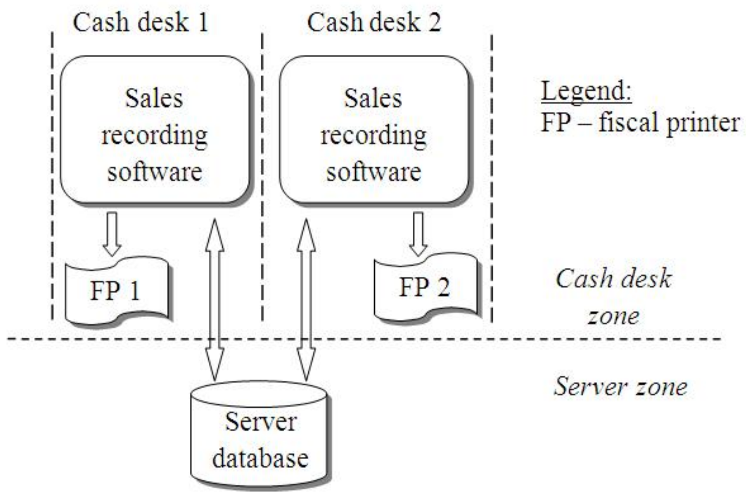

Figure 2. Second technology model of a software system for recording sales in cash desks zone.

In the second approach, data is maintained in a centralized database, which improves the operability of data. In this case, the manager receives updated information about sales. A significant advantage of the second approach is the absence of the need for: (1) the use of a local database and (2) transfer of sales transactions made at the end of the day to the server database.

It should be noted that the second technology model uses a server database. There is no need to install client side DBMS on local workstations. At each cash desk there is a computer with an installed operating system and application software that communicates with the server database (it derives price and the name of a specific item by its barcode and it records sale transactions in the database). The second technology model is a typical example of an online recording system of sales working in client-server mode.

For its reliability of great importance is the availability of secure network connection between the server (which stores the database) and separate workstations. The cost of implementing the second technology model (which is popular in business practice) requires fewer resources than the first one. Several saving of costs are achieved. As an example we mention costs for licenses of DBMS on individual client machines and the cost of software which periodically transfers the data from different workstations to the server. At the first glance, the second model has no disadvantages. That is why it is widely spread in practice. However, our research continues to seek improvements in the development of the second model. Later in the text we provide guidelines for improvement the model of information support of commercial processes.

3. Guidelines for improving the model of information support of commercial processes

3.1. The use of thin clients

The current status of information systems used in retail outlets is an adequate implication of the second approach. Despite many years of experience of the application of information systems in retail, we can seek guidelines to improve existing technological models. Some possibilities for cost reduction can be found in the following areas: reducing the cost of application software, system software, hardware and power supply. Further, the text puts forward concrete proposals to enhance the second model, in which some costs for both hardware and software may be saved.

It makes an impression that a centralized database (located on the server) is maintain in both approaches. As noted, an analogue of the centralized database cannot be recommended. In terms of software, desktop applications are mostly used in Bulgaria. It is typical for them that they are installed on each workstation. This feature requires the use of a personal computer on every work place. In order to reduce the cost of electricity it is possible to use *thin clients*.

Using thin client does not allow the installation of desktop applications. To be able to use software system for recording sales on hardware devices such as thin clients, there should be a change in the application software to shift from desktop applications to Web app to record sales. In this case, the technological model for recording of sales is as follows (Figure 3).

The recommended third technological model for sales recording is used in some European retail shops, but it is not popular in Bulgaria. The use of an intranet software system for recording sales allows the work of web applications within a corporate local area network. The application software is installed only on a server. It is accessible to workstations through a web browser.

The third technology model is done to reduce costs in the following areas: (1) energy (one thin client spends significantly less electricity than a desktop computer), (2) cost of installation and maintenance of a license for application software on each workstation (application software (software product for recording sales) only be installed on server) and (3) the maintenance costs of hardware thin clients have significantly fewer parts than a desktop computer hence their tendency to damage is lower than a personal computer. Indirectly labor costs are reduced because fewer people are needed to maintain the hardware and software in the third model.

In terms of costs, the application of the represented third technology model leads to cost savings for hardware, software and power supply. Hardware savings are in the following areas. In

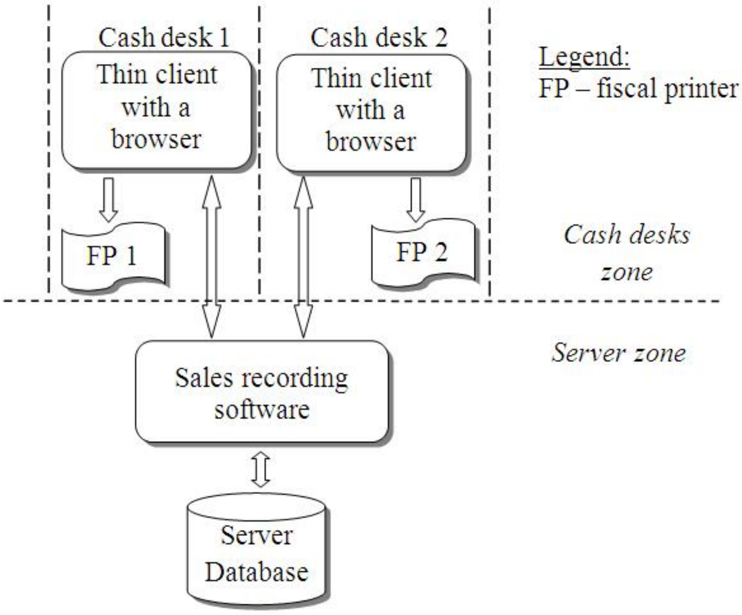

Figure 3. Third (enhanced) technology model of a software system for recording sales in cash desks zone.

the first and second model each workstation computer is configured with a hard disk, a central processing unit (CPU), RAM memory, a monitor, a keyboard and a mouse. The third approach suggests the installation of a thin client (which does not have a hard disk drive) and it is much more compact than a desktop computer. Furthermore, the thin client consumes significantly less power than a standard computer. In terms of software, the time for installation of application software is saved, because it is only installed on the server.

In all three approaches (three technological models) sales data are recorded in a database and a cash register receipt is printed. For the end user the technological model of the information system for recording of sales remains hidden, but the manager is interested in the costs and reliability of the software system.

3.2. Adoption of the enhanced model. The use of middle tier software for implementation the link between desktop applications and cash registers

Elaboration of the software, so that it is connected with a FP (fiscal printer) is not an easy task. When ASP do not provide specification of FPs, including protocol for communication and describing how to send commands to the FP, the form and content of the response, received from the FP, it is a difficult task to write sales software working with a FP. In this case, developers use a specialized software system that communicates with a FP. To monitor the communication between the PC and a FP specialized software may be used called sniffer to eavesdrop the communication between two devices.

When using a software system to record sales, it sends commands to the FP and the program sniffer captures packets from the computer to the FP and packets from the FP to the computer. Using software from the class of sniffers is considered as a hacking technique, but it can be used to establish communication between the operating system and the FP, in case there is no documentation for FPs.

Observations in business practices indicate that some of the POS systems for recording sales in Bulgaria are connected to fiscal printers (FP). Some software systems are not connected with FPs. In 2012 the situation on the FP market changed. Many businesses need to use a new type of FPs directly connected to the NRA (National Revenue Agency) by GPRS (because the taken by the Government of Bulgaria legislative initiative aimed at displaying the light of the informal sector). This situation has led to an increase in sales of FPs, but also an increase in competition both on the FP market and the software market.

In response to increasing competition some companies, selling cash registers, have published on their website the communication protocol for communication between software for registering sales and a cash register (or fiscal printer). In this case, software developers can download the specification, inspect it and test the connection (communication) with FP on COM port. For testing of the software it is recommended to use a non-fiscal printer.

The new situation provides tremendous opportunities for software providers (ASP) to extend the functionality of their software. It should be noted that a number of other AIS are built to input data and then a cash receipt is issued (as an example we note the payment of interest and fees on credit institutions). A wide range of software vendors can extend the functionality offered by their software so that they connect with a FP. The initiative for further developing the software can be both from the software vendor and from the end customer (the user of the software product). To connect sales recording software with a FP, developers can implement two approaches.

First, communicating directly to a COM port by sending hexadecimal commands directly to the FP (www.daisy.bg, 2013*a*), (www.datecs.bg, 2013*a*), (www.tremol.bg, 2013*a*). The FP returns response: (1) with a successfully executed command or (2) an error code. In this case, the AIS should use a pointer to a COM port that is in standby mode to receive messages from the cash register. After sending the command from the AIS to the FP, the AIS should wait for 100 ms, to get an answer from the FP. In the *first approach* programmer sends ”low” level commands form the AIS to the FP. To record a sale at one cash register, a series of hexadecimal commands should be sent. The reply by the FP has to be read.

The specification of some FPs describes the communication between a FP and a PC. The format of the messages between the Host (PC) and Slave (FP), between Slave and Host is given. A description of various types of commands, error codes and status flags for FP are also given. Because the format and content of commands is rather complex, we do not include an example. It has to be marked that only a high qualified programmer can make the communication between Host and Slave. He will write the software for communication between the sales software and the FP. He has to be very familiar with COM port communication. From his side it is required considerable effort and energy to connect the POS system to record sales with a FP, which means that the cost of the software system (Total Cost of Ownership TCO) is increased.

Second, by using intermediate software (middle-tier software), developed by the FP provider and available for: (1) downloading from the company’s website (selling FPs) and (2) free to use (www.daisy.bg, 2013*b*), (www.datecs.bg, 2013*b*), (www.tremol.bg, 2013*b*), (www.orgtechnica.bg, 2013). The middle-tier software stands between software for recording sales and the FP. In the second approach, the sales recording software prepares a text file with the sale and the software copies the text file to the folder where the middleware software is installed. It stays resident in RAM on the workstation. Every 100 ms it inspects for a text file with a sale. If a text file is found the middle tier software prints the sale the middle tier software sends the sale to the FP. After sending sales to the FP, the sales recording software deletes the text file.

The second approach is much easier to implement from the programmer's perspective. A programmer sends a set of commands as a text file. It is structured very similarly to the sale. After that the file is printed. Individual lines in the text file are filled with, items, quantities and prices. It is possible to generate a cash receipt for a sale by departments. In the second approach, the work to print a cash receipt to a FP is significantly easier than the first approach. The cost of developing the software for the second approach is lower than the first approach.

Writing the program logic in the second approach is "high" level. The number of lines of code (LOC) that must be written to communicate with FP is significantly less than the number of LOC for communication with the application of the first approach. The fewer lines of code to write, the likelihood of errors is less and less time to implement fixes (design, programming, debugging, testing and deployment) is shorter.

Most of the software products offered on the Bulgarian market which can communicate with a FP are typically desktop applications (Graphic User Interface - GUI applications). Despite the serious boom in the development of Internet technology, leading to the development of a number of web-based applications, a majority of the POS systems continue to be GUI applications. Most of the web-based e-commerce solutions do not offer a traditional connection with a cash register. If necessary to update the software (replacing EXE file on the server), the application software (installed on POS terminals) must be closed on all workstations. It means stopping work with the sales software system.

For now we can say that single software companies attempt to connect their web-based applications for tracking sales with FPs. The reason is quite simple. To print a cash receipt, desktop applications generate a text file (the second approach) or they communicate directly with the COM port (the first approach) on the local machine that is running a GUI application to track sales.

3.3. Linking a web application for registering sales with fiscal printers. Software aspects for the implementation the link

Proposals for improvement (third (advanced) technology model) can be adjusted in business practices. There is a problem with the communication of web application for reporting sales with cash registers.

Web applications are server applications. Web applications are run on a server. Web applications have access to hardware resources of the server. They do not have access to hardware resources on the client machines (workstations). Web applications have access to each user session. *The problem is how to create a text file on the workstation by the web application or how the web application communicates with a COM port on the client machine*. Most of the integrated development environments (IDEs) do not allow web applications (1) to generate a text file on the client machine and (2) cannot communicate with a COM port on the client machine.

As we know, web applications can be created through a number of IDEs. For this study we should seek an appropriate IDE for developing web applications that allows the generation of a text file on a client machine or sending commands from a server to the COM port (again on the client machine). Previous experience in web applications allows us to choose the Intraweb technology (or VCL for the web developed by Atozed software (www.atozedsoftware.com, 2013) and Embarcadero technologies (www.embarcadero.com, 2013)) for developing web applications. *Web applications created by Intraweb technology can generate a text file on the client machine*. The paper shows how a web application for recording of sales (installed on the server) can generate cash receipt on a FP connected to the client machine. The approach is new, innovative and it is still not popular in Bulgarian business practices. Its implementation will lead to a significant multiplier effect. In order to clarify our proposal, the third technology model of the software system for recording sales is further developed.

Suppose that at a POS terminal sales web-based application is used. By pressing a button (or link) "Save sale and print a cash receipt", the sale shall be recorded in a sales database (DB) by sending an INSERT request to the DB) and a cash receipt is printed. Saving data in a DB through a web form is described in most textbooks on development of web applications. That is why we will not present the information process on generating SQL clauses and sending them to the database. More interesting is to show how to connect a web application with FPs.

When entering sales in a web form, they are recorded in a tabular form and they can be seen on the screen of a workstation. The data from the tabular part of the document (of the sale) can be saved in a special type of variable: a list of strings (*TStringList*). When using the method *SaveToFile* the list of strings is stored on the server, which is not a suitable option in our case. Here most developers give up their assignment or leave the software company, because they do not know how to save a list of strings (containing information about a sale) on the client machine.

If a desktop application (running in client-server mode) is used, the *SaveToFile* method writes on the client machine. It means that there is no problem: (1) a sale to be recorded in the server database and (2) a text file may be generated on the client machine and it can be passed to the middleware software. And it prints a cash receipt on the FP, which is connected to the client machine.

Quite differently information processes are carried out in web application. As noted, web apps are server applications. It means that they are executed on the server. There is not a problem with writing to a server database. Web applications for recording sales are server applications. In this case, data is recorded on the server. *A significant problem remains how to generate a text file on the client machine from a web application*. If the web application calls the method *SaveToFile* for an object of type *TStringList*, strings (contained in the object type *TStringList*) are recorded on the server. This approach is working very embarrassing because a web application can write to a text file on the server. The web app can be associated with only one cash register and not at a cash desk, but in a separate room where the server is located. This approach is extremely uncomfortable and it is not applied in business practice. Therefore the research has to be continued in order to find a solution of the marked problem how to generate a text file on the client machine from a web application installed on a server.

Most software products for recording sales are desktop applications (GUI). Software applications from the class of web applications built to record sales in Bulgaria are not connected to FPs so far. Therefore, to solve the problem posed in this work, we suggest a specific approach for connecting a web application with FPs. To realize the information link (between the web application (which is located on the server) and FPs, which are at workstations) the web application for reporting sales has to generate a text file on the client machine. The technological solution (offered in this work) was developed in IDE Delphi using the technology Intraweb. A temporary file (with a unique name for the session) is generated on the server. Then the file is sent to the client machine. On closing the web application, all temporary files are deleted.

In the procedure that saves web form data in the database, the following variables are declared (Listing 1).

Listing 1. Declaration of variables.

```
aRow:Integer;
f:TextFile;
Line, File_Name : String;
```

The source code is as follows (Listing 2).

Listing 2: Source code in Delphi for generating a text file from the web application and sending it to the client machine.

```
// The name of the file is the session number + 'txt' extension.
// The file is saved on the server in a folder where the web
// application is started.
File_Name := ExtractFilePath( ParamStr( 0 ) ) + WebApplication.AppID +'.txt';
// The temporary file is deleted.
DeleteFile( File_Name );
// Assigning the pointer F to the name of the file, stored
// in the File_Name variable.
AssignFile( F, File_Name );
// The file is created and opened for appending data.
Rewrite( F );
// The sale is written in the text file on the server.
for aRow := 0 to ClientSideDataset.Data.Count - 1 do
  Begin // a cycle by rows of the table in the web form
    // a sale by department
    Line := 'E,1,______,_,__;;%.2f;;1;1;%d;0;0;'
   // Writing the sum of tax group (department) in the sales row
    // Line := Format( Line, [ Sum, Departament_sale ] );
    // Writing one row in the text file
    WriteLn( F, Line );
  End; // for
// End of the fiscal receipt (cash receipt)
Line := 'T,1,______,_,__;';
//Writing a marker for the end of the cash receipt in the text file
WriteLn( F, Line );
// Closing the file
CloseFile( F );
// Sending the file from the server to the workstation
  WebApplication.SendFile( File_Name, '', '' );
// Deleting the temporary file on the server
DeleteFile( File_Name );
```

4. Conclusion

This work proposes an advanced technology model of a software system for recording sales in retail outlets. The problem for connecting a web application with cash registers is solved. The proposed innovative approaches have significant multiplier effects because several costs are reduced energy costs, computer hardware and software costs. Preconditions for software enhancements of wide range of web applications are created. A new functionality of Web applications (to connect a web based system with fiscal printers) may be added.

Increasing the effectiveness of commercial enterprises depends not only on good managerial and logistics practice, but also on the costs for computer hardware and software. With the implementation of the proposed advanced technology model of companys information system a serious problem occurs how to connect a web app with cash registers. Because last literature sources do not provide a technological solution to the problem, the solution is given in this work. A "white spot" in the field of informatics is enlightened. Specific attention is given to the source software code in Pascal language (Delphi), describing how to connect a web application with cash registers.

As a result, the study concludes that many costs can be reduced. In particular it comes to energy, hardware and software costs. An argument proposal to streamline the existing process models of an information system in stores is made.

The most important factor in reducing the cost of adopting new technology models is the cost of providing information for commercial business processes. The proposed technology model and a software solution enable to make guidelines that can be perceived by businesses to streamline IT processes and reduce a number of expenses.

The result of the study showed that in times of crisis managers can find ways to reduce costs. Therefore, in parallel with the proposal to improve the technological model of an information system in the trading business a proposal for communication between a web application and cash registers is formulated and thoroughly described. The proposals for improvement do not cover all possible ways to reduce the cost of providing information for commercial processes. In the future it is necessary to explore new approaches to reduce costs and streamline IT processes. A new research should be made to look for other ways to reduce costs and other ways to improve the implementation of information process and in terms of the managerial process. These approaches (which we do have not covered in this work) may be the subject of a further systematic and thorough study.

The adaptation of the proposals in business practice makes the processes for maintenance of software systems simpler and easier. The ideas formulated and grounded in this paper may have broader continuity not only because of their innovativeness but mostly because of cost savings. As a guideline for future development of this work we may note the improvement of managerial processes in adaptation of new technological models of software systems in the trading business.

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Mean-Variance Portfolio Selection with Inflation Hedging Strategy: a Case of a Defined Contributory Pension Scheme

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Abstract

In this paper, we consider a mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. We establish the optimal wealth which involves a cash account and two risky assets for the pension plan member (PPM). The efficient frontier is obtained for the three asset classes which gives the PPM the opportunity to decide his or her own risk and wealth. It was found that inflation-linked bond is a suitable asset for hedging inflation risks in an investment portfolio.

Keywords: Mean-variance, inflation hedging, defined contribution, efficient frontier, optimal utility, expected wealth, inflation risks fighter.
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JEL: C61.

1. Introduction

A mean-variance optimization is a quantitative method that is adopted by fund managers, consultants and investment advisors to construct portfolios for the investors. When the market is less volatile, mean-variance model seems to be a better and more reasonable way of determining portfolio selection problem. One of the aims of mean-variance optimization is to find portfolio that optimally diversify risk without reducing the expected return and to enhance portfolio construction strategy. This method is based on the pioneering work of Markowitz (Markowitz, 1952, 1959). The optimal investment allocation strategy can be found by solving a mean and variance optimization problem.

There are extensive literature that exist on the area of accumulation phase of a DC pension plan and optimal investment strategies. For some of the literature, see for instance, (Cairns *et al.*,

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2006), (Di Giacinto *et al.*, 2011), (Haberman & Vigna, 2002), (Vigna, 2010), (Gao, 2008), (Nkeki, 2011), (Nkeki & Nwozo, 2012).

In the context of DC pension plans, the problem of finding the optimal investment strategy involving a riskiness asset and two distinct risky assets, and inflation hedging strategy under mean-variance efficient approach has not been reported in published articles. Bjarne Hϕjgaard and Elena Vigna (Hϕjgaard & Vigna, 2007) and Vigna (Vigna, 2010)assumed a constant flow of contributions into the pension scheme. This paper follows the same assumption.

In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by Richardson (Richardson, 1989) via the Martingale approach. (Li & Ng, 2000) solved a mean-variance optimization problem in a discrete-time multi-period framework. (Zhou & Li, 2000) considered a mean-variance in a continuous-time framework. They shown the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, Vigna (2010), (Bielecki *et al.*, 2005), (Hϕjgaard & Vigna, 2007), (Chiu & Li, 2006), (Josa-Fombellida & Rincn-Zapatero, 2008). In this paper, we study a mean-variance approach (MVA) to portfolio selection problem with inflation protection strategy in accumulation phase of a DC pension scheme. Our result shows that inflation-linked bond can be used to hedge inflation risk that is associated with the PPM's wealth. We found that our optimal portfolio is efficient in the mean-variance approach.

The remainder of this paper is organized as follows. In section 2, we present the financial market model problem. In section 3, we present the optimal portfolio and optimal expected terminal wealth of the PPM. The efficient frontier is presented in section 4. In section 5, some numerical examples were presented. Finally, section 6 concludes the paper.

2. The Problem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0, T]\}$, where $\mathcal{F}_t = \sigma(S(s), I(s) : s \leq t)$, where $S(t)$ is stock price process at time $s \leq t$, $I(t)$ is the inflation index at time $s \leq t$. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure and σ^S and σ_I are the volatility vectors of stock and volatility of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^I(t)$, respectively. μ is the appreciation rate for stock. Moreover, σ^S and σ_I are the volatilities for the stock and inflation-linked bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

We assume that the investor faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are tradeable. In this paper, we allow the stock price to be correlated to inflation. The dynamics of the underlying assets are given by (2.1) to (2.3)

$$dC(t) = rC(t)dt, C(0) = 1 \tag{2.1}$$

$$dS(t) = \mu S(t)dt + \sigma_1^S S(t)dW^I(t) + \sigma_2^S S(t)dW^S(t), S(0) = s_0 > 0 \tag{2.2}$$

$$dF(t, I(t)) = (r + \sigma_1\theta^I)F(t, I(t))dt + \sigma_I F(t, I(t))dW(t), F(0) = F_0 > 0 \tag{2.3}$$

where, r is the nominal interest rate, θ^I is the price of inflation risk, $C(t)$ is the price process of the cash account at time t, $S(t)$ is stock price process at time t, $I(t)$ is the inflation index at time t and has the dynamics: $dI(t) = E(q)I(t)dt + \sigma_I I(t)dW(t)$, where $E(q)$ is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $E(q) = r - R$). $F(t, I(t))$ is the inflation-indexed bond price process at time t and $\sigma_I = (\sigma_1, 0)$.
Then, the volatility matrix

$$\Sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix} \tag{2.4}$$

corresponding to the two risky assets and satisfies $det(\Sigma) = \sigma_1\sigma_2^S \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying

$$\theta := \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \theta^I \\ \frac{\mu - r - \theta^I\sigma_1^S}{\sigma_2^S} \end{pmatrix} \tag{2.5}$$

where θ^S is the market price of stock risks and θ^I is the market price of inflation risks (MPIR).

3. The Wealth Process

Let $X(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t. Let c be the contribution rate of PPM.

Definition 3.1. The portfolio process Δ is said to be self-financing if the corresponding wealth process $X(t)$, $t \in [0, T]$, satisfies

$$\begin{aligned} dX(t) &= \Delta^S(t)X(t)\frac{dS(t)}{S(t)} + \Delta^I(t)X(t)\frac{dF(t, I(t))}{F(t, I(t))} + (1 - \Delta^S(t) - \Delta^I(t))X(t)\frac{dC(t)}{C(t)} + cdt, \\ X(0) &= x_0. \end{aligned} \tag{3.1}$$

(3.1) can be re-written in compact form as follows:

$$\begin{aligned} dX(t) &= (X(t)(r + \Delta(t)A) + c)dt + X(t)(\Sigma\Delta'(t))'dW(t)], \\ X(0) &= x_0, \end{aligned} \tag{3.2}$$

where, $A = (\sigma_1\theta^I, \mu - r)'$ and $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix}$. The amount x_0 is the initial fund paid in by the PPM. This amount can be null, if the PPM has just joined the pension scheme without any transfer fund. The PPM enters the plan at initial time, 0 and contributes for T years, thereafter he or she retires and withdraws all his or her entitlement (or converts it into annuity). The aim of the PPM is pursued the two conflicting objectives of maximum expected terminal wealth together with minimum variance of the terminal wealth. PPM seeks to minimize the vector

$$[-E(X(T)), Var(X(T))].$$

Definition 3.2. (Hϕjgaard & Vigna, 2007) The mean-variance optimization problem is defined as

$$\begin{array}{l} Minimize(\Psi_1(\Delta(\cdot)); \Psi_2(\Delta(\cdot))) \equiv (-E(X(T)), Var(X(T))) \\ \text{subject to} \left\{ \begin{array}{l} \Delta(\cdot) \;\; admissible \\ X(\cdot), \Delta(\cdot) \;\; satisfy \;\; (3.2). \end{array} \right. \end{array} \tag{3.3}$$

An admissible strategy $\Delta^*(\cdot)$ is called an efficient strategy if there exists no admissible strategy $\Delta(\cdot)$ such that

$$\Psi_1(\Delta(\cdot)) \leq \Psi_1(\Delta^*(\cdot)), \Psi_2(\Delta(\cdot)) \leq \Psi_2(\Delta^*(\cdot)) \tag{3.4}$$

and at least one of the inequalities holds strictly. In this case, the point $(\Psi_1(\Delta(\cdot)), \Psi2(\Delta(\cdot))) \in \mathbf{R}^2$ is called an efficient point and the set of all efficient points is called the efficient frontier.

Hϕjgaard and Vigna (Hϕjgaard & Vigna, 2007) established that solving (3.3) will address the following problem

$$\min_{\Delta(\cdot)}[-E(X(T)) + \delta Var(X(T))], \tag{3.5}$$

where $\delta > 0$. (3.5) is not easy to tackle with standard stochastic control techniques, see (Hϕjgaard & Vigna, 2007). Zhou and Li (Zhou & Li, 2000) and Li and Ng (Li & Ng, 2000) shown that it is possible to transform (3.5) into a tractable one. They were able to show that (3.5) is equivalent to the following problem

$$\min_{\Delta(\cdot)} E[\delta X(T)^2 + \omega X(T)], \tag{3.6}$$

which is a linear-quadratic control problem. Zhou and Li (Zhou & Li, 2000) and Li and Ng (Li & Ng, 2000) further show that if $\Delta(\cdot)$ is a solution of (3.5), then it is a solution of (3.6) with

$$\omega^* = 1 + 2\delta E(X^*(T)). \tag{3.7}$$

Our aim now is to solve

$$\begin{array}{l} Minimize(\Psi(\Delta(\cdot)), \delta, \omega) \equiv E[\delta X(T)^2 + \omega X(T)] \\ \text{subject to} \left\{ \begin{array}{l} \Delta(\cdot) \;\; \text{admissible} \\ X(\cdot), \Delta(\cdot) \;\; \text{satisfy (3.2)} \end{array} \right. \end{array} \tag{3.8}$$

3.1. Optimal Portfolio Process

We now follow the approach presented by Zhou and Li (Zhou & Li, 2000) and Hϕjgaard and Vigna (Hϕjgaard & Vigna, 2007). Let $\eta = \frac{\omega^*}{2\delta}$ and $\Phi(t) = X(t) - \eta$. It therefore resulted that our problem is equivalent to solving

$$\min_{\Delta(\cdot)} E\left[\frac{1}{2}\delta\Phi(T)^2\right] = \min_{\Delta(\cdot)} \Psi(\Delta(\cdot); \delta), \tag{3.9}$$

where $\Phi(t)$ satisfies the stochastic differential equation

$$\begin{array}{l} d\Phi(t) = ((\Phi(t) + \eta)(\Delta(t)A + r) + c)dt + (\Phi(t) + \eta)(\Sigma\Delta'(t))'dW(t), \\ \Phi(0) = x_0 - \eta. \end{array} \tag{3.10}$$

We now adopt the dynamic programming approach to solve the standard stochastic optimal control problem (3.9) and (3.10). Let define the value function

$$U(t,\Phi) = \inf_{\Delta(\cdot)} E_{t,\Phi}\left[\frac{1}{2}\delta\Phi(T)^2\right] = \min_{\Delta(\cdot)} \Psi(\Delta(\cdot);\delta). \tag{3.11}$$

Then U which is assume to be a convex utility function of Φ, satisfies the Hamilton-Jacobi-Bellmann (HJB) equation

$$\begin{array}{l} \inf_{\Delta\in\mathbf{R}}\left\{U_t + ((\Phi+\eta)(\Delta(t)A+r)+c)U_\Phi + \frac{1}{2}(\Phi+\eta)^2\Sigma\Delta(t)\Sigma'\Delta'(t)U_{\Phi\Phi}\right\} = 0, \\ U(T,\Phi) = \frac{1}{2}\delta\Phi^2. \end{array} \tag{3.12}$$

Let $\mathcal{H}$ be the Hamiltonian such that

$$\mathcal{H} = ((\Phi+\eta)(\Delta(t)A+r)+c)U_\Phi + \frac{1}{2}(\Phi+\eta)^2\Sigma\Delta(t)\Sigma'\Delta'(t)U_{\Phi\Phi}. \tag{3.13}$$

Then,

$$\frac{\partial\mathcal{H}}{\partial\Delta(t)} = (\Phi+\eta)AU_\Phi + (\Phi+\eta)^2\Sigma\Sigma'\Delta'(t)U_{\Phi\Phi} = 0$$

Therefore,

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}AU_\Phi}{(\Phi+\eta)U_{\Phi\Phi}}. \tag{3.14}$$

Substituting (3.14) into (3.12), we obtain the following non-linear partial differential equation for the value function

$$U_t + (r(\Phi+\eta)+c)U_\Phi - \frac{1}{2}M\frac{U_\Phi^2}{U_{\Phi\Phi}} = 0, \tag{3.15}$$

where, $M = [(\Sigma\Sigma')^{-1}A]'A$. Let assume the solution of the form, see Hϕjgaard and Vigna (2007) and Vigna (2010),

$$U(t,\Phi) = P(t)\Phi^2 + Q(t)\Phi + R(t). \tag{3.16}$$

Finding the partial derivatives of U in (3.16) with respect to U_t, U_Φ and $U_{\Phi\Phi}$ and then substitute into (3.15), we have the following system of ordinary differential equations (ODEs):

$$\left.\begin{array}{l} P'(t) + (2r-M)P(t) = 0 \\ Q'(t) + 2(r\eta+c)P(t) + (r-M)Q(t) = 0 \\ R'(t) + (r+r\eta+c)Q(t) - \frac{1}{4}M\frac{Q(t)^2}{P(t)} = 0 \end{array}\right\} \tag{3.17}$$

with boundary conditions

$$P(T) = \frac{1}{2}\delta, Q(T) = 0, R(T) = 0.$$

Solving the system of ODEs (3.17) using the boundary conditions $P(T) = \frac{1}{2}\delta, Q(T) = 0, R(T) = 0$, we have

$$\left.\begin{array}{l} P(t) = \frac{\delta}{2}\exp[(2r+M)(T-t) \\ Q(t) = \frac{\delta(c+r\eta)}{2M+r}\exp[-(M-r)(T-t)](\exp[(2M+r)(T-t)]-1) \\ R(t) = \int_t^T\left((r+r\eta+c)Q(s) - \frac{1}{4}M\frac{Q(s)^2}{P(s)}\right)ds \end{array}\right\} \tag{3.18}$$

Hence, replacing the partial derivatives of U in (3.14), the optimal fraction of portfolio to be invested in the two risky assets at time time t, becomes

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{\Phi+\eta}G_{\Delta}(t), \tag{3.19}$$

where,

$$G_{\Delta}(t) = \Phi+\eta - \frac{\eta(2M+r) - (r\eta+c)(1-\exp[-(2M+r)(T-t)])}{2M+r}.$$

Now, replacing $\Phi+\eta$ with x in (3.19), we have

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{x}\left[x - \frac{\eta(2M+r) - (r\eta+c)(1-\exp[-(2M+r)(T-t)])}{2M+r}\right] \tag{3.20}$$

Simplifying (3.20), we have

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{x}\bar{G}(t), \tag{3.21}$$

where,

$$\bar{G}(t) = x - \frac{\eta(2M+\exp[-(2M+r)(T-t)])}{(2M+r)} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}.$$

3.2. Expected Optimal Wealth

In this subsection, we determine the expected wealth that will accrued to the PPM at the final time horizon. We also consider in this subsection, the second moment of the expected final wealth of the PPM. These will enable us to established the efficient frontier in the next section.

Substituting (3.20) into (3.2), we have that the evolution of wealth of the PPM under optimal control $X^*(t)$ is obtained as follows:

$$\begin{aligned} dX^*(t) &= \{(r-M)X^*(t) + \frac{\eta M(2M-\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{cM(1-\exp[-(2M+r)(T-t)])}{2M+r} + c\}dt - \Sigma^{-1}A\{X^*(t) \\ &- \frac{\eta(2M+\exp[-(2M+r)(T-t)])}{2M+r} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}\}dW(t). \end{aligned} \tag{3.22}$$

Then, applying Itô lemma to (3.22), we obtain the SDE that satisfies the evolution of $X^{*2}(t)$:

$$\begin{aligned} dX^{*2}(t) &= \{(2r-M)X^{*2}(t) + 2cX^*(t) + M[\frac{\eta(2M+\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}]^2\}dt - 2\Sigma^{-1}A\{X^{*2}(t) \\ &- \frac{\eta X^*(t)(2M-\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{cX^*(t)(1-\exp[-(2M+r)(T-t)])}{2M+r}\}dW(t) \end{aligned} \tag{3.23}$$

Taking the mathematical expectation on both sides of (3.22) and (3.23), we have the following expected value of the optimal wealth and the expected value of its square:

$$
\begin{aligned}
&dE(X^*(t)) = E[(r-M)X^*(t) + \frac{\eta M(2M - \exp[-(2M+r)(T-t)])}{2M+r} \\
&+ \frac{cM(1-\exp[-(2M+r)(T-t)])}{2M+r} + c]dt, \\
&E(X(0)) = x_0.
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
&dE(X^{*2}(t)) = E[(2r-M)X^{*2}(t) + 2cX^*(t) \\
&+ M\left(\frac{\eta(2M + \exp[-(2M+r)(T-t)])}{2M+r} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}\right)^2]dt, \\
&E(X^{*2}(t)) = x_0^2.
\end{aligned}
\tag{3.25}
$$

Solving (3.24) and (3.25), we have the following:

$$
\begin{aligned}
&E(X^*(t)) = x_0 \exp[-(M-r)t] + \frac{2M^2\eta}{(M-r)(2M+r)}(1+\exp[-(M-r)t]) \\
&+ \frac{c}{3(2M+r)} \exp[-2MT - r(T-t)](\exp[-Mt] - \exp[2Mt]) \\
&+ \frac{c(3M+r)}{(M-r)(2M+r)}(1-\exp[-(M-r)t]) \\
&- \frac{\eta \exp[-Mt - r(T-t)]}{3(2M+r)}(\exp[-3M(T-t)] - \exp[-2MT]),
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
&E(X^{*2}(t)) = x_0^2 \exp[-(M-2r)t] + \frac{c^2 \exp[-2r(T-t) - M(4T+t)](\exp[5Mt]-1)}{5(2M+r)^2} \\
&+ \frac{2c^2(3M+r)\exp[-(M-r)t]}{r(M-r)(2M+r)} - \frac{\eta^2 \exp[-2r(T-t) - M(4T+t)]}{5(2M+r)^2} \\
&+ \frac{12cM^2\eta}{(2M+r)^2(M^2-3Mr+2r^2)} - \frac{c^2(4M+r)(3M+2r)\exp[-Mt+2rt]}{r(M-2r)(2M+r)^2} + D_1(t) \\
&+ \frac{c^2(13M^2+9Mr+2r^2)}{(2M+r)^2(M^2-3Mr+2r^2)} + D_2(t) - \frac{2c\eta}{5(2M+r)^2} \\
&(\exp[-2r(T-t) - M(4T+t)] - \exp[-2(2M+r)(T-t)]) \\
&- \frac{2c\eta \exp[-(2MT+Mt) - r(T-t)]}{r(6M+3r)} - \frac{2c\eta(M(5+6M)+r)\exp[-(2M+r)(T-t)]}{3(3M-r)(2M+r)^2} \\
&+ \frac{4c(M(M+r+Mr))\eta \exp[-(M-r)t - r(T-t) - 2MT]}{r(3M-r)(2M+r)^2} + \\
&\frac{2cx_0}{r}\exp[-(M-r)t](1-\exp[rt]) \\
&- \frac{2c\eta \exp[-(M-r)t]}{r(r^2+Mr-2M^2)} - \frac{4M^2\eta}{(M-2r)(2M+r)^2}\left(M\eta + \frac{2c(M+r)}{r}\right)\exp[-(M-2r)t]
\end{aligned}
\tag{3.27}
$$

where,

$$\begin{aligned} D_1(t) &= \frac{2c^2 \exp[-r(T-t) - M(2T+t)]}{3r(3M-r)(2M+r)^2} \\ &\times (6M^2 + Mr(1 + 5\exp[3Mt]) - r^2(1 - \exp[3Mt]) - 6M(M-r)\exp[rt]), \end{aligned} \tag{3.28}$$

$$\begin{aligned} D_2(t) &= -\frac{4M^2\eta^2 \exp[-r(T-t) - M(2T+t)](\exp[3Mt] - \exp[rt])}{(3M-r)(2M+r)^2} \\ &+ \frac{\eta^2(\exp[-2(2M+r)(T-t)] + \frac{20M^3}{M-2r})}{5(2M+r)^2} \end{aligned} \tag{3.29}$$

At terminal time, that is, at $t = T$, we have:

$$\begin{aligned} E(X^*(T)) &= x_0 \exp[-(M-r)T] + \frac{2M^2\eta}{(M-r)(2M+r)}(1 + \exp[-(M-r)T]) \\ &+ \frac{c}{3(2M+r)}(\exp[-3MT] - 1) + \frac{c(3M+r)}{(M-r)(2M+r)}(1 - \exp[-(M-r)T]) \\ &- \frac{\eta \exp[-MT]}{3(2M+r)}(1 - \exp[-2MT]), \end{aligned} \tag{3.30}$$

$$\begin{aligned} E(X^{*2}(T)) &= x_0^2 \exp[-(M-2r)T] + \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1 - \exp[-(M-2r)T]) \\ &+ \frac{(\eta+c)^2}{5(2M+r)^2}(1 - \exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1 - \exp[-3MT + rT]) \\ &- \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\ &- \frac{2c}{r}\left(x_0(1 + \exp[rT]) + \frac{\eta}{r^2 + Mr - 2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\ &+ \frac{4c^2M^2 \exp[-3MT](1 - (1 - \frac{r}{M} + \frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\ &\left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2 + 9Mr + 2r^2)}{(M-r)(M-2r)(2M+r)^2} \\ &+ \frac{2c(c(M-r)\exp[-3MT] + 5M(c-\eta) + r(c-\eta))}{3(3M-r)(2M+r)^2} \\ &- \frac{2c\eta \exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r} + 1\right)\exp[rT]\right). \end{aligned} \tag{3.31}$$

Since $\eta = \frac{\omega^*}{2\delta}$ and ω^* is as defined in (3.7), we have that

$$\begin{aligned} \eta &= \frac{3(M-r)(2M+r)}{3(M-r)(2M+r) - 6M^2(1 + \exp[-(M-r)T]) + (M+r)\exp[-MT](1 - \exp[-2MT])} \\ &\times \left(\frac{1}{2\delta} + x_0 \exp[-(M-r)T] + \frac{c(\exp[-3MT] - 1)}{3(2M+r)} + \frac{c(3M+r)(1 - \exp[-(M-r)T])}{(M-r)(2M+r)}\right). \end{aligned} \tag{3.32}$$

Observe that η is a decreasing function of δ. Therefore, the expected optimal terminal wealth of the PPM can be re-express in terms of δ as follows:

$$\begin{aligned} E(X^*(T)) =& \\ &\times\left(1+\frac{6M^2(1+\exp[-(M-2r)T])-(M-r)\exp[-MT](1-\exp[-2MT])}{3(M-r)(2M+r)-6M^2(1+\exp[-(M-r)T])+(M+r)\exp[-MT](1-\exp[-2MT])}\right) \\ &\times\left(x_0\exp[-(M-r)T]+\frac{c(\exp[-3MT]-1)}{3(2M+r)}+\frac{c(3M+r)(1-\exp[-(M-r)T])}{(M-r)(2M+r)}\right) \\ &+\frac{6M^2(1+\exp[-(M-2r)T])-(M-r)\exp[-MT](1-\exp[-2MT])}{2\delta(3(M-r)(2M+r)-6M^2(1+\exp[-(M-r)T])+(M+r)\exp[-MT](1-\exp[-2MT]))} \end{aligned} \tag{3.33}$$

Observe that the expected optimal terminal wealth for the PPM is the sum of the wealth that invested would get for investing the whole portfolio always in both the riskless and the risky assets plus a term,

$$\frac{6M^2(1+\exp[-(M-2r)T])-(M-r)\exp[-MT](1-\exp[-2MT])}{2\delta(3(M-r)(2M+r)-6M^2(1+\exp[-(M-r)T])+(M+r)\exp[-MT](1-\exp[-2MT]))}.$$

This term depends both on the goodness of the risky assets with respect to the riskless asset and on the weight given to the minimization of the variance. Hence, the higher the value of M (which is the Sharpe ratio of the risky assets, stock and inflation-linked bond), the higher the expected optimal terminal wealth, everything else being equal. The higher the parameter given to the minimization of the variance of the terminal wealth, δ, the lower its mean.

4. The Efficient Frontier

We now establish the efficient frontier for the three classes of assets in the investment portfolio. From (3.30), we have that

$$E(X^*(T)) = x_0\exp[-(M-r)T]+\lambda, \tag{4.1}$$

where,

$$\begin{aligned} \lambda =& \frac{2M^2\eta}{(M-r)(2M+r)}(1+\exp[-(M-r)T]) \\ &+\frac{c}{3(2M+r)}(\exp[-3MT]-1)+\frac{c(3M+r)}{(M-r)(2M+r)}(1-\exp[-(M-r)T]) \\ &-\frac{\eta\exp[-MT]}{3(2M+r)}(1-\exp[-2MT]). \end{aligned} \tag{4.2}$$

$$E(X^{*2}(T)) = x_0^2\exp[-(M-2r)T]+\psi, \tag{4.3}$$

where,

$$
\begin{aligned}
\psi = {} & \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1-\exp[-(M-2r)T]) \\
+ {} & \frac{(\eta+c)^2}{5(2M+r)^2}(1-\exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1-\exp[-3MT+rT]) \\
- {} & \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\
- {} & \frac{2c}{r}\left(x_0(1+\exp[rT]) + \frac{\eta}{r^2+Mr-2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\
+ {} & \frac{4c^2M^2\exp[-3MT](1-(1-\frac{r}{M}+\frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\
\times {} & \left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2+9Mr+2r^2)}{(M-r)(M-2r)(2M+r)^2} \\
+ {} & \frac{2c(c(M-r)\exp[-3MT]+5M(c-\eta)+r(c-\eta))}{3(3M-r)(2M+r)^2} \\
- {} & \frac{2c\eta\exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r}+1\right)\exp[rT]\right).
\end{aligned} \tag{4.4}
$$

Therefore,

$$
\begin{aligned}
Var(X^*(T)) &= x_0^2\exp[-(M-2r)T] + \psi - (E(X^*(T)))^2 \\
&= x_0^2\exp[rT]\exp[-(M-r)T] + \psi - (E(X^*(T)))^2 \\
&= x_0\exp[rT](E(X^*(T)) - \lambda) + \psi - (E(X^*(T)))^2 \\
&= x_0\exp[rT]E(X^*(T) - \lambda x_0\exp[rT] + \psi - (E(X^*(T)))^2 \\
&= E(X^*(T))(x_0\exp[rT] - x_0\exp[-(M-r)T] - \lambda) + \psi - \lambda x_0\exp[rT] \\
&= E(X^*(T))(x_0\exp[rT](1-\exp[-MT]) - \lambda) + \psi - \lambda x_0\exp[rT].
\end{aligned}
$$

Therefore,

$$
E(X^*(T)) = \frac{\lambda x_0\exp[rT] - \psi}{x_0\exp[rT](1-\exp[-MT]) - \lambda} + \frac{\sigma^2(X^*(T))}{x_0\exp[rT](1-\exp[-MT]) - \lambda}. \tag{4.5}
$$

This show that the expected terminal wealth of the PPM is a function of its variance. The efficient frontier in the mean-variance diagram is a straight line with gradient $\frac{1}{x_0\exp[rT](1-\exp[-MT])-\lambda}$ which measures the rate at which the terminal wealth will increase or decrease as the variance increases by one unit. If $x_0\exp[rT](1-\exp[-MT]) < \lambda$, we have a negative gradient. If $x_0\exp[rT](1-\exp[-MT]) > \lambda$, we have a positive gradient. If $x_0\exp[rT](1-\exp[-MT]) = \lambda$, we have an infinite gradient. Note that if the gradient is negative, it implies that the mean will increase as the variance decreases. If the gradient is positive, it implies that the mean will increase as the variance increases. If the gradient is infinite, we have that the mean will tends to negative infinity. Observe that if the PPM entered the scheme with no initial endowment, then (4.5) will become

$$
E(X^*(T)) = \frac{\bar{\psi}}{\lambda} - \frac{\sigma^2(X^*(T))}{\lambda}. \tag{4.6}
$$

where,

$$\begin{aligned}
\bar{\psi} &= \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1-\exp[-(M-2r)T]) \\
&+ \frac{(\eta+c)^2}{5(2M+r)^2}(1-\exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1-\exp[-3MT+rT]) \\
&- \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\
&- \frac{2c}{r}\left(\frac{\eta}{r^2+Mr-2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\
&+ \frac{4c^2M^2\exp[-3MT](1-(1-\frac{r}{M}+\frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\
&\times \left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2+9Mr+2r^2)}{(M-r)(M-2r)(2M+r)^2} \\
&+ \frac{2c(c(M-r)\exp[-3MT]+5M(c-\eta)+r(c-\eta))}{3(3M-r)(2M+r)^2} \\
&- \frac{2c\eta\exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r}+1\right)\exp[rT]\right). \qquad (4.7)
\end{aligned}$$

In this case, the gradient of the mean-variance portfolio selection becomes $-\frac{1}{\lambda}$ and the intercept is $\frac{\bar{\psi}}{\lambda}$.

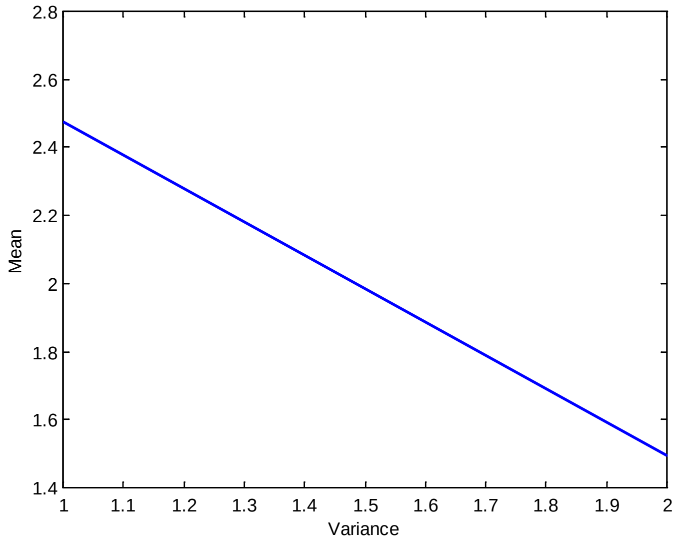

Figure 1: Efficient Frontier. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, and $\alpha = 0.05$.

Figure 1 shows the efficient frontier of portfolios in the mean-variance plan and reports the points $(\sigma^2(X^*(T)), E(X^*(T)))$ for each strategy under consideration. Observe that figure 1 has negative gradient of −0.979322 and intercept of 3.45372. Observe that the higher the variance, the lower the mean and vice versa. But, with that presents of inflation-linked bond as one of the risky assets, the variance could be minimized.

5. Numerical Example

Suppose a market involve a cash account with nominal annual interest rate 4%, an inflation-linked bond with a nominal annual appreciation rate $r + \sigma_I \theta^I$, where $r = 4\%$ is the nominal annual interest rate, $\sigma_I = 35\%$ is the inflation volatility and $\theta^I = 30\%$ is the market price of inflation risks, and a stock with a nominal annual appreciation rate 9.2% and a standard deviations arising from inflation and stock market 38% and 45% respectively. Suppose also that the following parameters (which have been defined earlier) take the values as follows: $c = 0.07$ million, $x_0 = 1$ million, $\delta = 0.05$ and $T = 5$ (years), we have the following results.

A PPM who contributes a constant flow of 0.07 million and have initial wealth $x_0 = 1$ million in the pension scheme and wishes to obtain an expected wealth between 0 − 2.5 million has a portfolio value in inflation-linked bond as obtain in figure 2 and stock as obtain in figure 3 for 5 year period. Under the same strategy but for a period of 30 years, we have the results for inflation-linked bond and stock in figure 4 and 5, respectively.

In particular, at the initial time $t = 0$, $\Delta^S(0, x_0) = 0.280635$ million and $\Delta^I(0, x_0) = -1.60143$ million. These imply that the inflation-linked bond needs to be shorten for an amount 1.60143 million and then invest into cash account which is already having an amount 0.617935 million together with the initial endowment 1 million. It implies that at $t = 0$, a total of 3.219365 million should be invested in cash account.

We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

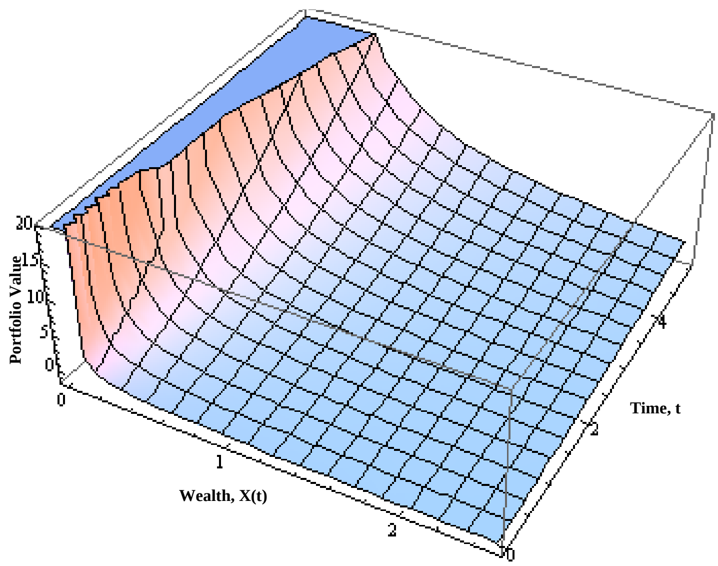

Figure 2: Portfolio Value in Inflation-linked Bond. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

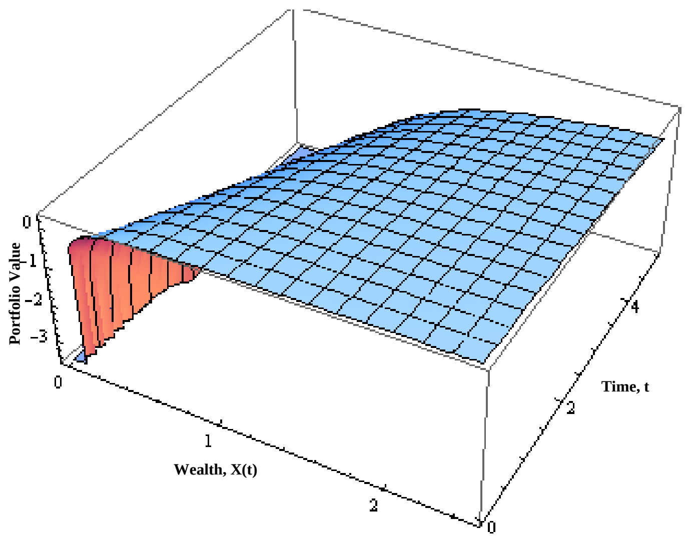

Figure 3: Portfolio Value in Stock. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

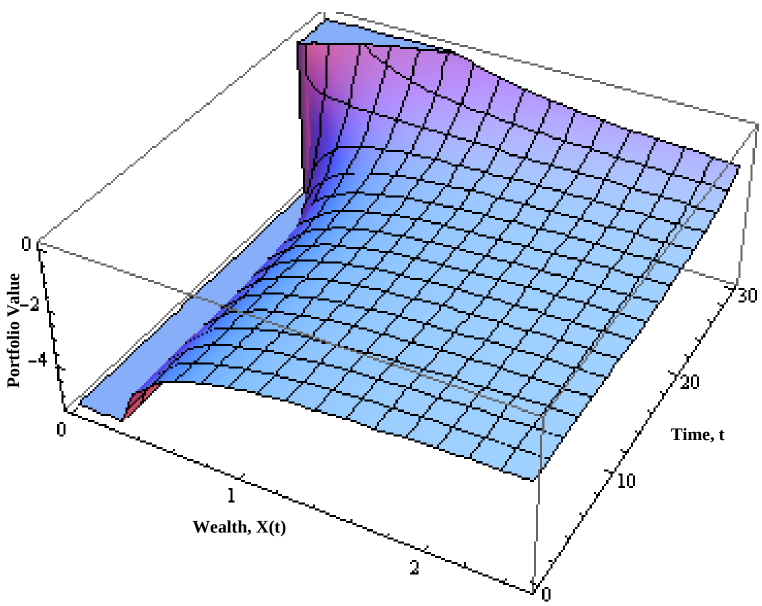

Figure 4: Portfolio Value in Inflation-linked Bond. We take $x_0 = 1$, $\delta = 0.05$, $T = 30$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

Table 1: EPMV at Different Value of c

c	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
0.07	2.11382	-0.370427	2.24252	1.23677
0.10	2.16595	-0.379561	2.40778	1.17607
0.18	2.30494	-0.403918	2.84850	1.02772
0.50	2.86089	-0.501344	4.61135	0.63131
1.00	3.72958	-0.653573	7.36580	0.64283

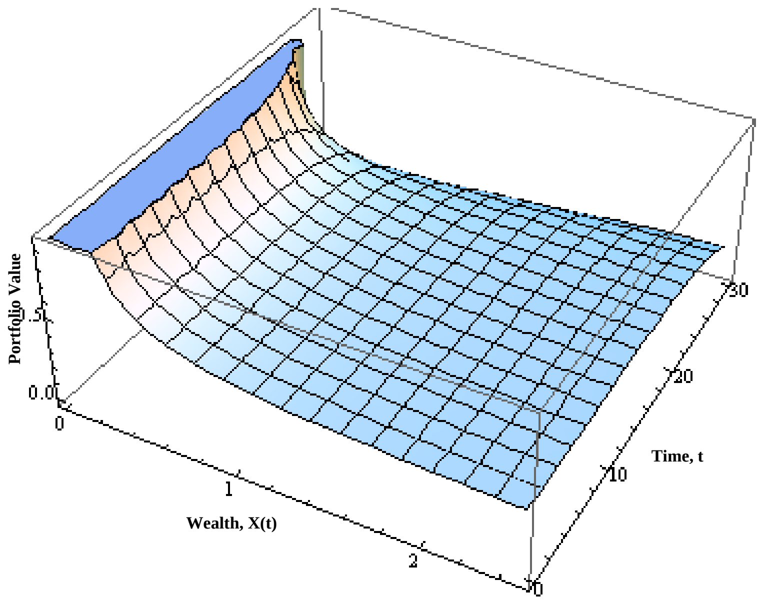

Figure 5: Portfolio Value in Stock. We take $x_0 = 1$, $\delta = 0.05$, $T = 30$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

where, EPMV stands for Expected Portfolio, Mean and Variance

Table 2: EPMV at Different Value of θ^I

θ^I	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
-0.40	1.04336	-0.62614	-0.73204	1.92241
-0.30	-0.0528	0.036369	0.63063	1.46643
-0.20	-0.7444	0.672109	2.08936	1.05585
-0.10	-0.1036	0.218662	1.95446	0.44016
0.12	-18.9559	-0.69169	-1.8832	161.417
0.20	-0.56135	0.052899	1.57551	0.00502
0.30	2.11382	-0.370427	2.24252	1.23677
0.40	0.36761	-0.081166	0.84008	1.92241

Table 3: EPMV at Different Value of x_0

x_0	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
1.00	2.11382	-0.370427	2.24252	1.236770
2.00	2.36116	-0.413770	3.02677	0.839136
2.24	2.42114	-0.424282	3.21697	0.823284
2.25	2.42299	-0.424606	3.22284	0.823295
3.00	2.60849	-0.457113	3.81103	0.976339

Table 4: EPMV at Different Value of δ

δ	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
5.00000	-1.34329	0.235398	1.1806	0.253598
0.50000	-1.02901	0.180323	1.27714	0.155583
0.05000	2.11382	-0.370427	2.24252	1.23677
0.00500	33.5421	-5.87793	11.8963	218.182
0.00050	347.825	-60.953	108.434	23001
0.00005	3490.66	-611.704	1073.81	2312170

5.1. Discussion of the Results in the Tables

From table 1, observe that as the contributions of the PPM increases, the portfolio value in inflation-linked bond increases while the portfolio value in stock decreases. Observe also that as the contributions increases, the expected terminal wealth increases and the variance decreases, which is an interesting result since the aim of an investor is to maximize wealth and minimize risks. The reason for this, is that the inflation risks in the investment profile have been hedged by the inflation-linked bond. This shows that inflation risks on the contributions of the PPM can be hedged by the inflation-linked bond. We conclude that the higher the contributions of the PPM, the higher the expected wealth and vice versa, which is an expected result. The expected optimal wealth (as in above) can be actualized only when the entire portfolio is invested in inflation-linked bond.

From table 2, we found that, when the market price of inflation risks, θ^I, is -0.40, the portfolio value in inflation-linked bond, $\Delta^I(T)$ at $T = 5$, is 1.04336 million and stock, $\Delta^S(T)$ is -0.62614 million. This means that the portfolio value in stock should be shorten by an amount 0.62614 million and invest it in inflation-linked bond. Observe also that when $\theta^I = -0.40$, the expected wealth is -0.73204 million and variance 1.92241 million. This shows negative expected wealth with high variance. Similar interpretation go to when $\theta^I = -0.30, -0.20$, and -0.10. Observe that at $\theta^I = 0.12$, $\Delta^I(T) = -18.9559$ million and $\Delta^S(T) = -0.69169$ million. This implies that that the entire portfolio values of the PPM should remain only in cash account. This is because the risks associated with the portfolio in stock and inflation-linked bond are very high. At $\theta^I = 0.20$, $\Delta^I(T) = -0.56135$ million and $\Delta^S(t) = 0.052899$ with expected wealth of 1.57551 million and variance of 0.00502. This means that the entire portfolio should remain in stock and cash account. Observe also that the PPM will have a higher expected wealth at $\theta^I = 0.30$. This occur when the entire portfolio is invested in inflation-linked bond.

From table 3, observe that the higher the initial endowment of the PPM, the higher the portfolio value in inflation-linked bond and the expected wealth, the lower the variance, which is an interesting result since the aim of an investor is to minimize risks and maximize wealth. The reason for the gradual reduction of the variance is because the inflation risks on the initial endowment has be hedged due to the presents of an inflation-linked bond in the investment profile. We therefore conclude that inflation-linked bond is an ”inflation risks fighter”.

From table 4, observe that the higher the weight given to the minimization of the variance, the lower the portfolio value in stock and inflation-linked bond, and vice versa, which is an expected result. Therefore, it is optimal to invest the entire portfolio into cash account when $\delta = +\infty$. We found that the lower the value of δ, higher the portfolio value in inflation-linked bond and expected wealth. This also lead to high variance.

6. Conclusion

In this paper, we have considered a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension scheme. The optimal portfolio and optimal expected terminal wealth for the pension plan member (PPM) were established. The efficient frontier was obtained for the three assets class. It was found that inflation-linked bond is an ”inflation risks fighter”.

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How many $SU(4)_L \otimes U(1)_Y$ Gauge models ?

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Abstract

We prove in this letter that the general method of solving gauge models with high symmetries proposed by Cotăescu several years ago can predict precisely two distinct classes of $SU(4)_L \otimes U(1)_Y$ electroweak models. Their fermion representations with respect to this gauge group are exactly obtained in rach case.

Keywords: 3-4-1 gauge models, electric charge assignment.
PACS numbers: 12.10.Dm, 12.60.Fr, 12.60.Cn.

1. Introduction

One of the most stringent topics in modern theoretical particle physics is to find the proper extension of the Standard Model (SM) able to accommodate (or even to predict) the new and richer observed phenomenology at colliders or in cosmology such as: (i) neutrino oscillation, (ii) 126 GeV Higgs signal at CERN-LHC, (iii) new Z'gauge boson. etc. More than a decade ago, Cotăescu (Cotăescu, 1997) proposed a general method for solving chiral gauge models of the type $SU(3)_c \otimes SU(N)_L \otimes U(1)_Y$ that undergo a spontaneous symmetry breaking (SSB) in its electroweak sector. Based on a particular parametrization of the scalar sector leading to an unusual Higgs mechanism to accomplish the SSB, the method established itself as a successful tool in investigating the phenomenology of interest at present facilities (CERN-LHC, Tevatron, LEP etc). Also, it can give some estimates of the expected processes.

We focus in this letter on the classification job the method supplies in the case of the $SU(3)_c \otimes SU(4)_L \otimes U(1)_Y$ gauge models, subject to a sustained research (R. Foot & Tran, 1994), (Pisano & Pleitez, 1995), (Doff & Pisano, 1999), (Doff & Pisano, 2001), (Fayyazuddin & Riazuddin, 2004), (W. A. Ponce & Sanchez, 2004), (L. A. Sanchez & Ponce, 2004), (Ponce & Sanchez, 2004), (L. A. Sanchez & Zuluaga, 2008), (Riazuddin & Fayyazuddin, 2008), (Palcu, 2009*c*), (Palcu,

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2009*a*), (Nisperuza & Sanchez, 2009), (Palcu, 2009*e*), (Palcu, 2009*d*), (Villada & Sanchez, 2009), (Palcu, 2009*b*), (Jaramillo & Sanchez, 2011), (Palcu, 2012) lately. More precisely, we obtain all the classes allowed by the general method when applied to this particular gauge group. Of course, in all these models the $SU(3)_c$ group is the color group of chromodynamics and it remains vector-like as usual, contributing in the case at hand only to the cancellation of the axial anomaly. Therefore it will be no longer mentioned, as the extension takes place only in the electroweak sector.

The paper is organized as follows: Sec. 2 briefly displays the main results of the general method with a special emphasise on the charge operators which are worked out in detail, while in Sec.3 our conclusions are sketched.

2. Charge operators in $SU(4)_L \otimes U(1)_Y$ models

To begin with, we present some general results of the method involved here with a particular focus on the charge operators and their concrete expressions.

2.1. Main results of the general method

2.1.1. Irreducible representations of $SU(N)_L \otimes U(1)_Y$

When constructing a gauge model, one must consider proper fermion representations of the $SU(N)_L \otimes U(1)_Y$ gauge group. usually, these are the fundamental irreducible unitary representations (irreps) $\mathbf{n}$ and $\mathbf{n}^*$ of the $SU(N)$group. They supply different classes of tensors of ranks (r, s) as direct products like $(\otimes \mathbf{n})^r \otimes (\otimes \mathbf{n}^*)^s$. These tensors exhibit r lower and s upper indices for which the notation $i, j, k, \cdots = 1, \cdots, n$. The irrep ρ of $SU(N)$ by indicating its dimension, $\mathbf{n}_\rho$. The $su(n)$ Lie algebra can have different parameterizations , but we prefer here a hybrid basis (see Ref. (Cotăescu, 1997)) consisting of $n-1$ diagonal generators of the Cartan sub-algebra, $D_{\hat{i}}$, labeled by indices $\hat{i}, \hat{j}, ...$ ranging from 1 to $n-1$, and the generators $E^i_j = H^i_j/\sqrt{2}$, $i \neq j$, related to the off-diagonal real generators H^i_j. We got thus the elements $\xi = D_{\hat{i}}\xi^{\hat{i}} + E^i_j \xi^j_i \in su(n)$ now parameterized by $n-1$ real parameters, $\xi^{\hat{i}}$, and by $n(n-1)/2$ c-number ones, $\xi^i_j = (\xi^j_i)^*$, for $i \neq j$. That is a suitable choice since the parameters ξ^i_j can be directly associated to the c-number gauge fields due to the factor $1/\sqrt{2}$ which gives their correct normalization. In addition, this basis ensures a convenient trace orthogonality relations:

$$Tr(D_{\hat{i}}D_{\hat{j}}) = \frac{1}{2}\delta_{\hat{i}\hat{j}}, \quad Tr(D_{\hat{i}}E^i_j) = 0\,, \quad Tr(E^i_j E^k_l) = \frac{1}{2}\delta^i_l \delta^k_j\,. \tag{2.1}$$

If one deals with different irreps, ρ of the $su(n)$ algebra one denotes $\xi^\rho = \rho(\xi)$ for each $\xi \in su(n)$ such that the corresponding basis-generators of the irrep ρ become $D^\rho_{\hat{i}} = \rho(D_{\hat{i}})$ and $E^{\rho\, i}_j = \rho(E^i_j)$.

2.1.2. Fermion sector

The $U(1)_Y$ transformations corresponding to the new hypercharge are simply phase factor multiplications. Therefore - once the coupling constants g for $SU(n)_L$ and g' for the $U(1)_Y$ are established - the transformation rule of the fermion tensor L^ρ with respect to the whole gauge group yields:

$$L^\rho \to U(\xi^0, \xi)L^\rho = e^{-i(g\xi^\rho + g' y_{ch}\xi^0)}L^\rho \tag{2.2}$$

where $\xi = \in su(n)$ and y_{ch} is the chiral hypercharge defining the irrep of the $U(1)_Y$ group parametrized by ξ^0. In order to simplify the notations, the general method used to deal with the character $y = y_{ch}g'/g$ instead of the chiral hypercharge y_{ch}. This small mathematical artifice does not alter at all the results. The irreps of the whole gauge group $SU(n)_L \otimes U(1)_Y$ are uniquely determined by identifying the dimension of the $SU(n)$ tensor and its character y for particular representations $\rho = (\mathbf{n}_\rho, y_\rho)$ of interest in each case.

2.1.3. Electric and neutral charges

In order to introduce specific interaction among fermions, a proper mechanism to conceive couplings must be set up. This goal is achieved by postulating the covariant derivatives in the manner: $D_\mu L^\rho = \partial_\mu L^\rho - ig(A^a_\mu T^\rho_a + y_\rho A^0_\mu)L^\rho$. Here T^ρ_a are generators (regardless they are diagonal or off-diagonal) defining the $su(n)$ algebra, expressed in the representation ρ. The gauge fields in our notation are $A^0_\mu = (A^0_\mu)^*$ and $A_\mu = A^+_\mu \in su(n)$ respectively.

The charge spectrum of the general method is essentially related to the problem of finding the basis of the physical neutral bosons after separating the electromagnetic massless A^{em}_μ. It corresponds to the residual $U(1)_{em}$ symmetry, that is to the one-dimensional subspace of the parameters ξ^{em} in the parameter space $\{\xi^0, \xi^{\hat{i}}\}$ of the whole Cartan sub-algebra. It is uniquely determined by the $n-1$ - dimensional unit vector ν and the angle θ giving the subspace equations $\xi^0 = \xi^{em}\cos\theta$ and $\xi^{\hat{i}} = \nu_{\hat{i}}\xi^{em}\sin\theta$.

The remaining massive neutral gauge fields $A'^{\hat{i}}_\mu$ will exhibit non-diagonal mass matrix successively the SSB via a proper Higgs mechanism (whose details we will overpass here). The mass basis can be reached by resorting to a $SO(n-1)$ transformation, namely $A'^{\hat{i}}_\mu = \omega^{\hat{i}}_{\cdot\,\hat{j}} Z^{\hat{j}}_\mu$ where $Z^{\hat{i}}_\mu$ are the physical neutral bosons with well-defined masses. Explicitly, this $SO(n-1)$ transformation works in the manner:

$$
\begin{aligned}
A^0_\mu &= A^{em}_\mu \cos\theta - \nu_{\hat{i}}\omega^{\hat{i}}_{\cdot\,\hat{j}} Z^{\hat{j}}_\mu \sin\theta, \\
A^{\hat{k}}_\mu &= \nu^{\hat{k}} A^{em}_\mu \sin\theta + \left(\delta^{\hat{k}}_{\hat{i}} - \nu^{\hat{k}}\nu_{\hat{i}}(1-\cos\theta)\right)\omega^{\hat{i}}_{\cdot\,\hat{j}} Z^{\hat{j}}_\mu.
\end{aligned}
\tag{2.3}
$$

It connects the gauge basis $(A^0_\mu, A^{\hat{i}}_\mu)$ to the physical one $(A^{em}_\mu, Z^{\hat{i}}_\mu)$. This transformation ω is called the generalized Weinberg transformation (gWt).

At this stage, one can easily identify the charges of the fermions involved with respect to the above determined physical bosons. The spinor multiplet L^ρ acquires the following electric charge matrix:

$$Q^\rho = g\left[(D^\rho \cdot \nu)\sin\theta + y_\rho \cos\theta\right], \tag{2.4}$$

and $n-1$ neutral charge matrices:

$$Q^\rho(Z^{\hat{i}}) = g\left[D^\rho_{\hat{k}} - \nu_{\hat{k}}(D^\rho \cdot \nu)(1-\cos\theta) - y_\rho \nu_{\hat{k}} \sin\theta\right]\omega^{\hat{k}}_{\cdot\,\hat{i}} \tag{2.5}$$

each corresponding to the $n-1$ neutral physical fields, $Z^{\hat{i}}_\mu$.

2.2. $SU(4)_L \otimes U(1)_Y$ gauge group

In the particular $SU(4)_L \otimes U(1)_Y$ gauge model one has to properly identify the diagonal generators and set up the possible options for the versor ν. For our purpose, the standard generators T_a of the $su(4)$ algebra are the Hermitian diagonal generators of the Cartan sub-algebra, namely $D_1 = T_3 = \frac{1}{2}\text{diag}(1,-1,0,0)$, $D_2 = T_8 = \frac{1}{2\sqrt{3}}\text{diag}(1,1,-2,0)$, and $D_3 = T_{15} = \frac{1}{2\sqrt{6}}\text{diag}(1,1,1,-3)$.

We will prefer in the following to denote the irreps of the gauge group by $\rho = (\mathbf{n}_\rho, y^\rho_{ch})$ indicating the genuine chiral hypercharge y_{ch} instead of y. Hence, the multiplets of the 3-4-1 model under consideration here - subject to anomaly cancellation in order to keep renormalizable the whole theory - will be denoted by $(\mathbf{n}_{color}, \mathbf{n}_\rho, y^\rho_{ch})$. The condition $e = g\sin\theta_W$ established in the SM is valid throughout.

In principle, there will be three distinct cases in choosing the versors. They are:

- versors $\nu_1 = 1$, $\nu_2 = 0$, $\nu_3 = 0$,
- versors $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 0$,
- versors $\nu_1 = 0$, $\nu_2 = 0$, $\nu_3 = 1$.

2.2.1. Class A ($\nu_1 = 1$, $\nu_2 = 0$, $\nu_3 = 0$)

The lepton quadruplet obeys the fundamental irrep of the gauge group $\rho = (\mathbf{4}, 0)$. Eq. (2.4) yields:

$$Q^{(4,0)} = eT_3^{(4)}\frac{\sin\theta}{\sin\theta_W}, \tag{2.6}$$

which denotes the lepton representation $\begin{pmatrix} e^c_\alpha, & e_\alpha, & \nu_\alpha, & N_\alpha \end{pmatrix}^T_L \sim (\mathbf{4}, 0)$ if and only if $\sin\theta = 2\sin\theta_W$ holds.

In the quark sector there are two families ($i = 1, 2$) transforming similarly under the gauge group $\begin{pmatrix} J_i, & u_i, & d_i, & D_i \end{pmatrix}^T_L \sim (\mathbf{4}^*, -1/3)$ and a third one transforming as $\begin{pmatrix} J_3, & d_3, & u_3, & U_3 \end{pmatrix}^T_L \sim (\mathbf{4}, +2/3)$. Their electric charge operators will take, respectively, the forms

$$Q^{(4^*,-\frac{1}{3})} = e\left[T_3^{(4^*)}\frac{\sin\theta}{\sin\theta_W} - \frac{1}{3}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right], \tag{2.7}$$

$$Q^{(4,+\frac{2}{3})} = e\left[T_3^{(4)}\frac{\sin\theta}{\sin\theta_W} + \frac{2}{3}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right], \tag{2.8}$$

Now, in order to get the known electric charges of the quarks one must enforce the coupling match:

$$\frac{g'}{g} = \frac{\sin\theta_W}{\sqrt{1 - 4\sin^2\theta_W}}. \tag{2.9}$$

The anomaly-free content in the fermion sector of this class of 3-4-1 models stands:

Lepton families

$$f_{\alpha L} = \begin{pmatrix} e^c_\alpha \\ e_\alpha \\ \nu_\alpha \\ N_\alpha \end{pmatrix}_L \sim (\mathbf{1}, \mathbf{4}, 0) \tag{2.10}$$

Quark families

$$Q_{iL} = \begin{pmatrix} J_i \\ u_i \\ d_i \\ D_i \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}^*, -1/3) \qquad Q_{3L} = \begin{pmatrix} J_3 \\ -d_3 \\ u_3 \\ U_3 \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}, +2/3) \tag{2.11}$$

$$(d_{3L})^c, (d_{iL})^c, (D_{iL})^c, \sim (\mathbf{3}, \mathbf{1}, +1/3) \tag{2.12}$$

$$(u_{3L})^c, (u_{iL})^c, (U_{3L})^c \sim (\mathbf{3}, \mathbf{1}, -2/3) \tag{2.13}$$

$$(J_{3L})^c \sim (\mathbf{3}, \mathbf{1}, -5/3) \qquad (J_{iL})^c \sim (\mathbf{3}, \mathbf{1}, +4/3) \tag{2.14}$$

with $\alpha = 1, 2, 3$ and $i = 1, 2$.

The capital letters J label some exotic quarks included in each family. They exhibit exotic electric charges $\pm 4/3$ and $\pm 5/3$.

Based on Eq.(2.5) one can compute the neutral charges for the above model A. They are presented in Table 1. Obviously, the SM fermions exhibit the same neutral charges as they do in the SM framework with respect to the Z boson.

2.2.2. Class B ($\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 0$)

Due to $T_8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2, 0)$ there is no room for a plausible electric charge operator, since there is only 0 and $\pm e$ allowed in the lepton quadruplets. Therefore, this case must be ruled out from phenomenological reasons.

2.2.3. Class C ($\nu_1 = 0$, $\nu_2 = 0$, $\nu_3 = -1$)

In this case, one can assign two different chiral hypercharges $-\frac{1}{4}$ and $-\frac{3}{4}$ respectively for the lepton quadruplet. Hence, we get two sub-cases leading to two different versions of this class. The coupling matching yields the same relation in both sub-cases.

The lepton sector's electric charge operator for the first choice stands as

$$Q^{(4^*, -\frac{1}{4})} = e\left[-T_{15}^{(4^*)} \frac{\sin\theta}{\sin\theta_W} - \frac{1}{4}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right], \tag{2.15}$$

for the first sub-case. This leads to the lepton representation $\begin{pmatrix} e_\alpha, & \nu_\alpha, & N_\alpha, & N'_\alpha \end{pmatrix}^T_L \sim (\mathbf{4}^*, -\frac{1}{4})$ including two new kinds of neutral leptons (N_α, N'_α) possibly right-handed neutrinos.

Table 1. Coupling coefficients of the neutral currents in 3-4-1 in Model A

Particle\Coupling($e/\sin 2\theta_W$)	$Z\bar{f}f$	$Z'\bar{f}f$	$Z''\bar{f}f$
$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	1	$-\frac{\sqrt{1-4\sin^2\theta_W}}{\sqrt{3}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
e_L, μ_L, τ_L	$2\sin^2\theta_W - 1$	$-\frac{\sqrt{1-4\sin^2\theta_W}}{\sqrt{3}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
e_R, μ_R, τ_R	$-2\sin^2\theta_W$	$\frac{2\sqrt{1-4\sin^2\theta_W}}{\sqrt{3}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
$N_{eL}, N_{\mu L}, N_{\tau L}$	0	0	$-\sqrt{\frac{3}{2}}\cos\theta_W$
u_L, c_L	$1 - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{\sqrt{3}}\left(\frac{1-2\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}\right)$	$-\frac{\cos\theta_W}{\sqrt{6}}$
d_L, s_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$\frac{1}{\sqrt{3}}\left(\frac{1-2\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}\right)$	$-\frac{\cos\theta_W}{\sqrt{6}}$
t_L	$1 - \frac{4}{3}\sin^2\theta_W$	$-\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{1-4\sin^2\theta_W}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
b_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$-\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{1-4\sin^2\theta_W}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
T_L	$-\frac{4}{3}\sin^2\theta_W$	$-\left(\frac{4}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	$-\sqrt{\frac{3}{2}}\cos\theta_W$
D_{1L}, D_{2L}	$\frac{2}{3}\sin^2\theta_W$	$\left(\frac{2}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	$\sqrt{\frac{3}{2}}\cos\theta_W$
u_R, c_R, t_R, T_R	$-\frac{4}{3}\sin^2\theta_W$	$\left(\frac{4}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	0
d_R, s_R, b_R, D_{iR}	$+\frac{2}{3}\sin^2\theta_W$	$-\left(\frac{2}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	0
J_{1L}, J_{2L}	$\frac{8}{3}\sin^2\theta_W$	$-\left(\frac{2}{\sqrt{3}}\right)\frac{1-5\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	$-\frac{\cos\theta_W}{\sqrt{6}}$
J_{1R}, J_{2R}	$\frac{8}{3}\sin^2\theta_W$	$-\left(\frac{8}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	0
J_{3L}	$-\frac{10}{3}\sin^2\theta_W$	$\left(\frac{2}{\sqrt{3}}\right)\frac{1-6\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	$\frac{\cos\theta_W}{\sqrt{6}}$
J_{3R}	$-\frac{10}{3}\sin^2\theta_W$	$\left(\frac{10}{\sqrt{3}}\right)\frac{\sin^2\theta_W}{\sqrt{1-4\sin^2\theta_W}}$	0

For the second choice, the lepton electric charge operator is represented as

$$Q^{(4,-\frac{1}{4})} = e\left[-T_{15}^{(4)}\frac{\sin\theta}{\sin\theta_W} - \frac{3}{4}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right], \tag{2.16}$$

allowing for lepton families such as $\left(\begin{array}{cccc}\nu_\alpha, & e_\alpha^-, & E_\alpha^-, & E_\alpha'^-\end{array}\right)_L^T \sim (\mathbf{4}, -\frac{3}{4})$. A phenomenological analysis in this sub-case must assume some new kind of charged lepton $(E_\alpha^-, E_\alpha'^-)$. possibly very heavy.

After a little algebra the coupling matching for both sub-cases arises:

$$\frac{g'}{g} = \frac{\sin\theta_W}{\sqrt{1-\frac{3}{2}\sin^2\theta_W}}. \tag{2.17}$$

For the quark sector the electric charge operator takes the following representations

$$Q^{(4^*,\frac{5}{12})} = e\left[-T_{15}^{(4^*)}\frac{\sin\theta}{\sin\theta_W} + \frac{5}{12}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right] \tag{2.18}$$

$$Q^{(4,-\frac{1}{12})} = e\left[-T_{15}^{(4)}\frac{\sin\theta}{\sin\theta_W} - \frac{1}{12}\left(\frac{g'}{g}\right)\frac{\cos\theta}{\sin\theta_W}\right] \tag{2.19}$$

2.2.4. Fermion content of Model C1

A natural fermion outcome occurs in this first choice, namely:

Lepton families

$$f_{\alpha L} = \begin{pmatrix} e_\alpha \\ \nu_\alpha \\ N_\alpha \\ N'_\alpha \end{pmatrix}_L \sim (\mathbf{1}, \mathbf{4}^*, -\tfrac{1}{4}) \qquad (e_{\alpha L})^c \sim (\mathbf{1}, \mathbf{1}, 1) \tag{2.20}$$

Quark families

$$Q_{iL} = \begin{pmatrix} u_i \\ d_i \\ D_i \\ D'_i \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}, -1/12) \qquad Q_{3L} = \begin{pmatrix} -d_3 \\ u_3 \\ U \\ U' \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}^*, 5/12) \tag{2.21}$$

$$(d_{3L})^c, (d_{iL})^c, (D_{iL})^c, (D'_{iL})^c \sim (\mathbf{3}, \mathbf{1}, +1/3) \tag{2.22}$$

$$(u_{3L})^c, (u_{iL})^c, (U_L)^c, (U'_L)^c \sim (\mathbf{3}, \mathbf{1}, -2/3) \tag{2.23}$$

with $\alpha = 1, 2, 3$ and $i = 1, 2$.

Based on Eq.(2.5) one can compute the neutral charges for model C1. They are presented in Table 2. The outcome is suitable too: the SM fermions exhibit the same neutral charges as they do in the SM framework with respect to the Z boson.

Table 2. Coupling coefficients of the neutral currents in 3-4-1 in Model C1

Particle\Coupling($e/\sin 2\theta_W$)	$Z\bar{f}f$	$Z'\bar{f}f$	$Z''\bar{f}f$
$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	1	$\frac{1-3\sin^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
e_L, μ_L, τ_L	$2\sin^2\theta_W - 1$	$\frac{1-3\sin^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
$N_{eL}, N_{\mu L}, N_{\tau L}$	0	$-\frac{3\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
$N'_{eL}, N'_{\mu L}, N'_{\tau L}$	0	$-\frac{3\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$
u_L, c_L	$1 - \frac{4}{3}\sin^2\theta_W$	$\frac{2-9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
d_L, s_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$\frac{2-9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
t_L	$1 - \frac{4}{3}\sin^2\theta_W$	$\frac{2+9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
b_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$\frac{2+9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
$u_R, c_R, t_R, U_{1R}, U'_{iR}$	$-\frac{4}{3}\sin^2\theta_W$	$\frac{4\sin^2\theta_W}{3\sqrt{2-3\sin^2\theta_W}}$	0
$d_R, s_R, b_R, D_{iR}, D'_{iR}$	$+\frac{2}{3}\sin^2\theta_W$	$-\frac{2\sin^2\theta_W}{3\sqrt{2-3\sin^2\theta_W}}$	0
D_{1L}, D_{2L}	$\frac{2}{3}\sin^2\theta_W$	$\frac{-4+9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$
D'_{1L}, D'_{2L}	$\frac{2}{3}\sin^2\theta_W$	$\frac{-4+9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
U_{3L}	$-\frac{4}{3}\sin^2\theta_W$	$\frac{8-9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
U'_{3L}	$-\frac{4}{3}\sin^2\theta_W$	$\frac{8-9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$

2.2.5. Fermion content of Model C2

Some strange charged fermions occur in the second choice, but it is still plausible since these could come very massive.

Lepton families

$$f_{\alpha L} = \begin{pmatrix} \nu_\alpha \\ e^-_\alpha \\ E^-_\alpha \\ E'^-_\alpha \end{pmatrix}_L \sim (\mathbf{1}, \mathbf{4}, -3/4) \qquad (e_{\alpha L})^c, (E_{\alpha L})^c, (E'_{\alpha L})^c \sim (\mathbf{1}, \mathbf{1}, 1) \tag{2.24}$$

Quark families

$$Q_{iL} = \begin{pmatrix} d_i \\ -u_i \\ U_i \\ U'_i \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}^*, 5/12) \qquad Q_{3L} = \begin{pmatrix} u_3 \\ d_3 \\ D \\ D' \end{pmatrix}_L \sim (\mathbf{3}, \mathbf{4}, -1/12) \tag{2.25}$$

$$(d_{3L})^c, (d_{iL})^c, (D_L)^c, (D'_L)^c \sim (\mathbf{3}, \mathbf{1}, +1/3) \tag{2.26}$$

$$(u_{3L})^c, (u_{iL})^c, (U_{iL})^c, (U'_{iL})^c \sim (\mathbf{3}, \mathbf{1}, -2/3) \tag{2.27}$$

with $\alpha = 1, 2, 3$ and $i = 1, 2$.

Based on Eq.(2.5) one can compute the neutral charges for model C1. They are presented in Table 3. As in the previous cases the SM fermions come out with the same neutral charges as they do in the SM framework with respect to the Z boson.

3. Concluding remarks

In this letter we obtained all the possible 3-4-1 models allowed by the general method of solving gauge models with high symmetries that undergo a spontaneous symmetry breaking. All in all, they are three different 3-4-1 models: one belonging to the Class A and two to the Class C. For the three classes of 3-4-1 gauge models the neutral charges (couplings to neutral bosons of the model) are obtained along with the electric corresponding charges. If we restrict ourself to non-exotic electric charges,then only Class C survives. Even more, if heavy charged leptons are still unobserved experimentally, then only subclass A1 remains to be further analyzed from phenomenological point of view. However, all fermion contents are anomaly-free and hence the theoretical models they account for are renormalizable. After some algebraic computations for all the representations involved therein this statement is proved immediately. Therefore, the phenomenological predictions in this promising framework can be valuable indeed. Furthermore, the phenomenology (to be confirmed at present facilities) can be analyzed in detail once each particular model is taken into consideration.

Table 3. Coupling coefficients of the neutral currents in 3-4-1 in Model C2

Particle\Coupling($e/\sin 2\theta_W$)	$Z\bar{f}f$	$Z'\bar{f}f$	$Z''\bar{f}f$
$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	1	$\frac{1-3\sin^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
e_L, μ_L, τ_L	$2\sin^2\theta_W - 1$	$\frac{1-3\sin^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
$E_{eL}, E_{\mu L}, E_{\tau L}$	0	$-\frac{3\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$
$E'_{eL}, E'_{\mu L}, E'_{\tau L}$	0	$-\frac{3\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
u_L, c_L	$1 - \frac{4}{3}\sin^2\theta_W$	$\frac{2-9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
d_L, s_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$\frac{2-9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
t_L	$1 - \frac{4}{3}\sin^2\theta_W$	$\frac{2+9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
b_L	$-1 + \frac{2}{3}\sin^2\theta_W$	$\frac{2+9\cos^2\theta_W}{2\sqrt{2-3\sin^2\theta_W}}$	0
$u_R, c_R, t_R, U_{1R}, U'_{iR}$	$-\frac{4}{3}\sin^2\theta_W$	$\frac{4\sin^2\theta_W}{3\sqrt{2-3\sin^2\theta_W}}$	0
$d_R, s_R, b_R, D_{iR}, D'_{iR}$	$+\frac{2}{3}\sin^2\theta_W$	$-\frac{2\sin^2\theta_W}{3\sqrt{2-3\sin^2\theta_W}}$	0
D_{3L}	$\frac{2}{3}\sin^2\theta_W$	$\frac{-4+9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$
D'_{3L}	$\frac{2}{3}\sin^2\theta_W$	$\frac{-4+9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
U_{1L}, U_{2L}	$-\frac{4}{3}\sin^2\theta_W$	$\frac{8-9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$-\cos\theta_W$
U'_{1L}, U'_{2L}	$-\frac{4}{3}\sin^2\theta_W$	$\frac{8-9\cos^2\theta_W}{6\sqrt{2-3\sin^2\theta_W}}$	$\cos\theta_W$

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