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Semitopological Vector Spaces and Hyperseminorms

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Abstract

In this paper, we introduce and study semitopological vector spaces. The goal is to provide an efficient base for developing the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces. Semitopological vector spaces are more general than conventional topological vector spaces, which proved to be very useful for solving many problems in functional analysis. To study semitopological vector spaces, hypermetrics and hyperpseudometrics are introduced and it is demonstrated that hyperseminorms, studied in previous works of the author, induce hyperpseudometrics, while hypernorms induce hypermetrics. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. We also show that semitopological vector spaces are closely related to systems of hyperseminorms. Then defining boundedness and continuity relative to associated systems of hyperseminorms, we study relations between relative boundedness and relative continuity for mappings of vector spaces with systems of hyperseminorms and systems of hypernorms.

Keywords: Functional analysis, topological vector space, norm, seminorm, hyperseminorm, boundedness, continuity, extrafunction.
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1. Introduction

The concept of a real or complex extrafunction essentially extends the concept of a real or complex function, encompassing, in particular, the concept of a distribution, i.e., distributions are a kind of extrafunctions (Burgin, 2012). Extrafunctions have many advantages in comparison with functions and distributions. For instance, integration of extrafunctions is more powerful than integration of functions allowing integration of a much larger range of functions as it is demonstrated in (Burgin, 2012).

At the same time, spaces of extrafunctions have a more sophisticated structure in comparison with spaces of functions, which are topological vector spaces and have a highly advanced theory (cf., for example, (Bourbaki, 1953-1955); (Robertson & Robertson, 1964); (Riez & Sz.-Nagy,

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1955); (Rudin, 1991); (Grothendieck, 1992); (Kolmogorov & Fomin, 1999)). In particular, it has been demonstrated that topological vector spaces provide an efficient context for the development of integration and are very useful for solving many problems in functional analysis in general (Choquet, 1969); (Edwards & Wayment, 1970); (Shuchat, 1972); (Kurzweil, 2000). In addition, locally convex topological vector spaces offer a convenient structure for studies of summation, which is integration of functions on natural numbers (Pietsch, 1965).

In this paper we introduce and study semitopological vector spaces, operators in these spaces and their mappings. It provides a base for the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces. Semitopological vector spaces are more general than conventional topological vector spaces. To study semitopological vector spaces, hypermetrics and hyperpseudometrics are introduced and it is demonstrated that hyperseminorms induce hyperpseudometrics, while hypernorms induce hypermetrics. Norms are special cases of hypernorms, while seminorms are special cases of hyperseminorms. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. We also show that semitopological vector spaces are closely related to systems of hyperseminorms.

An essential property of operators in mathematics is continuity (cf. (Dunford & Schwartz, 1958); (Rudin, 1991); (Kolmogorov & Fomin, 1999)). One of the central results of functional analysis is the theorem that establishes equivalence between continuity and boundedness for linear operators. Here we extend the concepts of boundedness and continuity for operators and mappings of semitopological vector spaces with systems of hyperseminorms and seminorms, differentiating between different types of boundedness and continuity and making these concepts relative to systems of hyperseminorms and seminorms. Then we study these concepts, proving a series of theorems, which establish equivalence between a type of relative continuity and the corresponding type of relative boundedness for linear operators in semitopological vector spaces with systems of hyperseminorms or seminorms. Classical results describing continuous operators in convex spaces become direct corollaries of theorems proved in this paper. In conclusion, several problems for further research are formulated.

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2. Semitopological vector spaces

The concept of a semitopological vector space is an extension of the concept of a topological vector space.

Definition 2.1. *A semitopological vector space* L over a field $\mathbf{F}$ is a vector space over $\mathbf{F}$ with a topology in which addition is continuous, while scalar multiplication by elements from $\mathbf{F}$ is continuous with respect to L, i.e., the scalar multiplication mapping $m : \mathbf{F} \times L \to L$ is continuous in the second coordinate.

When the multiplication mapping $m : \mathbf{F} \times L \to L$ is continuous, then L is a topological vector space over the field $\mathbf{F}$. Some authors (cf., for example, (Rudin, 1991)) additionally demand that the point $\mathbf{0}$ in a topological vector space is closed. This condition results in the Hausdorff topology in topological vector spaces.

In what follows, **F** stands either for the field $\mathbb{R}$ of all real numbers or for the field $\mathbb{C}$ of all complex numbers or for a subfield of $\mathbb{C}$ that contains $\mathbb{R}$, while **0** denotes the zero element of any vector space.

Semitopological vector spaces are closely related to hypernorms and hyperseminorms.

Let $\mathbb{R}_\omega$ be the set of all real hypernumbers and $\mathbb{R}^+_\omega$ be the set of all non-negative real hypernumbers (Burgin, 2012).

Definition 2.2. a) A mapping $q : L \to \mathbb{R}^+_\omega$ is called a *hypernorm* if it satisfies the following conditions:

N1 . For any x from L, $q(x) = 0$ if and only if $x = \mathbf{0}$.

N2 . $q(ax) = |a| \cdot q(x)$ for any x from L and any number a from **F**.

N3 . (the triangle inequality or subadditivity).

$$q(x + y) \leq q(x) + q(y) \quad \text{for any } x \text{ and } y \text{ from } L$$

b) A vector space L with a norm is called a *hypernormed vector space* or simply, a *hypernormed space*.

c) The real hypernumber $q(x)$ is called the *hypernorm* of an element x from the hypernormed space L.

Note that *norms* in vector spaces coincide with hypernorms that take values only in the set of real numbers.

Example 2.1. As it is proved in (Burgin, 2012), the set of all real hypernumbers $\mathbb{R}_\omega$ is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If α is a real hypernumber, i.e.,$\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ with $a_i \in \mathbb{R}$ for all $i \in \omega$, then $||\alpha|| = \mathrm{Hn}(|a_i|)_{i\in\omega}$.

Note that this hypernorm coincides with the conventional norm on real numbers but it is impossible get the same topology by means of a conventional finite norm.

Example 2.2. As it is proved in (Burgin, 2002), the set of all complex hypernumbers $\mathbb{C}_\omega$ of all complex hypernumbers is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If α is a complex hypernumber,i.e., $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ with $a_i \in \mathbb{C}$ for all $i \in \omega$, then $||\alpha|| = \mathrm{Hn}(|a_i|)_{i\in\omega}$.

Note that this hypernorm coincides with the conventional norm on complex numbers but it is impossible get the same topology by means of a conventional finite norm.

There are hypernormed spaces that are not normed spaces.

Example 2.3. The set $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space where the hypernorm $|| \cdot ||$ is defined by the following formula:

If $f : \mathbb{R} \to \mathbb{R}$, then $||f|| = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = max\{|f(x)|; a_i = [-i, i]\}$.

At the same time, it is known that $C(\mathbb{R}, \mathbb{R})$ is not a normed space (Robertson & Robertson, 1964).

There are natural relations between hypernorms and semitopological vector spaces.

Theorem 2.1. *Any hypernormed space is a Hausdorff semitopological vector space.*

Proof. Let us consider a vector space L with a hypernorm q. Taking an element x from L and a positive real number k, we define the neighborhood $O_k x$ of x by the following formula

$$O_k x = \{y \in L;\ q(x - y) < k\}.$$

At first, we show that the system of so defined neighborhoods determines a topology in L. To do this, it is necessary to check the following neighborhood axioms (Kuratowski, 1966):

NB1. Any neighborhood of a point $x \in X$ contains this point.

NB2. For any two neighborhoods $O_1 x$ and $O_2 x$ of a point $x \in X$, there is a neighborhood Ox of x that is a subset of the intersection $O_1 x \cap O_2 x$.

NB3. For any neighborhood Ox of a point $x \in X$ and a point $y \in Ox$, there is a neighborhood Oy of y that is a subset of Ox.

Let us consider a point x from X.

NB1: The point x belongs to $O_k x$ because $q(x - x) = q(\mathbf{0}) = 0 < k$ for any positive real number k.

NB2: Taking two positive real numbers k and h, we see that the intersection $O_k x \cap O_h x = O_l x$ also is a neighborhood of x where $l = \min\{k, h\}$.

NB3: Let $y \in O_k x$. Then $q(x - y) < k$ and by properties of real numbers, there is a positive real number t such that $q(x - y) < k - t$. Then $O_t x \subseteq O_k x$. Indeed, if $z \in O_t x$, then $q(y - z) < t$. Consequently,

$$q(x - z) = q((x - y) + (y - z)) \leq q(x - y) + q(y - z) < (k - t) + t = k.$$

It means that $z \in O_k x$.

Thus, we have a topology in L, and this topology is Hausdorff because any hypernorm separates points, i.e., if $x \neq y$, then $q(x - y) \neq 0$.

Now we show that addition is continuous and scalar multiplication is continuous in the second coordinate with respect to this topology.

Let us consider a sequence $\{x_i;\ i = 1, 2, 3, \ldots\}$ that converges to x, a sequence $\{y_i;\ i = 1, 2, 3, \ldots\}$ that converges to y, and the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \ldots\}$. Convergence of these two sequences means that for any $k > 0$, there are a natural number n such that $q(x_i - x) < k$ for any $i > n$ and a natural number m such that $q(y_i - y) < k$ for any $i > m$. Then by properties of a hypernorm, we have

$$q(z_i - (x + y)) = q((x_i + y_i) - (x + y)) = q((x_i - x) + (y_i - y)) \leq q(x_i - x) + q(y_i - y) < k + k = 2k$$

when $i > \max\{n, m\}$. As k is an arbitrary positive real number, this means that the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \dots\}$. converges to $x + y$. Consequently, addition is continuous in L.

In addition, for any number a from $\mathbf{F}$, we have

$$q(u_i - ax) = q(ax_i - ax) = q(a(x_i - x)) \leq |a|q(x_i - x) < |a|k$$

where $u_i = ax_i$. As k is an arbitrary positive real number and $|a|$ is a constant, this means that the sequence $\{u_i = ax_i;\ i = 1, 2, 3, \dots\}$ converges to ax. Consequently, scalar multiplication is continuous in the second coordinate. Theorem is proved. □

Hypernormed spaces are also hypermetric spaces.

Definition 2.3. a) A mapping $\mathbf{d} : X \times X \to \mathbb{R}^+_\omega$ is called a *hypermetric* (or a *hyperdistance function*) in a set X if it satisfies the following axioms:

M1. For any x and y from X, $\mathbf{d}(x, y) = 0$ if and only if $x = y$.

M2. (Symmetry). $\mathbf{d}(x, y) = \mathbf{d}(y, x)$ for all $x, y \in X$.

M3. (the triangle inequality or subadditivity).

$$\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y) \text{ for all } x, y, z \in X.$$

b) A set X with a hypermetric $\mathbf{d}$ is called a *hypermetric space*.

c) The real hypernumber $\mathbf{d}(x, y)$ is called the *distance* between x and y in the hypermetric space X.

Note that the distance between two elements in a hypermetric space can be a real number, finite hypernumber or infinite hypernumber. When the distance between two elements of X is always a real number, $\mathbf{d}$ is a metric.

Lemma 2.1. *a) A hypernorm q in a vector space L induces a hypermetric $\mathbf{d}_q$ in this space.*

b) If q is a norm in L, then $\mathbf{d}_q$ is a metric.

Indeed, if $q : X \to R^+_\omega$ is a hypernorm in L and x and y are elements from L, then we can define $\mathbf{d}_q(x, y) = q(x - y)$. Properties of a hypernorm imply that $\mathbf{d}_q$ satisfies all axioms M1- M3. The statement (b) directly follows from definitions.

Theorem 2.1 and Lemma 2.1 imply the following result.

Corollary 2.1. *$\mathbb{R}_\omega$ and $\mathbb{C}_\omega$ are hypermetric spaces.*

It is interesting to find what hypermetrics in vector spaces are induced by hypernorms and what metrics in vector spaces are induced by norms. To do this, let us consider additional properties of hypermetrics and metrics.

Definition 2.4. A hypermetric (metric) in a vector space L is called *linear* if it satisfies the following axioms:

LM1. $\mathbf{d}(x+z, y+z) = \mathbf{d}(x, y)$ for any $x, y, z \in L$.

LM2. $\mathbf{d}(ax, ay) = |a| \cdot \mathbf{d}(x, y)$ for all $x, y \in L$ and $a \in \mathbf{F}$.

Example 2.4. Let us take the space of all real numbers $\mathbb{R}$ as the space L. The natural metric in this space is defined as $\mathbf{d}(x, y) = |x - y|$. This metric is linear. Indeed,

$$\mathbf{d}(x+z, y+z) = |(x+z) - (y+z)| = |x - y| = \mathbf{d}(x, y)$$

and

$$\mathbf{d}(ax, ay) = |ax - ay| = |a(x - y)| = |a| \cdot |x - y| = |a| \cdot \mathbf{d}(x, y).$$

Example 2.5. Let us take the two-dimensional real vector space $\mathbb{R}^2$ as the space L. The natural metric in this space is defined by the conventional formula

$$\text{If} \quad x = (x_1, x_2) \text{ and } y = (y_1, y_2), \text{ then } \mathbf{d}(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

This metric is also linear. Indeed,

$$\mathbf{d}(x+z, y+z) = \sqrt{((x_1 + z_1) - (y_1 + z_1))^2 + ((x_2 + z_2) - (y_2 + z_2))^2} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \mathbf{d}(x, y)$$

and

$$\mathbf{d}(ax, ay) = \sqrt{(ax_1 - ay_1)^2 + (ax_2 - ay_2)^2} = \sqrt{a^2(x_1 - y_1)^2 + a^2(x_2 - y_2)^2} =$$
$$= |a|\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = |a| \cdot \mathbf{d}(x, y).$$

Example 2.6. Let us take the two-dimensional real vector space $\mathbb{R}^2$ as the space L. The natural metric in this space is defined by the conventional formula

$$\text{If } x = (x_1, x_2) \text{ and } y = (y_1, y_2), \text{ then } \mathbf{d}(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

This metric is not linear. Indeed, let us take $x = (3, 3)$, $y = (1, 1)$, and $a = 2$. Then $\mathbf{d}(x, y) = 8$, while $\mathbf{d}(2x, 2y) = 32$.

These examples show that there are linear metrics (hypermetrics) in vector spaces and there are metrics (hypermetrics) in vector spaces that are not linear. The majority of popular metrics are induced by norms and thus, they are linear as the following result demonstrates.

Theorem 2.2. *A hypermetric* $\mathbf{d}$ *is induced by a hypernorm if and only if* $\mathbf{d}$ *is linear.*

Proof. Necessity. Let us consider a vector space L with a hypernorm q. By Lemma 2.1, it induces the hypermetric $\mathbf{d}_q(x, y) = q(x-y)$. Then $\mathbf{d}_q(x+z, y+z) = q((x+z)(y+z)) = q(x-y) = \mathbf{d}_q(x, y)$, i.e., Axiom LM1 is true. In addition, $\mathbf{d}_q(ax, ay) = q(ax-ay) = q(a(x-y)) = |a| \cdot q(x-y) = |a| \cdot \mathbf{d}_q(x, y)$, i.e., Axiom LM2 is also true.

Necessity. Let us consider a vector space L with a linear hypermetric $\mathbf{d}$. We define the hypernorm $q_{\mathbf{d}}$ by the following formula

$$q_{\mathbf{d}}(x) = \mathbf{d}(\mathbf{0}, x).$$

We show that $q_{\mathbf{d}}$ is a hypernorm. Indeed, $q_{\mathbf{d}}(\mathbf{0}) = \mathbf{d}(\mathbf{0}, \mathbf{0}) = 0$. Besides, if $q_d(x) = \mathbf{d}(\mathbf{0}, x) = 0$, then $x = 0$ by Axiom **M1**. This gives us Axiom **N1** for $q_{\mathbf{d}}$.

In addition,

$$q_{\mathbf{d}}(ax) = \mathbf{d}(\mathbf{0}, ax) = \mathbf{d}(a\mathbf{0}, ax) = \mathbf{d}(a(\mathbf{0}, x)) = |a| \cdot \mathbf{d}(\mathbf{0}, x) = |a| \cdot q_{\mathbf{d}}(x)$$

by Axiom LM2. This gives us Axiom N2 for $q_{\mathbf{d}}$.

Likewise, by Axioms M3 and LM1, we have

$$q_{\mathbf{d}}(x + y) = \mathbf{d}(\mathbf{0}, x + y) \leq \mathbf{d}(\mathbf{0}, x) + \mathbf{d}(x, x + y) = \mathbf{d}(\mathbf{0}, x) + \mathbf{d}(\mathbf{0}, y) = q_{\mathbf{d}}(x) + q_{\mathbf{d}}(y).$$

This gives us the triangle inequality (Axiom N3) for $q_{\mathbf{d}}$.

Theorem is proved. □

Corollary 2.2. *A metric* **d** *is induced by a norm if and only if* **d** *is linear.*

Taking only a part of the hypernorm properties, we come to the concept of a hyperseminorm.

Definition 2.5. a) A mapping $q : L \to \mathbb{R}^+_\omega$ is called a *hyperseminorm* if it satisfies the following conditions:

N2. $q(ax) = |a| \cdot q(x)$ for any x from L and any number a from $\mathbb{R}$.

N3. (the triangle inequality or subadditivity).

$$q(x + y) \leq q(x) + q(y) \text{ for any } x \text{ and } y \text{ from } L.$$

b) A vector space L with a norm is called a *hyperseminormed vector space* or simply, a *hyperseminormed space*.

c) The real hypernumber $q(x)$ is called the *hyperseminorm* of an element x from the hyperseminormed space L.

d) A set $X \subseteq L$ is called q - bounded if there is a positive real number h such that for any element a from X, the inequality $q(a) < h$ is true.

e) A set $X \subseteq L$ is called weakly q - *bounded* if there is a positive real hypernumber α such that for any element a from X, the inequality $q(a) < \alpha$ is true.

Note that any seminorm is a hyperseminorm that takes values only in the set of real numbers.

Proposition 2.1. *If $q : L \to \mathbb{R}$ is a hyperseminorm, then it has the following properties:*

(1) $q(x) \geq 0$ *for any* $x \in L$.

(2) $q(x - y) = q(y - x)$ *for any* $x, y \in L$.

(3) $q(\mathbf{0}) = 0$.

(4) $|q(x)q(y)| = q(x - y)$ *for any* $x, y \in L$.

(5) $q(x) - q(y) \leq q(x + y)$ *for any* $x, y \in L$.

Proof. (1) By Axiom N3, we have

$$q(x) + q(-x) \geq q(x + (-x)) = q(\mathbf{0}).$$

At the same time, by N2, we have $q(\mathbf{0}) = 0 \cdot q(\mathbf{0}) = 0$ and $q(-x) = q(x)$. This gives us

$$q(x) + q(-x) = q(x) + q(x) = 2q(x) \geq q(x + (-x)) = q(\mathbf{0}) = 0$$

and thus, $q(x) \geq 0$.

(2) By Axiom N2, we have

$$q(x - y) = q(-(y - x)) = |-1| \cdot q(y - x) = q(y - x).$$

(3) By Axiom N2, we have

$$q(\mathbf{0}) = q(0 \cdot \mathbf{0}) = |0| \cdot q(\mathbf{0}) = 0.$$

(4) By Axiom N3, we have

$$q(x) = q(x - y + y) \leq q(x - y) + q(y).$$

Thus,

$$q(x) - q(y) \leq q(x - y).$$

As q is symmetric (property (2)), we have

$$q(y) - q(x) \leq q(x - y).$$

Consequently,

$$|q(x) - q(y)| = q(x - y).$$

Property (5) is a consequence of property (4).

Proposition is proved. □

There are intrinsic relations between hyperseminorms and semitopological vector spaces.

Theorem 2.3. *Any hyperseminormed space is a semitopological vector space, which is Hausdorff if and only if it is a hypernormed space.*

Proof. Let us consider a vector space L with a hyperseminorm q. Taking an element x from L and a positive real number k, we define the neighborhood $O_k x$ of x by the following formula

$$O_k x = \{y \in L; q(x - y) < k\}.$$

To show that the system of so defined neighborhoods determines a topology in L, we check the neighborhood axioms (Kuratowski, 1966).

NB1: The point x belongs to $O_k x$ because by Proposition 1, $q(x - x) = q(\mathbf{0}) = 0 < k$ for any positive real number k.

NB2: Taking two positive real numbers k and h, we see that the intersection $O_k x \cap O_h x = O_l x$ is also a neighborhood of x where $l = \min\{k, h\}$.

NB3: Let $y \in O_k x$. Then $q(x - y) < k$ and by properties of real numbers, there is a positive real number t such that $q(x - y) < k - t$. Then $O_t x \subseteq O_k x$. Indeed, if $z \in O_t x$, then $q(y - z) < t$. Consequently,

$$q(x - z) = q((x - y) + (y - z)) \leq q(x - y) + q(y - z) < (k - t) + t = k.$$

It means that $z \in O_k x$.

Now we show that addition is continuous and scalar multiplication is continuous in the second coordinate with respect to this topology.

Let us consider a sequence $\{x_i;\ i = 1, 2, 3, \dots\}$ that converges to x, a sequence $\{y_i;\ i = 1, 2, 3, \dots\}$ that converges to y, and the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \dots\}$. Convergence of these two sequences means that for any $k > 0$, there are a natural number n such that $q(x_i - x) < k$ for any $i > n$ and a natural number m such that $q(y_i - y) < k$ for any $i > m$. Then by properties of a hyperseminorm, we have

$$q(z_i - (x + y)) = q((x_i + y_i) - (x + y)) = q((x_i - x) + (y_i - y)) \leq q(x_i - x) + q(y_i - y) < k + k = 2k,$$

when $i > \max\{n, m\}$. As k is an arbitrary positive real number, this means that the sequence $\{z_i = x_i + y_i;\ i = 1, 2, 3, \dots\}$ converges to $x + y$. Consequently, addition is continuous in L.

In addition, for any number a from $\mathbf{F}$, we have

$$q(u_i - ax) = q(ax_i - ax) = q(a(x_i - x)) = |a| q(x_i - x) < |a| k,$$

where $u_i = ax_i$. As k is an arbitrary positive real number and $|a|$ is a constant, this means that the sequence $\{u_i = ax_i;\ i = 1, 2, 3, \dots\}$ converges to ax. Consequently, scalar multiplication is continuous in the second coordinate.

By Theorem 2.2, if q is a hypernorm, then the space L is Hausdorff. At the same time, if q is not a hypernorm, then there are x and y from L such that $x \neq y$ but $q(x - y) = 0$. According to definition, these points x and y cannot be separated in the topology defined above. Thus, the space L is not Hausdorff.

Theorem is proved. □

Hyperseminormed spaces are also hyperpseudometric spaces.

Definition 2.6. A *hyperpseudometric* in a set X is a mapping $\mathbf{d} : X \times X \to \mathbb{R}^+_\omega$ that satisfies the following axioms:

P1. $\mathbf{d}(x, y) = 0$ if $x = y$,
i.e., the distance between an element and itself is equal to zero.

M2. (Symmetry). $d(x, y) = d(y, x)$ for all $x, y \in X$,
i.e., the distance between x and y is equal to the distance between y and x.

M3. (the triangle inequality or subadditivity).

$$\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y) \text{ for all } x, y, z \in X.$$

When the distance between two elements of X is always a real number, $\mathbf{d}$ is a *pseudometric* (Kuratowski, 1966).

Note that although it would look natural, we do not use terms semimetric and hypersemimetric because according to the mathematical convention, semimetric is defined by a distance that satisfies only axioms M1 and M2.

Lemma 2.2. *a) A hyperseminorm in a vector space L induces a hyperpseudometric in this space.*

b) If q is a seminorm in L, then $\mathbf{d}_q$ *is a pseudometric.*

Indeed, if $q : X \times \mathbb{R}^+_\omega$ is a hyperseminorm in L and x and y are elements from L, then we can define $\mathbf{d}_q(x, y) = q(x - y)$. Properties of a hyperseminorm imply that $\mathbf{d}_q$ satisfies all axioms P1, M2 and M3. In addition, if q takes values only in $\mathbb{R}$, then the same is true for $\mathbf{d}_q$, i.e., $\mathbf{d}_q$ is a pseudometric.

It is interesting to find what hyperpseudometrics in vector spaces are induced by hyperseminorms and what pseudometrics in vector spaces are induced by seminorms. To do this, let us consider additional properties of hypermetrics and metrics.

Definition 2.7. A hyperpseudometric (metric) in a vector space L is called *linear* if it satisfies the Axioms LM1 and LM2.

Examples 2.4 - 2.6 show that there are linear pseudometrics (hyperpseudometrics) in vector spaces and there are pseudometrics (hyperpseudometrics) in vector spaces that are not linear. The majority of popular pseudometrics are induced by seminorms and thus, they are linear as the following result demonstrates.

Theorem 2.4. *A hyperpseudometric* $\mathbf{d}$ *is induced by a hyperseminorm if and only if* $\mathbf{d}$ *is linear. Proof is similar to the proof of Theorem 2.2.*

Corollary 2.3. *A pseudometric* $\mathbf{d}$ *is induced by a seminorm if and only if* $\mathbf{d}$ *is linear. We define the kernel* Ker q *of a hyperseminorm q in L as*

$$\operatorname{Ker} q = \{x \in L;\ q(x) = 0\}.$$

Theorem 2.5. *The kernel* Ker q *of a hyperseminorm q in L is a vector subspace of L.*

Indeed, if $q(x) = 0$ and $a \in \mathbf{F}$, then by Axiom N2,

$$q(ax) = |a| \cdot q(x) = |a| \cdot 0 = 0$$

i.e., $ax \in \operatorname{Ker} q$. In addition, $q(x) = 0$ and $q(y) = 0$, then by Axiom N3,

$$q(x + y) \leq q(x) + q(y) = 0 + 0 = 0$$

and $q(x + y) = 0$ because by Proposition 2.1, $q(x + y) \geq 0$.

Theorem 2.5 allows factorization of the hyperseminormed space L by its subspace $\operatorname{Ker} q$, obtaining the quotient space L_q . The hyperseminorm q induces the hypernorm p_q in the space L_q . This gives us the natural projection $\tau : L \to L_q$, which preserves the hyperseminorm q.

Example 2.7. Let us consider the set $C^\infty(\mathbb{R}, \mathbb{R})$ of all smooth real functions. The following seminorms are considered in is the set $C^\infty(\mathbb{R}, \mathbb{R})$. For each point $a \in \mathbb{R}$, and $f \in C^\infty(\mathbb{R}, \mathbb{R})$, we define

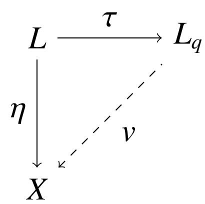
$$q_k(f) = (f(a))^2 + (f'(a))^2 + (f''(a))^2 + \ldots + (f^{(k)}(a))^2.$$

The factorization of the space by its subspace $\operatorname{Ker} q$ is called the k - th order jet space $J^k_a(\mathbb{R}, \mathbb{R})$ of $C^\infty(\mathbb{R}, \mathbb{R})$ at the point a. Jet spaces were introduced by Ehresmann (Ehresmann, 1952, 1953) and have various applications in the theory of differential equations and differential relations, as well as in the theory of manifolds (Gromov, 1986), (Krasilshchik *et al.*, 1986).

It is possible to get the same quotient space using the following seminorm

$$m_k(f) = \max\{|f(a)|, |f'(a)|, |f''(a)|, \ldots, |f^{(k)}(a)|\}.$$

Let us consider a Hausdorff space X that is a quotient space of L with the projection $\eta : L \to X$, preserves the hyperseminorm q. Then it is possible to define a projection $\nu : L_q \to X$ preserves the hyperseminorm q and for which $\eta = \nu\tau$, i.e., the following diagram is commutative:



This gives us the following result.

Theorem 2.6. *a) L_q is the largest Hausdorff quotient space of the topological space L that preserves the hyperseminorm q.*

b) L_q is the largest quotient space of the topological space L in which the hyperseminorm q induces the hypernorm p_q.

It is possible to define two basic operators in a vector space L.

1. If z is an element from **L**, then the translation operator T_z is defined by the formula:

$$T_z(x) = x + z \text{ where } x, z \in L.$$

2. If $a \neq 0$ is an element from **F**, then the multiplication operator M_a is defined by the formula:

$$M_a(x) = ax \text{ where } x \in L.$$

Proposition 2.2. *Operators T_z and M_a are homeomorphisms of the semitopological vector space L.*

Proof. The axioms of a vector space imply that T_z and M_a are one-to-one mappings and their inverses are T_{-z} and M_{-a} , respectively. As addition is continuous in L, the operator T_z is also continuous. As scalar multiplication is continuous with respect to L, the operator M_a is also continuous. Proposition is proved. □

Corollary 2.4. *The topology of a semitopological vector space L is translation-invariant, or simply invariant, i.e., a subset A from L is open if and only if any its translation $A + a$ is open.*

As a result, such a topology is completely determined by any local base and thus, by any local base at **0**.

Let us consider two subsets K and C of a semitopological vector space L.

Theorem 2.7. *If C is compact, K is closed and $K \cap C = \varnothing$, then* **0** *has a neighborhood V such that*

$$(K + V) \cap (C + V) = \varnothing.$$

Proof Proof is similar to the proof of Theorem 1.10 from (Rudin, 1991) because it uses only the first property of semitopological vector spaces.

As topological vector spaces are special cases of semitopological vector spaces, Theorem 1.10 from (Rudin, 1991) is a corollary of Theorem 2.7.

In a topological space X, the weakest separation axiom is $\mathbf{T}_0$ (Kelly, 1955) where:

$$\mathbf{T}_0 \text{ (the } \textit{Kolmogorov Axiom}). \; \forall x, y \in X\, (\exists Ox(y \notin Ox) \vee \nexists Oy(x \notin Oy)).$$

Lemma 2.3. *In a topological space X, all points are closed if and only if X satisfies the axiom* $\mathbf{T}_0$.

Proof. Sufficiency. If X satisfies the axiom $\mathbf{T}_0$ and x is a point from X, then each point from the complement Cx of x has a neighborhood that does not contain x. Thus, all these neighborhoods are subsets of Cx. By definition, Cx is an open set (Kuratowski, 1966) and consequently, its complement x is a closed set.

Necessity. If $x, y \in X$ and the point x is closed, then y belongs to the complement Cx of x, which is open as the complement of a closed set (Kuratowski, 1966). Thus, y has a neighborhood Oy that is a subset of Cx. Consequently, Oy does not contain x. As points x and y are arbitrary, X satisfies the axiom $\mathbf{T}_0$.

Lemma is proved. □

We remind (Alexandroff, 1961) that T_3 - *spaces*, or *regular spaces*, are topological spaces in which satisfy Axiom $\mathbf{T}_3$:

$\mathbf{T}_3$ *For every point a and closed set B, there exist disjoint open sets which separately contain a and B.*

It means that points and closed sets are separated.

Note that there are semitopological vector spaces in which not all points are closed. The space $\mathbb{R}^\omega$ of all sequences of real numbers is an example of such a semitopological vector space. Moreover, in $\mathbb{R}^\omega$, there are no closed points.

As a point is a compact space, Theorem 2.5 implies the following result.

Corollary 2.5. *Every semitopological vector space L in which all points are closed is a regular space.*

Lemma 2.3 and Corollary 2.5 imply the following result.

Corollary 2.6. *In semitopological vector spaces, L axiom* $\mathbf{T}_0$ *implies axiom* $\mathbf{T}_3$.

As any regular space is a Hausdorff space (Alexandroff, 1961), we have the following result.

Corollary 2.7. *Every semitopological vector space L in which all points are closed is a Hausdorff space.*

Lemma 2.3 and Corollary 2.7 imply the following result.

Corollary 2.8. *In semitopological vector spaces, L axiom* $\mathbf{T}_0$ *implies axiom* $\mathbf{T}_2$.

As both sets $K + V$ and $C + V$ in Theorem 2.7 are open, the closure of $K + V$ does not intersect $C + V$, while the closure of $C + V$ does not intersect $K + V$. As any point a from L is a compact space, we can take $K = \{a\}$. Applying Theorem 2.7 to this situation, we obtain the result, which has a considerable interest according to (Rudin, 1991).

Corollary 2.9. *Any neighborhood O_a of any point a in a semitopological vector space L contains the closure of some neighborhood V_a of the same point a.*

As topological vector spaces are special cases of semitopological vector spaces, Theorem 1.11 from (Rudin, 1991) is a corollary of Corollary 2.9.

3. Mappings of hyperseminormed vector spaces

Let us consider a hyperseminormed vector space L, i.e., a vector space L with a system of hyperseminorms Q, a hyperseminormed vector space M with a system of hyperseminorms P, a hyperseminorm q from Q, a hyperseminorm p from P, and a subset V of the space L.

Vector spaces with systems of hyperseminorms (of hypernorms) will be called *polyhyperseminormed spaces (polyhypernormed spaces)* because vector spaces over $\mathbb{R}$ with systems of norms or seminorms are called *polynormed spaces* (see (Helemski, 1989); (Dosi, 2011)).

Definition 3.1. a) An operator (mapping) $A : L \to M$ is called (q, p) - *bounded at a point a* from L if for any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

b) An operator (mapping) $A : L \to M$ is called (q, p) - *bounded* if it is (q, p) - bounded at all points of L.

c) An operator (mapping) $A : L \to M$ is called V - *uniformly* (q, p) - *bounded* if for any positive real number k, there is a positive real number h such that for any element a from V and any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called *uniformly* (q, p) - bounded in V if for any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

Note that when the set V contains only one point (say a), then V - uniform (q, p) - boundedness coincides with (q, p) - boundedness at the point a.

Definitions imply the following result.

Lemma 3.1. *Any uniformly (q, p) - bounded in L operator is L - uniformly (q, p) - bounded and any L - uniformly (q, p) - bounded operator is (q, p) - bounded.*

At the same time, as the following example demonstrates, there are (q, p) - bounded operators that are not L - uniformly (q, p) - bounded.

Example 3.1. Let us take $L = M = \mathbb{R}$ and assume that q and p are both equal to the absolute value, while $A(x) = x^2$. This mapping (operator) is (q, p) - bounded but not L - uniformly (q, p) - bounded.

However, for linear operators, the inverse of Lemma 3.1 is also true.

Proposition 3.1. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (q, p) - bounded.

(2) A is uniformly (q, p) - bounded in L.

(3) For some point a, A is uniformly (q, p) - bounded at the point a.

(4) A is L - uniformly (q, p) - bounded.

Proof. Implications (2) ⇒ (1) ⇒ (3) directly follow from definitions. So, we need to prove only (3) ⇒ (2), namely, if $A : L \to M$ is (q, p) - bounded at a point a from L, then it is uniformly (q, p) - bounded.

Let us consider another point b from L and assume that $q(b - c) < k$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a)) = q(b - c) < k.$$

As A is (q, p) - bounded at a, there is a positive real number h such that $p(A(a) - A(d)) < h$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)) = p(A(a - d)) = p(A(a) - A(d)) < h.$$

This shows that A is (q, p) - bounded at the point b because c is an arbitrary point for which $q(b - c) < k$. Thus, A is uniformly (q, p) - bounded in L because for a fixed number k, we have the same number h for all points in L.

In addition, we see that by definition, properties (2) and (4) always coincide.

Proposition is proved. □

Corollary 3.1. *A linear operator (mapping) A is (q, p) - bounded if and only if it is (q, p) - bounded at* $\mathbf{0}$.

The above proof of Proposition 3.1 gives us the following result.

Corollary 3.2. *Any (q, p) - bounded linear operator (mapping) $A : L \to M$ is L - uniformly (q, p) - bounded.*

These results show that for linear operators, the concepts of a (q, p) - bounded at a point operator and of a (q, p) - bounded operator coincide.

For operators that are not linear, these results are true as the following examples demonstrate.

Example 3.2. Let us assume that $L = M = \mathbb{R}_\omega$ is the space of all real hypernumbers (cf. Example 2.1), while both hyperseminorms q and p are both equal to the absolute value $\| \cdot \|$ of real hypernumbers. Actually the absolute value $\| \cdot \|$ is a norm in the space $\mathbb{R}_\omega$ (Burgin, 2012).

For the operator A, we define $A(x) = x$ for all real hypernumbers x but the hypernumber $\nu = \mathrm{Hn}(i)_{i \in \omega}$ and put $A(\nu) = 1$. Then $\|\nu - (\nu + 1)\| = 1$ but $\|A(\nu) - A(\nu + 1)\| = \|1 - (\nu + 1)\| = \|\nu\| = \nu$ and this hypernumber is larger than any positive real number (Burgin, 2012). Thus, operator A is (q, p) - bounded at any real number but it is not (q, p) - bounded at the hypernumbers ν.

This shows that an operator can be (q, p) - bounded at one point and not (q, p) - bounded at another point of L.

Example 3.3. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space (cf. Example 2.1) where the hypernorm $\| \cdot \|$ is defined by the following formula:

If $f : \mathbb{R} \to \mathbb{R}$, then $\|f\| = \mathrm{Hn}(a_i)_{i \in \omega}$ where $a_i = max\{|f(x)|;\ a_i \in [-i, i]\}$.

We define $A(f) = f$ for all real functions f but the function $\nu(x) = x^2$ and put $A(x^2) = e(x)$ where $e(x) = 1$ for all $x \in \mathbb{R}$. This operator A is (q, p) - bounded at any constant function from L but it is not (q, p) - bounded at ν. At the same time, taking $u(x) = x^2 + 1$, we have $\|\nu - u\| = 1$, while $\|A(\nu) - A(u)\| = \|e - u\| = \mathrm{Hn}(i)_{i \in \omega}$ and this hypernumber is larger than any positive real number (Burgin, 2011).

This also shows that an operator can be (q, p) - bounded at one point and not (q, p) - bounded at another point of L.

However, for norms and seminorms, we do not need additional conditions to establish the result of Proposition 3.1.

Proposition 3.2. *If q is a seminorm, then an operator (mapping) $A : L \to M$ is (q, p) - bounded if and only if it is (q, p) - bounded, at least, at one point.*

Proof. Let us consider two points a and c from L and assume that an operator $A : L \to M$ is (q, p) - bounded at the point a. Then taking a point b such that $q(c - b) < u$ where u is a positive real number.

As q is a seminorm, $q(a - c)$ is equal to some positive real number w. Thus, by properties of seminorms, we have

$$q(a - b) = q(a - c + c - b) \leq q(a - c) + q(c - b) < w + u.$$

As the operator A is (q, p) - bounded at the point a and $q(a-c) < w+1$, we have a positive real number h such that $p(A(a) - A(b)) < h$ and a positive real number k such that $p(A(a) - A(c)) < k$. Consequently,

$$p(A(c) - A(b)) \leq p(A(a) - A(c)) + p(A(a) - A(b)) < k + h.$$

As b is an arbitrary point from L, A is (q, p) - bounded at the point c.

As c is an arbitrary point from L, the operator A is (q, p) - bounded.

Proposition is proved. □

Proposition 3.2 implies the following results.

Corollary 3.3. *The concepts of a (q, p) - bounded at a point operator and of a (q, p) - bounded operator coincide when q is a seminorm.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Corollary 3.4. *When q is a seminorm, an operator (mapping) A is (q, p) - bounded if and only if it is (q, p) - bounded at* $\mathbf{0}$.

The above proof of Proposition 3.2 gives us the following result.

Corollary 3.5. *If q is a seminorm, then any (q, p) - bounded operator (mapping) $A : L \to M$ is L - uniformly (q, p) - bounded.*

Proposition 3.3. *If q is a seminorm and there is a (q, p) - bounded operator (mapping) A of the linear space L onto the linear space M, then p is a finite hyperseminorm.*

Proof. Let us take a point u from M. As A is a projection (surjection), there are points a and b such that $A(a) = \mathbf{0}$ and $A(b) = u$. As q is a seminorm, $q(b - a)$ is less than some positive real number w. As the operator A is (q, p) - bounded, there is a positive real number h such that $p(A(a) - A(b)) < h$

$$p(u) = p(u - \mathbf{0}) = p(A(b) - A(a)) < h.$$

As u is an arbitrary point from M, the hyperseminorm p is finite.

Proposition is proved. □

Note that a finite hyperseminorm is not always a seminorm and a finite hypernorm is not always a norm.

Definition 3.2. (Burgin, 2012). A real hypernumber is called *monotone* is it has a monotone representative.

For instance, all real numbers are monotone hypernumbers (Burgin, 2012). At the same time, all finite monotone real hypernumbers are real numbers (Burgin, 2012). Thus, Proposition 3.3 implies the following result.

Corollary 3.6. *If q is a seminorm, there is a (q, p) - bounded operator (mapping)A of the linear space L onto the linear space M and all values of p are monotone hypernumbers, then p is a seminorm.*

Definitions imply the following results.

Lemma 3.2. *If $W \subseteq V \subseteq L$, then any V - uniformly (q, p) - bounded operator is W - uniformly (q, p) - bounded and any uniformly (q, p) - bounded in V operator is uniformly (q, p) - bounded in W.*

Lemma 3.3. *Any V - uniformly (q, p) - bounded operator is (q, p) - bounded in V.*

Let us consider a binary relation u between the system of hyperseminorms Q, the system of hyperseminorms P and a subset V of the space L.

Definition 3.3. a) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *bounded at a point a* from L if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - bounded at the point a.

b) An operator (mapping) $A : L \to M$ is called V - *uniformly* (Q, u, P) - *bounded* if for any hyperseminorms q and p with $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any element a from V and any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (Q, u, P) - *bounded* in V if for any hyperseminorms q and p with $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *bounded* if it is (Q, u, P) - bounded at all points of L.

It means that an operator (mapping) A is (Q, u, P) - bounded if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - bounded.

Note that when the set V contains only one point (say a), then V - uniform (Q, u, P) - boundedness coincides with (Q, u, P) - boundedness at the point a.

Lemma 3.1 implies the following result.

Lemma 3.4. *Any uniformly (Q, u, P) - bounded operator in L is L - uniformly (Q, u, P) - bounded, while any L - uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.1 demonstrates that there are (Q, u, P) - bounded operators that are not L - uniformly (Q, u, P) - bounded.

However, for linear operators, the inverse of Lemma 3.4 is also true because Proposition 3.1 implies the following result.

Proposition 3.4. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (Q, u, P) - bounded.

(2) A is uniformly (Q, u, P) - bounded in L.

(3) For some point a, A is uniformly (Q, u, P) - bounded at the point a.

(4) A is L - uniformly (Q, u, P) - bounded.

Corollary 3.7. *A linear operator (mapping) A is (Q, u, P) - bounded if and only if it is (Q, u, P) - bounded at* **0**.

Corollary 3.2 implies the following result.

Corollary 3.8. *Any (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - bounded.*

These results show that for linear operators, the concepts of a (Q, u, P) - bounded at a point operator and a (Q, u, P) - bounded operator coincide.

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Examples 3.2 and 3.3 demonstrate that there are operators that are (Q, u, P) - bounded at one point and not (Q, u, P) - bounded at another point.

However, for norms and seminorms, we do not need additional conditions to establish the result of Proposition 3.4. We remind that the definability domain of the relation u is defined as

$$Du = \{q;\ \text{there is a pair } (q, p) \text{ that belongs to } u\}.$$

Then Proposition 3.2 implies the following result.

Proposition 3.5. *If all q from the definability domain Du of u are seminorms, then an operator (mapping) $A : L \to M$ is (Q, u, P) - bounded if and only if it is (Q, u, P) - bounded, at least, at one point.*

Proposition 3.5 implies the following result.

Corollary 3.9. *The concepts of (Q, u, P) - bounded at a point operators and (Q, u, P) - bounded operator coincide when all q from the definability domain Du of u are seminorms.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Corollary 3.10. *When all q from the definability domain Du of u are seminorms, an operator (mapping) A is (Q, u, P) - bounded if and only if it is (Q, u, P) - bounded at $\mathbf{0}$.*

The above proof of Proposition 3.2 gives us the following result.

Corollary 3.11. *If all q from the definability domain Du of u are seminorms, then any (Q, u, P) - bounded operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - bounded.*

Proposition 3.3 implies the following result.

Proposition 3.6. *If all q from the definability domain Du of u are seminorms and there is a (Q, u, P) - bounded operator (mapping) A of the linear space L onto the linear space M, then all p from the range $Rg\ u$ of u are finite hyperseminorms.*

Corollary 3.12. *If all q from the definability domain Du of u are seminorms and there is a (Q, u, P) - bounded operator (mapping) A of the linear space L onto the linear space M, and all values of all p from the range $Rg\ u$ are monotone hypernumbers, then all such p are seminorms.*

Definitions imply the following results.

Lemma 3.5. *If $W \subseteq V \subseteq L$, then any V - uniformly (Q, u, P) - bounded operator is W - uniformly (q, p) - bounded and any uniformly (Q, u, P) - bounded in V operator is uniformly (q, p) - bounded in W.*

Lemma 3.6. *Any V - uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded in V.*

Let us take a subset V of the space L.

Definition 3.4. a) An operator (mapping) $A : L \to M$ is called *uniformly (Q, u, P) - bounded at a point a* from L if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

b) An operator (mapping) $A : L \to M$ is called *u - uniformly (Q, u, P) - bounded* if it is uniformly (Q, u, P) - bounded at all points of L.

c) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *bounded* in V if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any elements a and b from V, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

d) An operator (mapping) $A : L \to M$ is called uV - *uniformly* (Q, u, P) - *bounded* in V if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, and any elements a from V and b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

Asking whether any (Q, u, P) - bounded at a point operator (mapping) is uniformly (Q, u, P) - bounded at the same point, we find that the answer is negative.

Example 3.4. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions. It is possible (Burgin, 2012) for all real numbers x, to define seminorms $q_{ptx} = p_{ptx}$ by the following formula

$$q_{ptx}(f) = p_{ptx}(f) = |f(x)|.$$

We define $A(f) = xf(x)$ for all real functions f and $u = \{(q_{ptx}, p_{ptx}); x \in \mathbb{R}\}$. Taking the function $f(x) = x$ as the point a from L, we see that $A(f) = x^2$. Thus, taking some positive real number k, e.g., $k = 1$, the corresponding h from Definition 3.2 always exists but it grows with the growth of x. For instance, when $k = 1$, we have

$$q_{pt1}(f - g) < 1 \text{ implies } p_{pt1}(A(f) - A(g)) = p_{pt1}(xf - xg) < 1.$$

At the same time, $q_{pt10}(f - g) < 1$ does not imply $p_{pt10}(A(f) - A(g)) < 1$. It only implies $p_{pt10}(A(f) - A(g)) = p_{pt10}(xf - xg) < 10$. This means that for any pair (q_{ptx}, p_{ptx}) of seminorms and a number k, we need to find a specific number h to satisfy Definition 3.3 a. Consequently, the operator A is (Q, u, P) - bounded at f but it is not uniformly (Q, u, P) - bounded at f.

The same example shows that there are (Q, u, P) - bounded operators that are not uniformly (Q, u, P) - bounded.

It is also possible to ask whether Propositions 3.4 and 3.5 remain true for uniformly (Q, u, P) - bounded operators. In this case, the answer is positive.

Proposition 3.7. *If all q from the definability domain Du of the relation u are seminorms, then an operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - bounded if and only if it is uniformly (Q, u, P) - bounded, at least, at one point.*

Indeed, Proposition 3.7 is a direct corollary of Proposition 3.5 because any uniformly (Q, u, P) - bounded at a point operator is (Q, u, P) - bounded at the same point and any uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.

Proposition 3.7 implies the following result.

Corollary 3.13. *The concepts of uniformly (Q, u, P) - bounded at a point operators and uniformly (Q, u, P) - bounded operators coincide when all q from the definability domain Du of u are seminorms.*

Note that Examples 3.2 and 3.3 show this is not true for the general case of hyperseminorms.

Proposition 3.8. *If all q from the definability domain Du of u are seminorms and there is a uniformly* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, then all p from the range Rg u of u are finite hyperseminorms.*

Indeed, Proposition 3.8 is a direct corollary of Proposition 3.6 because any uniformly (Q, u, P) - bounded operator is (Q, u, P) - bounded.

Corollary 3.14. *If all q from the definability domain Du of u are seminorms and there is a uniformly* (Q, u, P) *- bounded operator (mapping) A of the linear space L onto the linear space M, and all values of all p from the range Rg u are monotone hypernumbers, then all such p are seminorms.*

Definitions imply the following results.

Lemma 3.7. *a) Any uniformly* (Q, u, P) *- bounded at a point a operator A is* (Q, u, P) *- bounded at the point* a.

b) Any u-uniformly (Q, u, P) *- bounded operator A is* ((Q, u, P) *- bounded.*

Lemma 3.8. *Any u-uniformly* (Q, u, P) *- bounded in L operator is u-uniformly* (Q, u, P) *- bounded.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that hyperseminorms q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.1 demonstrates that there are u-uniformly (Q, u, P) - bounded operators that are not uniformly (Q, u, P) - bounded because if Q has only one hyperseminorm q, P also has only one hyperseminorm p and u is a complete relation, then any (Q, u, P) - bounded operator is u-uniformly (Q, u, P) - bounded.

However, for linear operators, this is impossible as Proposition 3.1 allows us to prove the following result.

Proposition 3.9. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is u-uniformly (Q, u, P) *- bounded.*

(2) A is u-uniformly (Q, u, P) *- bounded in L.*

(3) For some point a, A is uniformly (Q, u, P) *- bounded at the point a.*

Proof. Implications (2) ⇒ (1) ⇒ (3) directly follow from definitions. So, we need to prove only (3) ⇒ (2), namely, if $A : L \to M$ is uniformly (Q, u, P) - bounded at a point a from L, then it is uniformly (Q, u, P) - bounded in L.

Let us consider another point b from L, take two hyperseminorms q and p with $(q, p) \in u$, and assume that $q(b - c) < k$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a))) = q(b - c) < k.$$

As A is uniformly (Q, u, P) - bounded at a, it is also (q, p) - bounded at a. Thus, there is a positive real number h such that $p(A(a) - A(d)) < h$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)))) = p(A(a - d)) = p(A(a) - A(d)) < h.$$

This shows that A is (q, p) - bounded at the point b because c is an arbitrary point for which $q(b - c) < k$ and thus, A is u-uniformly (Q, u, P) - bounded because q and p are arbitrary hyperseminorms with $(q, p) \in u$. In addition, A is uniformly (q, p) - bounded in L because for a fixed number k, we have the same number h for all points in L.

Proposition is proved. □

Corollary 3.15. *A linear operator (mapping) $A : L \to M$ is u-uniformly (Q, u, P) - bounded if and only if it is uniformly (Q, u, P) - bounded at* $\mathbf{0}$.

Corollary 3.2 implies the following result.

Corollary 3.16. *Any u-uniformly (Q, u, P) - bounded linear operator (mapping) A is u-uniformly (Q, u, P) - bounded in L.*

These results show that for linear operators, different types of uniformly bounded operators coincide.

Proposition 3.10. *If the relation u is finite, then an operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - bounded (at a point a) if and only if it is (Q, u, P) - bounded (at the point a).*

Proof. As any uniformly (Q, u, P) - bounded (at a point a) operator is (Q, u, P) - bounded (at the same point), we need only to show that when the relation u is finite, a (Q, u, P) - bounded (at a point a) operator $A : L \to M$ is uniformly (Q, u, P) - bounded (at the point a). At first, we consider local boundedness.

Indeed, by Definition 3.3, for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p)-bounded at the point a, that is, by Definition 3.1, the following condition is true:

Condition 1. For any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < k$ implies the inequality $p(A(b) - A(a)) < h$.

This number h can be different for different pairs (q, p), but because u is finite, there is only a finite number of these pairs. So, we can take

$$l = max\{h; h \text{ satisfies Condition 1 for a pair } (q, p) \in u\}$$

and this number l will satisfy the condition from Definition 3.4. Thus, the operator A is uniformly (Q, u, P) - bounded at the point a.

The global case is proved in a similar way.

Proposition is proved. □

Corollary 3.17. *If systems of hyperseminorms Q and P are finite, then an operator (mapping) A is uniformly* (Q, u, P) *- bounded (at a point a) if and only if it is* (Q, u, P) *- bounded (at the point a).*

Now let us study different types of continuity in polyhyperseminormed vector spaces.

Definition 3.5. a) An operator (mapping) $A : L \to M$ is called (q, p) - *continuous at a point a* from L if for any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

b) An operator (mapping) $A : L \to M$ is called (q, p) - *continuous* if it is (q, p) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (q, p) - *continuous in* $V \subseteq L$ if for any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called V - *uniformly* (q, p) - *continuous* if for any positive real number k, there is a positive real number h such that for any element a from $V \subseteq L$ and any element b from L, the inequality $q(b - a) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that when the set V contains only one point (say a), then V - *uniform* (q, p) - *continuity* coincides with (q, p) - continuity at the point a. Besides, to be L - *uniformly* (q, p) - *continuous* or to be *uniformly* (Q, u, P) - *continuous* in L means the same for all operators.

Definitions imply the following results.

Lemma 3.9. *For any* $V \subseteq L$*, any* V *- uniformly* (q, p) *- continuous operator is* (q, p) *- continuous in* V*.*

Lemma 3.10. *Any* L *- uniformly* (q, p) *- continuous operator is* (q, p) *- continuous.*

At the same time, as the following example demonstrates, there are (q, p) - continuous operators that are not L - uniformly (q, p) - continuous.

Example 3.5. Let us take $L = M = \mathbb{R}$ and assume that q and p are both equal to the absolute value, while $A(x) = x^2$. This mapping (operator) is (q, p) - continuous but not L - uniformly (q, p) - continuous.

However, for linear operators, the inverse of Lemma 3.9 is also true.

Proposition 3.11. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (q, p) *- continuous.*

(2) A is uniformly (q, p) *- continuous in L.*

(3) For some point a, A is uniformly (q, p) *- continuous at the point a.*

(4) A is L - uniformly (q, p) - continuous.

Proof. Implications (2) ⇒ (1) ⇒ (3) directly follow from definitions. So, we need to prove only (3) ⇒ (2), namely, if $A : L \to M$ is (q, p) - continuous at a point a from L, then it is uniformly (q, p) - continuous in L.

Let us consider a positive real number k. Then because A is (q, p) - continuous at the point a, there is a positive real number h, such that the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Let us take another point b from L and assume that $q(b-c) < h$ for some c from L. Then taking $d = c - (b - a)$, we have

$$q(a - d) = q(a - (c - (b - a))) = q(b - c) < h.$$

As A is (q, p) - continuous at a, we have $p(A(a) - A(d)) < k$. As A is linear operator, we have

$$p(A(b) - A(c)) = p(A(b - c)) = p(A(a - (c - (b - a)))) = p(A(a - d)) = p(A(a) - A(d)) < k.$$

This shows that A is (q, p) - continuous at the point b because c is an arbitrary point for which $q(b - c) < h$. Thus, A is uniformly (q, p) - continuous in L because for a fixed number k, we have the same number h for all points in L.

In addition, we see that by definition, properties (2) and (4) always coincide.

Proposition is proved. □

Corollary 3.18. *A linear operator (mapping) A is (q, p) - continuous if and only if it is (q, p) - continuous at* $\mathbf{0}$.

The above proof of Proposition 3.4 gives us the following result.

Corollary 3.19. *Any (q, p) - continuous linear operator (mapping) $A : L \to M$ is L - uniformly (q, p) - continuous.*

These results show that for linear operators, the concepts of (q, p) - continuous at a point operators and (q, p) - continuous operators coincide.

For operators that are not linear, these results are not true as the following examples demonstrate.

Example 3.6. Let us take $L = M = \mathbb{R}_\omega$ (cf. Example 2.1) and assume that q and p are both equal to the absolute value $\|\cdot\|$ of real hypernumbers. We define $A(x) = x$ for all real hypernumbers x but the hypernumber $\nu = Hn(i)_{i\in\omega}$ and put $A(\nu) = 1$. Then $\|\nu - (\nu + 1)\| = 1$ but $\|A(\nu) - A(\nu + 1)\| = \|1 - (\nu + 1)\| = \|\nu\| = \nu$ and this hypernumber is larger than any positive real number (Burgin, 2012). Thus, operator A is (q, p) - continuous at any real number but it is not (q, p) - continuous at ν.

This shows that an operator can be (q, p) - continuous at one point and not (q, p) - continuous at another point of L and thus not (q, p) - continuous in L, as well as not L - uniformly (q, p) - continuous.

Example 3.7. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions is a hypernormed space (cf. Example 2.1) where the hypernorm $\|\cdot\|$ is defined by the following formula:

$$\text{If } f : \mathbb{R} \to \mathbb{R}, \text{ then } \|f\| = Hn(a_i)_{i\in\omega} \text{ where } a_i = max\{|f(x)|; a_i \in [-i, i]\}.$$

We define $A(f) = f$ for all real functions f but the function $v(x) = x^2$ and put $A(x^2) = e(x)$ where $e(x) = 1$ for all $x \in \mathbb{R}$. This operator A is (q, p) - continuous at any constant function from L, but it is not (q, p) - continuous at v. At the same time, taking $u(x) = x^2 + 1$, we have $\|v - u\| = 1$, while $\|A(v) - A(u)\| = \|e - u\| = Hn(i)_{i\in\omega}$ and this hypernumber is larger than any positive real number (Burgin, 2012).

This also shows that an operator can be (q, p) - continuous at one point and not (q, p) - continuous at another point of L and thus not (q, p) - continuous in L, as well as not L - uniformly (q, p) - continuous.

Definitions imply the following result.

Lemma 3.11. *If $W \subseteq V \subseteq L$, then any V - uniformly (q, p) - continuous operator is W - uniformly (q, p) - continuous.*

Now let us consider continuity with respect to a binary relation u between systems of hyperseminorms.

Definition 3.6. a) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *continuous at a point* a from L if for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - continuous at the point a.

b) An operator (mapping) $A : L \to M$ is called (Q, u, P) - *continuous* if it is (Q, u, P) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called *uniformly* (Q, u, P) - *continuous in* $V \subseteq L$ if for any hyperseminorms q and p such that $(q, p) \in u$ and any positive real number k, there is a positive real number h such that for any elements a and b from V, the inequality $q(a-b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called V - *uniformly* (Q, u, P) - *continuous* if for any hyperseminorms q and p such that $(q, p) \in u$ and for any positive real number k, there is a positive real number h such that for any element a from $V \subseteq L$ and any element b from L, the inequality $q(b - a) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that to be L - *uniformly* (Q, u, P) - *continuous* or to be *uniformly* (Q, u, P) - *continuous in* L means the same for all operators.

Lemma 3.10 implies the following result.

Lemma 3.12. *Any uniformly (Q, u, P) - continuous in L operator is (Q, u, P) - continuous.*

At the same time, taking $L = M = \mathbb{R}$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value and $u = \{(q, p)\}$, we see that Example 3.5 demonstrates that there are (Q, u, P) - continuous operators that are not L - uniformly (Q, u, P) - continuous.

However, for linear operators, the inverse of Lemma 3.12 is also true as Proposition 3.11 implies the following result.

Proposition 3.12. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is (Q, u, P) - continuous.

(2) A is uniformly (Q, u, P) - continuous in L.

(3) For some point a, A is uniformly (Q, u, P) - continuous at the point a.

(4) A is L - uniformly (Q, u, P) - continuous.

Corollary 3.20. *A linear operator (mapping) A is (Q, u, P) - continuous if and only if it is (Q, u, P) - bounded at* $\mathbf{0}$.

Corollary 3.19 implies the following result.

Corollary 3.21. *Any (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - continuous.*

These results show that for linear operators, the concepts of (Q, u, P) - continuous at a point operators and (Q, u, P) - continuous operators coincide.

At the same time, taking $L = M = \mathbb{R}_\omega$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value of real hypernumbers and $u = \{(q, p)\}$, we see that Example 3.6 demonstrates that there are operators that are (Q, u, P) - continuous at one point and not (Q, u, P) - continuous at another point. A similar situation is also presented in Example 3.7.

Definitions and Lemma 3.10 imply the following result.

Lemma 3.13. *If $W \subseteq V \subseteq L$, then any V - uniformly (Q, u, P) - continuous operator is W - uniformly (Q, u, P) - continuous.*

Lemma 3.9 imply the following result.

Lemma 3.14. *For any $V \subseteq L$, any V - uniformly (Q, u, P) - continuous operator is (Q, u, P) - continuous in V.*

Let us study relations between relative continuity and relative boundedness.

Theorem 3.1. *A linear operator (mapping) $A : L \to M$ is (Q, u, P) - continuous if and only if it is (Q, u, P) - bounded.*

Proof. Sufficiency. Let us consider a (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ and suppose that A is not (Q, u, P) - continuous. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not (q, p) - continuous. By Corollary 3.9, A is not (q, p) - continuous at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < 1/n$ while $p(A(x_n)) > k$.

Let us consider the set $Z = \{z_n; n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q(n \cdot x_n) = n \cdot q(x_n) < 1,$$

i.e., Z is a q - bounded set. At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A(n \cdot x_n)) = n \cdot p(A(x_n)) > kn.$$

Thus, the image of Z is not a p - bounded set and A is not a (Q, u, P) - bounded operator. This contradicts our assumption and by *reductio ad absurdum*, A is (Q, u, P) - continuous.

Necessity. Let us consider a (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ and suppose that A is not (Q, u, P) - bounded. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not (q, p) - bounded. By Corollary 3.3, A is not (q, p) - bounded at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n;\ n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n);\ n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.5 and shows A is not a (Q, u, P) - continuous operator. Thus, we have a contradiction with our assumption that A is a (Q, u, P) - continuous operator. By *reductio ad absurdum*, A is (Q, u, P) - bounded.

Theorem is proved. □

Corollary 3.22. *A linear operator (mapping) $A : L \to M$ is (q, p) - continuous if and only if it is (q, p) - bounded.*

Corollary 3.22 implies the following result.

Corollary 3.23. *A linear operator (mapping) $A : L \to M$ is L - uniformly (Q, u, P) - continuous if and only if it is L - uniformly (Q, u, P) - bounded.*

As topology of topological vector spaces is determined by system of seminorms (Rudin, 1991), Theorem 3.1 gives us the following classical result ((Dunford & Schwartz, 1958); (Rudin, 1991)).

Corollary 3.24. *A linear mapping A of a topological vector space L into a topological vector space M is continuous if and only if it is bounded.*

As for linear operators (mappings) continuity at a point coincides with continuity and boundedness at a point coincides with boundedness, we have the following results.

Corollary 3.25. *A linear operator (mapping) $A : L \to M$ is (q, p) - continuous at a point a if and only if it is (q, p) - bounded at a.*

Corollary 3.26. *A linear operator (mapping) $A : L \to M$ is (Q, u, P) - continuous at a point a if and only if it is (Q, u, P) - bounded at a.*

Let us take a vector subspace V of L and consider uniform (Q, u, P) - continuity in V.

Theorem 3.2. *A linear operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - continuous in V if and only if it is uniformly (Q, u, P) - bounded in V.*

Proof. Sufficiency. Let us consider a vector subspace V of L and a uniformly (Q, u, P) - bounded in V linear operator (mapping) $A : L \to M$ and suppose that A is not uniformly (Q, u, P) - continuous in V. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not uniformly (q, p) - continuous. Consequently, there is a positive real number k such that for any natural number n, there are elements x_n and y_n from V for which $q(x_n - y_n) < 1/n$ while $p(A(x_n) - A(y_n)) > k$.

Let us consider two sets $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ and $U = \{u_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ and $u_n = n \cdot y_n$ for all $n = 1, 2, 3, \ldots$ As V is a vector subspace of L, then Z and U are subsets of V. Besides,

$$q(z_n - u_n) = q(n \cdot x_n - n \cdot y_n) = q(n \cdot (x_n - y_n)) = n \cdot q(x_n - y_n) < 1.$$

It means that the set $\{z_n - u_n;\ n = 1, 2, 3, \ldots\}$ is q - bounded.

At the same time, as A is a linear operator, we have

$$p(A(z_n - u_n)) = p(A(n \cdot x_n - n \cdot y_n)) = n \cdot p(A(x_n) - A(y_n)) > kn.$$

It means that the set $\{A(z_n - u_n);\ n = 1, 2, 3, \ldots\}$ is not p - bounded. Thus, A is not a uniformly (Q, u, P) - bounded in V operator. This contradicts our assumption and by *reductio ad absurdum*, A is uniformly (Q, u, P) - continuous in V.

Necessity. Let us consider a uniformly (Q, u, P) - continuous in V linear operator (mapping) $A : L \to M$ and suppose that A is not uniformly (Q, u, P) - bounded in V. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not uniformly (q, p) - bounded in V. By Corollary 3.3, A is not (q, p) - bounded in V at $\mathbf{0}$ as V is a vector subspace of L. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from V for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n;\ n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n);\ n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.5 and shows A is not a uniformly (Q, u, P) - continuous in V operator. Thus, we have a contradiction with our assumption that A is a uniformly (Q, u, P) - continuous in V operator. By *reductio ad absurdum*, A is uniformly (Q, u, P) - bounded in V.

Theorem is proved. □

Corollary 3.27. *For any vector subspace V of L, a linear operator (mapping) $A : L \to M$ is uniformly (q, p) - continuous in V if and only if it is uniformly (q, p) - bounded in V.*

As before, V is a vector subspace of L and we study V - uniform (Q, u, P) - continuity.

Theorem 3.3. *A linear operator (mapping) $A : L \to M$ is V - uniformly (Q, u, P) - continuous if and only if it is V - uniformly (Q, u, P) - bounded.*

Proof. Sufficiency. Let us consider a vector subspace V of L and a V - uniformly (Q, u, P) - bounded linear operator (mapping) $A : L \to M$ and suppose that A is not V - uniformly (Q, u, P) - continuous. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not V - uniformly (q, p) - continuous. Consequently, there is a positive real number k such that for any natural number n, there are elements x_n from V and y_n from L for which $q(x_n - y_n) < 1/n$ while $p(A(x_n) - A(y_n)) > k$.

Let us consider two sets $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ and $U = \{u_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = n \cdot x_n$ and $u_n = n \cdot y_n$ for all $n = 1, 2, 3, \ldots$ As V is a vector subspace of L, then Z is a subset of V. Besides,

$$q(z_n - u_n) = q(n \cdot x_n - n \cdot y_n) = q(n \cdot (x_n - y_n)) = n \cdot q(x_n - y_n) < 1.$$

It means that the set $\{z_n - u_n;\ n = 1, 2, 3, \ldots\}$ is q - bounded.

At the same time, as A is a linear operator, we have

$$p(A(z_n - u_n)) = p(A(n \cdot x_n - n \cdot y_n)) = n \cdot p(A(x_n) - A(y_n)) > kn.$$

It means that the set $\{A(z_n - u_n);\ n = 1, 2, 3, \ldots\}$ is not p - bounded. Thus, A is not a V - uniformly (Q, u, P) - bounded operator. This contradicts our assumption and by *reductio ad absurdum*, A is V - uniformly (Q, u, P) - continuous.

Necessity. Let us consider a V - uniformly (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ and suppose that A is not V - uniformly (Q, u, P) - bounded. It means that for some pair $(q, p) \in u$ of hyperseminorms q and p, the operator A is not V - uniformly (q, p) - bounded. By Corollary 3.3, A is not (q, p) - bounded at $\mathbf{0}$. Consequently, there is a positive real number k such that for any natural number n, there is an element x_n from L for which $q(x_n) < k$ while $p(A(x_n)) > n$.

Let us consider the set $Z = \{z_n;\ n = 1, 2, 3, \ldots\}$ where $z_n = (1/n) \cdot x_n$ for all $n = 1, 2, 3, \ldots$ Then

$$q(z_n) = q((1/n) \cdot x_n) = (1/n) \cdot q(x_n) < k/n.$$

It means that the sequence $\{z_n; n = 1, 2, 3, \ldots\}$ q - converges to $\mathbf{0}$.

At the same time, as A is a linear operator, we have

$$p(A(z_n)) = p(A((1/n) \cdot x_n)) = (1/n) \cdot p(A(x_n)) > k.$$

It means that the sequence $\{A(z_n); n = 1, 2, 3, \ldots\}$ does not p - converge to $\mathbf{0}$. This violates conditions from Definition 3.6 and shows A is not a V - uniformly (Q, u, P) - continuous operator. Thus, we have a contradiction with our assumption that A is a V - uniformly (Q, u, P) - continuous operator. By *reductio ad absurdum*, A is V - uniformly (Q, u, P) - bounded.

Theorem is proved. □

Corollary 3.28. *For any subset V of L, a linear operator (mapping) $A : L \to M$ is V - uniformly (q, p) - continuous if and only if it is V - uniformly (q, p) - bounded.*

Let us take a subset V of the space L.

Definition 3.7. a) An operator (mapping) $A : L \to M$ is called uniformly (Q, u, P) - *continuous at a point a* from L if for any positive real number k, there is a positive real number h such that for any hyperseminorms q and p with $(q, p) \in u$, for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

b) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *continuous* if it is uniformly (Q, u, P) - continuous at all points of L.

c) An operator (mapping) $A : L \to M$ is called u - *uniformly* (Q, u, P) - *continuous in V* if for any positive real number k, there is a positive real number h such that for any elements a and b from V and any hyperseminorms q and p with $(q, p) \in u$, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

d) An operator (mapping) $A : L \to M$ is called uV - *uniformly* (Q, u, P) - *continuous* if for any positive real number k, there is a positive real number h such that for any elements a from V and b from L, and any hyperseminorms q and p with $(q, p) \in u$, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

Note that to be uL - *uniformly* (Q, u, P) - *continuous* or to be u - *uniformly* (Q, u, P) - *continuous in L* means the same for all operators.

It it possible to ask a question how u - uniform (Q, u, P) - continuity is connected to (Q, u, P) - continuity. The following example and Lemma 3.5 clarify this situation.

Example 3.8. Let us take $L = M = C(\mathbb{R}, \mathbb{R})$, while the space $C(\mathbb{R}, \mathbb{R})$ of all continuous real functions. It is possible (Burgin, 2012) for all real numbers x, to define seminorms $q_{ptx} = p_{ptx}$ by the following formula

$$q_{ptx}(f) = p_{ptx}(f) = |f(x)|.$$

We define $A(f) = xf(x)$ for all real functions f and $u = \{(q_{ptx}, p_{ptx}); x \in \mathbb{R}\}$. Taking the function $f(x) = x$ as the point a from L, we see that $A(f) = x^2$. Thus, taking some positive real number k, e.g., $k = 1$, the corresponding h from Definition 3.2 always exists but it decreases with the growth of x. For instance, when $k = 1$, we have

$$q_{pt1}(f - g) < 1 \text{ implies } p_{pt1}(A(f) - A(g)) = p_{pt1}(xf - xg) < 1.$$

At the same time, $q_{pt10}(f - g) < 1$ does not imply $p_{pt10}(A(f) - A(g)) < 1$. It only implies $p_{pt10}(A(f) - A(g)) = p_{pt10}(xf - xg) < 10$. To have $p_{pt10}(A(f) - A(g)) < 1$, we need $q_{pt10}(f - g) < 0.1$.

It means that for any pair (q_{ptx}, p_{ptx}) of seminorms and a number k, we need to find a specific number h to satisfy Definition 3.7.a. Consequently, the operator A is (Q, u, P) - continuous at $f = x$ but it is not uniformly (Q, u, P) - continuous at f.

The same example shows that there are (Q, u, P) - continuous operators that are not u - uniformly (Q, u, P) - continuous.

Definitions imply the following result.

Lemma 3.15. *a) Any uniformly (Q, u, P) - continuous at a point a operator A is (Q, u, P) - continuous at the point a.*

b) Any u - uniformly (Q, u, P) - continuous operator A is (Q, u, P) - continuous.

Lemma 3.16. *Any u - uniformly (Q, u, P) - continuous in L operator is u - uniformly (Q, u, P) - continuous.*

For linear operators, the inverse of Lemma 3.15 is also true.

Proposition 3.13. *The following conditions are equivalent for a linear operator (mapping) A:*

(1) A is u - uniformly (Q, u, P) - continuous.

(2) A is u - uniformly (Q, u, P) - continuous in L.

(3) For some point a, A is uniformly (Q, u, P) - continuous at the point a.

(4) A is uL - uniformly (Q, u, P) - continuous.

Corollary 3.29. *A linear operator (mapping) A is u - uniformly (Q, u, P) - continuous in L if and only if it is (Q, u, P) - continuous at* **0**.

Corollary 3.20 implies the following result.

Corollary 3.30. *Any u - uniformly (Q, u, P) - continuous linear operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous in L.*

These results show that for linear operators, the concepts of uniformly (Q, u, P) - continuous at a point operators and u - uniformly (Q, u, P) - continuous operators coincide.

At the same time, taking $L = M = \mathbb{R}_\omega$, $Q = \{q\}$, $P = \{p\}$, and assuming that q and p are both equal to the absolute value of real hypernumbers and $u = \{(q, p)\}$, we see that Example 3.6 demonstrates that there are operators that are (Q, u, P) - continuous at one point and not (Q, u, P) - continuous at another point. A similar situation is also presented in Example 3.7.

Definitions and Lemma 3.9 imply the following result.

Lemma 3.17. *If $W \subseteq V \subseteq L$, then any u - uniformly (Q, u, P) - continuous in V operator is u - uniformly (Q, u, P) - continuous in W.*

For finite relations u, different concepts of uniform continuity coincide.

Proposition 3.14. *If the relation u is finite, then, an operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous (u - uniformly (Q, u, P) - continuous at a point a) if and only if it is (Q, u, P) - continuous ((Q, u, P) - continuous at a point a).*

Proof. As any u - uniformly (Q, u, P) - continuous (u - uniformly (Q, u, P) - continuous at a point a) operator is (Q, u, P) - continuous ((Q, u, P) - continuous at the same point), we need only to show that when the relation u is finite, a (Q, u, P) - continuous (at a point a) operator $A : L \to M$ is uniformly (Q, u, P) - continuous (at the point a). At first, we consider local boundedness.

Indeed, by Definition 3.6, for any hyperseminorms q and p such that $(q, p) \in u$, the operator (mapping) A is (q, p) - continuous at the point a, that is, by Definition 3.4, the following condition is true:

Condition 2. For any positive real number k, there is a positive real number h such that for any element b from L, the inequality $q(a - b) < h$ implies the inequality $p(A(b) - A(a)) < k$.

This number h can be different for different pairs (q, p), but because u is finite, there is only a finite number of these pairs. So, we can take

$$l = min\{h : h \text{ satisfies Condition 2 for a pair } (q, p) \in u\},$$

and this number l will satisfy the condition from Definition 3.7 Thus, the operator A is u - uniformly (Q, u, P) - continuous at the point a.

The global case is proved in a similar way.

Proposition is proved. □

Corollary 3.31. *If systems of hyperseminorms Q and P are finite, then an operator (mapping) A is uniformly (Q, u, P) - continuous if and only if it is (Q, u, P) - continuous.*

There are connections between uniform with respect to systems of hyperseminorms continuity and uniform boundedness that are similar to the connections between nonuniform with respect to systems of hyperseminorms continuity and nonuniform boundedness described in Theorems 3.1 - 3.3. Namely, we have the following results.

Theorem 3.4. *A linear operator (mapping) $A : L \to M$ is uniformly (Q, u, P) - continuous at a point a if and only if it is uniformly (Q, u, P) - bounded at a.*

Proof is similar to the proof of Theorem 3.1.

Let us take a vector subspace V of the space L.

Theorem 3.5. *A linear operator (mapping) $A : L \to M$ is u - uniformly (Q, u, P) - continuous in V if and only if it is u - uniformly (Q, u, P) - bounded in V.*

Proof is similar to the proof of Theorem 3.2.

Theorem 3.6. *A linear operator (mapping) $A : L \to M$ is uV - uniformly (Q, u, P) - continuous if and only if it is uV - uniformly (Q, u, P) - bounded.*

Proof is similar to the proof of Theorem 3.3.

4. Conclusion

Semitopological vector spaces are introduced and studied. Semitopological vector spaces are more general than conventional topological vector spaces, which have been very useful for solving many problems in functional analysis. Thus, we come to the following problems.

Problem 1. Study topology in semitopological vector spaces.

Problem 2. Study applications of semitopological vector spaces.

In addition, hypernorms and hyperseminorms are introduced and studied. In this paper, it is demonstrated that hyperseminormed and hypernormed spaces are semitopological vector spaces.

These results bring us to the following problems.

Problem 3. Study what kinds of topology it is possible to define with systems of seminorms, hypernorms or hyperseminorms.

It is proved (cf. (Rudin, 1991)) that systems of seminorms characterize locally convex spaces and thus, there are topological vector spaces topology in which is not defined by systems of seminorms. It is possible to ask if the same is true for semitopological vector spaces. Namely, we have the following problem.

Problem 4. Is the topology in a semitopological vector space always defined by a system of seminorms?

In this paper, hypermetrics and hyperpseudometrics are also introduced and it is demonstrated that hyperseminorms induce hyperpseudometrics, while hypernorms induce hypermetrics. Sufficient and necessary conditions for a hyperpseudometric (hypermetric) to be induced by a hyperseminorm (hypernorm) are found. Hyperpseudometrics and hypermetrics define definite topologies in vector spaces.

Problem 5. Study what kinds of topology it is possible to define with hyperpseudometrics and hypermetrics.

In this paper, boundedness and continuity are defined relative to systems of hyperseminorms or hypernorms. Inclusion of hyperseminorm sets is reflected in the strength of corresponding topologies, namely, the larger is the set Q of hyperseminorms (hypernorms), the weaker topology it defines. In such a way, we obtain a definite scalability of spaces ((Burgin, 2004); (Burgin, 2006)) with systems of hyperseminorms (hypernorms), coming to the following problem.

Problem 6. Study scalability of topological spaces defined by systems of hyperseminorms and hypernorms.

Topological vector spaces provide an efficient context for the development of integration (Choquet, 1969); (Edwards & Wayment, 1970); (Shuchat, 1972); (Kurzweil, 2000).

Problem 7. Study integration in semitopological (polyhyperseminormed) vector spaces.

At the same time, integration and hyperintegration in bundles with a hyperspace base are defined and studied in (Burgin, 2010) where the hyperspace is built by means of seminorms. The goal of this paper is to provide a base for developing the theory of extrafunction spaces in an abstract setting of algebraic systems and topological spaces, where integration plays an important role (Burgin, 2012). So, we naturally come to the following problem.

Problem 8. Study integration and hyperintegration in bundles with a hyperspace base where the hyperspace is built by means of hyperseminorms.

It is possible to define norms and seminorms with values not only in number or hypernumber spaces but in more general spaces, e.g., operator spaces.

Problem 9. Study vector spaces that have norms or/and seminorms with values in general spaces.

Problem 10. Study continuity of non-linear operators in (mappings of) polyhyperseminormed (semitopological) vector spaces.

Here we have proved (Theorem 2.3) that any hyperseminormed vector space is a semitopological vector space. It would be interesting to find if a more general statement is also true.

Problem 11. Is any polyhyperseminormed vector space a semitopological vector space?

Thus, the theory of semitopological vector spaces opens many new opportunities for research in mathematics.

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Visual Motif Patterns in Separation Spaces

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Abstract

This article introduces descriptive separation spaces useful in the discovery of what are known as motif patterns. The proposed approach presents the separation axioms in terms of descriptive proximities. Asymmetries arise naturally in the form of the separation of neighbourhoods of descriptively distinct points in what are known as Leader uniform topological spaces. A practical application of the proposed approach is given in terms of visual motif patterns, identification of nearness structures and pattern stability analysis in digital images.

Keywords: Descriptive proximity, near sets, visual motif patterns, separation spaces.
2010 MSC: Primary 26A21, Secondary 26A24, 54D35, 54A20, 54E99, 18B30.

1. Introduction

This article introduces separation spaces, useful in the study of set patterns. Various forms of separation in topological spaces are defined by what are known as separation axioms. The main purpose of a separation axiom is to make the points and sets in a space topologically distinguishable (Thron, 1966, §14.1). The earliest of such spaces comes from F. Hausdorff, where distinct points belong to disjoint neighbourhoods (Hausdorff, 1957*a*, §40.II). In this article, traditional separation spaces are extended to description-based separation spaces. The practical benefit of considering descriptive separation spaces is the generation of multiple patterns that are descriptively distinguishable. In a Hausdorff space, for example, a pair of descriptively distinct points become generators of distinguishable set patterns.

A form of set pattern (Grenander, 1993, §17.5) of particular interest in an approach to pattern recognition is given in terms of what are known as descriptive motif patterns. A *descriptive motif pattern* is a collection of sets such that each member of the collection is descriptively close to a motif. A *motif* is a set with members that are near one or more members of other sets. Motifs

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are a particular form set pattern generators. Visual motif patterns are found in pictures, geometric structures, and digital images. A *visual motif pattern* is a particular form of descriptive motif pattern that is a collection of sets such that each member of the collection is visually close to a set that is a motif. Visual motif patterns have a number of important applications (Naimpally & Peters, 2013; Peters, 2013*a*).

The study of visual patterns includes a consideration of S. Leader's uniform topology[1] in a metric space (Leader, 1959) and its extension to descriptive uniform topologies that provide a basis for new forms of asymmetric spaces. A *descriptive uniform topology* is determined by finding the collection of all sets that are descriptively near a given set.

Set descriptions result from the introduction of feature vectors that describe members of sets such as sets of pixels in digital images. These considerations lead in a straightforward way to a form of topology of digital images with considerable practical importance in solving image analysis and image classification problems. Since we are interested in patterns in separation spaces, we introduce stability criteria for the generation of multiple set patterns. A visual pattern is considered *stable*, provided the members of the pattern do not wander away from the pattern generator, neither spatially nor descriptively.

2. Preliminaries

Let X be a nonempty set of points, $\mathcal{P}(X)$ the powerset of X, $\mathcal{P}^2(X)$ the set of all collections of subsets of X. A single point $x \in X$ is denoted by a lowercase letter, a subset $A \in \mathcal{P}(X)$ by an uppercase letter, collection of subsets in $\mathcal{P}^2(X)$ by a round letter such as $\mathcal{B} \in \mathcal{P}^2(X)$. The *closure* of a subset $A \in \mathcal{P}(X)$ (denoted by cl$A$) is defined by

$$\text{cl}A = \{x \in X : x\ \delta\ A\},$$

i.e., cl$A$ is the set of all points x in X that are near A. Let δ on a nonempty set X denote a spatial nearness (proximity) relation. For $A, B \in \mathcal{P}(X)$, $A\ \delta\ B$ (reads A is spatially near B), provided $A \cap B \neq \emptyset$, *i.e.*, the intersection of A and B is not empty (cl$A$ and cl$B$ have at least one point in common). The spatial proximity (nearness) relation δ is defined by

$$\delta = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \text{cl}A \cap \text{cl}B \neq \emptyset\}.$$

$A\ \underline{\delta}\ B$ (reads A far (remote) from B), provided cl$A$ and cl$B$ have no points in common such that $\underline{\delta} = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta$. Sets that are far from each other relative to the locations of the points in the sets (the points in one set are not among the points of the other set) are called *spatially remote* sets. The complement of a set $C \in \mathcal{P}(X)$ is denoted by C^c.

In the study of patterns, a descriptive form of EF-proximity is useful (Peters & Naimpally, 2012). Let X be a nonempty set endowed with a descriptive proximity relation δ_Φ, $x \in X$, $A, B \in \mathcal{P}(X)$, and let $\Phi = \{\phi_1, \ldots, \phi_i, \ldots, \phi_n\}$, a set of probe functions $\phi_i : X \to \mathbb{R}$ that represent features

[1]Metric space uniformity is logically equivalent to EF-proximity and the axioms given by Efremovič (Efremovič, 1951) (see Theorem 1.15, one of the most beautiful results in set-theoretic topology (Naimpally & Peters, 2013, §1.11, p. 27)). Many thanks to Som Naimpally for pointing this out.

of each x, where $\phi_i(x)$ equals a feature value of x. Let $\Phi(x)$ denote a feature vector for the object x, *i.e.*, a vector of feature values that describe x, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_i(x), \ldots, \phi_n(x)) .$$

A feature vector provides a description of an object. Let $A, B \in \mathcal{P}(X)$. Let $\mathcal{Q}(A), \mathcal{Q}(B)$ denote sets of descriptions of points in A, B, respectively. For example,

$$\mathcal{Q}(A) = \{\Phi(a) : a \in A\} .$$

The expression $A\ \delta_\Phi\ B$ reads A *is descriptively near* B. The descriptive proximity of A and B is defined by

$$A\ \delta_\Phi\ B \Leftrightarrow \mathcal{Q}(\mathrm{cl}A) \cap \mathcal{Q}(\mathrm{cl}B) \neq \emptyset.$$

Descriptive remoteness of A and B (denoted by $A\ \underline{\delta}_\Phi\ B$) is defined by

$$A\ \underline{\delta}_\Phi\ B \Leftrightarrow \mathcal{Q}(\mathrm{cl}A) \cap \mathcal{Q}(\mathrm{cl}B) = \emptyset.$$

Early informal work on the descriptive intersection of disjoint sets based on the shapes and colours of objects in the disjoint sets is given by N. Rocchi (Rocchi, 1969, p.159). The *descriptive intersection* $\underset{\Phi}{\cap}$ of A and B is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \mathcal{Q}(\mathrm{cl}A) \text{ and } \Phi(x) \in \mathcal{Q}(\mathrm{cl}B)\} .$$

The descriptive intersection will be nonempty, provided there is at least one element of cl$A$ with a description that matches the description of a least one element of cl$B$. That is, a nonempty descriptive intersection of sets A and B is a set containing $a \in \mathrm{cl}A$ and $b \in \mathrm{cl}B$ such that $\Phi(a) = \Phi(b)$. Observe that A and B can be disjoint and yet $A \underset{\Phi}{\cap} B$ can be nonempty. In finding subsets $A, B \in \mathcal{P}(X)$ that are descriptively near, one considers descriptive intersection of the closure of A and the closure of B. That is, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B$ implies $A\ \delta_\Phi\ B$. The descriptive proximity (nearness) relation δ_Φ is defined by

$$\delta_\Phi = \left\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) : \mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B \neq \emptyset\right\}.$$

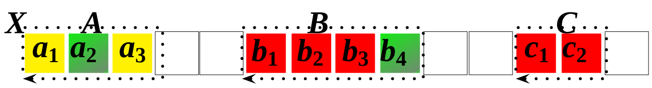

Figure 1. $\Phi = \{\text{colour probe fns}\}$, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B = \{a_2, b_4\}$, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}C = \emptyset$.

Example 2.1. Descriptive intersection of disjoint sets

The coloured and white squares in Figure 1 represent cells in a weave. A *cell* in a fabric is that part of a weave strand that overlaps another weave strand. The parallel strands of each layer in a weave are perpendicular to those strands in the other layer, making the cells square (Thomas, 2009). Choose Φ to be a set of probe functions representing weave cell colours. Let the set of cells X in Figure 1 be endowed with δ_Φ. Notice that sets $A, B \in \mathcal{P}(X)$ are disjoint but the descriptive intersection is nonempty. That is, $\mathrm{cl}A \underset{\Phi}{\cap} \mathrm{cl}B = \{a_2, b_4\}$. Similarly, for $B, C \in \mathcal{P}(X)$, $\mathrm{cl}B \underset{\Phi}{\cap} \mathrm{cl}C = \{b_1, b_2, b_3, c_1, c_2\}$. ■

The descriptive remoteness of A and B (denoted by $A \underline{\delta}_{\Phi} B$) such that $\underline{\delta}_{\Phi} = \mathcal{P}(X) \times \mathcal{P}(X) \setminus \delta_{\Phi}$ is defined by

$$A \underline{\delta}_{\Phi} B \Leftrightarrow \text{cl}A \underset{\Phi}{\cap} \text{cl}B = \emptyset.$$

Example 2.2. Descriptively remote disjoint sets
Choose Φ to be a set of probe functions representing weave cell colours. In Figure 1, sets $A, C \in \mathcal{P}(X)$ are disjoint. In addition, there are no cells in A with descriptions that resemble cells in C. Hence, the descriptive intersection is empty. That is, $A \underline{\delta}_{\Phi} C$ (A and C are remote), since $\text{cl}A \underset{\Phi}{\cap} \text{cl}C = \emptyset$. ■

2.1. Descriptive EF-proximity

A binary relation δ_{Φ} is a *descriptive EF-proximity*, provided the following axioms are satisfied for $A, B, C \in \mathcal{P}^2(X)$.

($\boldsymbol{EF_{\Phi}}$.1) $A \delta_{\Phi} B$ implies $A \neq \emptyset, B \neq \emptyset$.
($\boldsymbol{EF_{\Phi}}$.2) $A \underset{\Phi}{\cap} B \neq \emptyset$ implies $A \delta_{\Phi} B$.
($\boldsymbol{EF_{\Phi}}$.3) $A \delta_{\Phi} B$ implies $B \delta_{\Phi} A$ (descriptive symmetry).
($\boldsymbol{EF_{\Phi}}$.4) $A \delta_{\Phi} (B \cup C)$, if and only if, $A \delta_{\Phi} B$ or $A \delta_{\Phi} C$.
($\boldsymbol{EF_{\Phi}}$.5) Descriptive Efremovič axiom:

$$A \underline{\delta}_{\Phi} B \text{ implies } A \underline{\delta}_{\Phi} C \text{ and } B \underline{\delta}_{\Phi} C^c \text{ for some } C \in \mathcal{P}(X).$$

The structure (X, δ_{Φ}) is a *descriptive EF-proximity space* (or, briefly, *descriptive EF space*).

Theorem 2.1. *Let $(X, \delta), (X, \delta_{\Phi})$ be spatial and descriptive EF-spaces, respectively, with nonempty sets $A, B \in \mathcal{P}(X)$, $A \cap B \neq \emptyset$. Then $A \cap B \subseteq A \underset{\Phi}{\cap} B$.*

Proof. Let $A, B \in \mathcal{P}(X)$ and assume $A \cap B \neq \emptyset$. If $x \in A \cap B$, then, by definition, $\Phi(x) \in \mathcal{Q}(A)$ and $\Phi(x) \in \mathcal{Q}(B)$. By assumption $x \in A \cap B \subseteq A \cup B$. Then, $x \in A \underset{\Phi}{\cap} B$. Hence, $A \cap B \subseteq A \underset{\Phi}{\cap} B$. □

Descriptive EF-proximity is useful in describing, analysing and classifying the parts within a single digital image or the parts in either near or remote sets in separate digital images. The basic approach to the study of set patterns introduced in this article reflects recent work on descriptively near sets (see, *e.g.*, (Peters & Naimpally, 2012; Peters, 2012; Peters *et al.*, 2013)). Applications of descriptive EF-proximity are numerous (see, *e.g.*, (Naimpally & Peters, 2013; Peters, 2013*c*)).

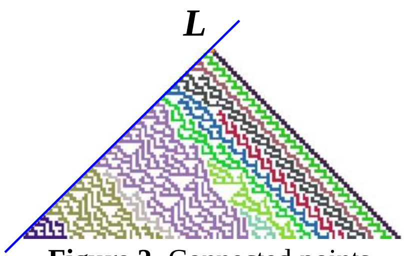

Figure 2. Connected points.

2.2. Shape set patterns

Shape descriptors are useful in representing, extracting and quantifying shape information from images. In general, a digital image *shape descriptor* is an expression that describes, identifies or indexes an image region. Shape descriptors are usually mathematical expressions used to extract image region shape feature values. In this section, we briefly consider picture points in terms of adjacency, connectedness, and edges.

Let $p, q \in \mathbb{Z} \times \mathbb{Z}$ be points in a grid X. Points p and q are *spatially adjacent*, provided they are joined by an edge (Klette & Rosenfeld, 2004). For example, pairs of magenta pixels in the grid in Figure 2 are spatially adjacent, since each pair of magenta pixels is joined by an edge.

Remark 2.1. Points vs. cells.
Points are the standard elements in standard topological spaces. In some discrete cases, the base elements are *cells* (indivisible collections of points) (Düntsch & Vakarelov, 2007). ■

In keeping with an interest in descriptive proximity, points p, q in a grid are *descriptively adjacent*, provided p, q have matching descriptions and there is an edge connecting p, q such that the description of the points on a connecting edge match the descriptions of p, q. For example, for the blue line $L \subset X$ along the northwest edge of the weave in Figure 2, each pair of pixels $p, q \in L$ are descriptively adjacent but pixels below L are not descriptively adjacent to any pixel in L, since L contains only blue pixels in Figure 2. Let p, q be magenta points, then the descriptive closure of L is descriptively far from p, q and the closure of any point $r \in L$ is descriptively far from either p or q, *i.e.*,

$$\Phi = \{\phi : \phi(x) = \text{ colour brightness of } x \text{ for } x \in X\},$$
$$\mathrm{cl}_\Phi L = \left\{x \in X : x \underset{\Phi}{\cap} L \neq \emptyset\right\},$$
$$\mathrm{cl}_\Phi r = \{y \in X : \Phi(y) = \Phi(r)\},$$
$$\mathrm{cl}_\Phi L \ \underline{\delta}_\Phi \ \{p, q\},$$
$$\mathrm{cl}_\Phi r \ \underline{\delta}_\Phi \ p,$$
$$\mathrm{cl}_\Phi r \ \underline{\delta}_\Phi \ q.$$

Descriptive adjacency is the heartbeat (main influence) in the study of visual motif patterns in pictures that are descriptive proximity spaces (Peters *et al.*, 2013; Peters, 2013*a*,*c*; Peters & Naimpally, 2012; Naimpally & Peters, 2013) (for the underlying near set theory, see, also, (Peters, 2013*b*; Henry, 2010)).

A 2D digital image (also called a picture) is defined on a finite, rectangular array of point samples called a *grid*. An element of a grid is a point sample or pixel. In terms of a digitized optical sensor value, a *point sample* (briefly, point) is a single number in a greyscale image or a set of 3 numbers in a colour image (Smith, 1995). In a 2D model of an image, a pixel is a point sample that exists only at a point in the plane. For a colour image, each pixel is defined by three point samples, one for each colour channel.

Let M be a set of grid points in a picture and let

$$S = p_0, p_1, \ldots, p_{i-1}, p_i, \ldots, p_n$$

be a sequence of points in M. The sequence S is called a *path*. Further, let $p = p_0, q = p_n$. Then M is *connected*, provided, for all points $p, q \in M$, point p_i is adjacent to p_{i-1} in a path between p

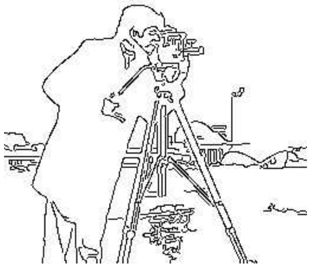

Figure 3. Sample straight edges.

and q in M. Maximally connected subsets of M are called *connected components* of M (Klette & Rosenfeld, 2004, §1.1.4).

The set of points in L in Figure 2 are connected and the remainder of the points in this weave are also connected. Let $X = L \cup W$, where W is the set of points in the threads in the weave in Figure 2. The set X is not a connected component, since there are pairs of points in separate threads with no path between the points. However, taken separately, any thread W containing pixels with the same colour is a connected component.

Let M be a grid that is connected and let points $p, q \in M$. A path between p and q defines an *edge*. A path between p and q defines a *straight edge*, provided every point in the path has the same gradient orientation. The penultimate example of a picture edge is a straight line segment such as the edges along the contour of the camera tripod legs in Figure 3. Hence, straight edges in a picture are distinguished from ridges, valleys and, in general, arcs, where the points in the paths defining non-straight edges have unequal gradient orientation.

A *shape set pattern* is a set pattern that results from the choices of shape descriptors used in comparing descriptions of picture elements. For example, the pairs of points along the diagonal in the northeast corner of Figure 2 are both spatially adjacent (each pair points along the upper northeast diagonal are joined by an edge) and descriptively adjacent (each pair points p, q along the diagonal are joined by an edge containing points that descriptively match p, q). Spatial adjacency and descriptive adjacency shape descriptors are important in separating spatially connected points from descriptively connected points in a picture and deriving spatial and descriptive set patterns in pictures.

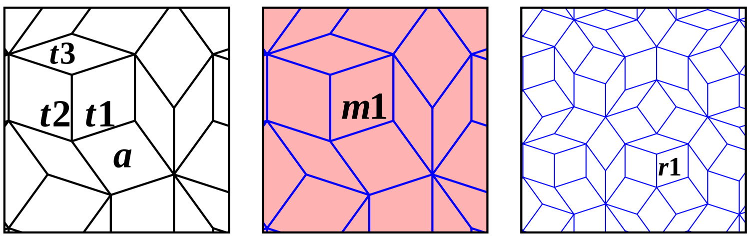

Figure 4. Spatial $\mathfrak{P}(t1) = \{t1, t2, t3, a\}$ & descriptive $\mathfrak{P}_{\Phi}(t1) = \{t1, m1, r1\}$.

Example 2.3. Descriptive penrose tiling shape pattern.
Choose Φ to be a set of probe functions representing shape features such as connected, edge gradient, and edge gradient orientation as well as colour and intensity features. Also, for example, choose tile $t1$ in Figure 4 as a shape pattern generator[2]. Tile $t1$ in the penrose tiling in Figure 4 is descriptively near tiles $m1$ (in the middle tiling) and $r1$ (in the righthand tiling) as well as a number of other unlabelled tiles that are descriptively near some part of $t1$. In generating descriptive shape patterns, we use the descriptive closure of a set A in a picture X (denoted by $\mathrm{cl}_\Phi A$), defined by

$$\mathrm{cl}_\Phi A = \left\{ x \in X : \{x\}\ \delta_\Phi\ A,\ i.e., \{x\} \underset{\Phi}{\cap} A \neq \emptyset \right\}.$$

In effect, $x\ \mathrm{cl}_\Phi A$ for $x \in X$ means $\Phi(x) \in \mathcal{Q}(A)$. Then

$$\mathfrak{P}_\Phi(t1) = \{t1, m1, r1, \ldots\}.$$

For example, $\mathrm{cl}_\Phi t1\ \cap\ \mathrm{cl}_\Phi m1 \neq \emptyset$, since the gradient orientation of edges along the border of $t1$ match the gradient orientation of the edges along the border of $m1$. Similarly, $\mathrm{cl}_\Phi t1\ \cap\ \mathrm{cl}_\Phi r1 \neq \emptyset$, and so on. ■

3. Descriptive uniform topology on digital images

It was S. Leader who pointed out in 1959 that it is possible to determine what he called a uniform topology in a metric space (Leader, 1959). By introducing a metric on a nonempty set of points, one obtains a metric space. Then a topology in the metric space results from observing which points are close to each given set of points. A point x in a set X is *close* to a set A, provided the distance between x and A is zero. A *digital uniform topology* in a metric space on a digital image is determined by observing which sets of pixels are close to a given set of pixels.

A useful alternative form of uniform topology (called a discrete uniform topology) arises in a proximity space by defining the nearness of sets in terms of set intersection. A discrete uniform topology in a proximity space is determined by observing which sets have nonempty intersection with a given set. In a discrete uniform topology, sets that are close to a given set are called *near sets*.

A descriptive form of either the Leader form of uniform topology or discrete uniform topology arises when the nearness of sets is based on the descriptions of members of one set matching the descriptions of members of another set. A *descriptive uniform topology* in a metric space is determined by finding which sets are descriptively close to each given set. In a descriptive uniform topology, nonempty disjoint sets can be descriptively near each other. The introduction of a uniform topology in a metric space or discrete uniform topology or descriptive uniform topology on a digital image provides a basis for the study of visual patterns in a image. In the sequel, it is assumed that each of the traditional separation spaces is defined in the context of a discrete uniform topology and that each descriptive separation space is defined in the context of a descriptive uniform topology on a nonempty set. From an application point-of-view, the focus in this article is on the introduction of uniform topologies that provide a basis for the introduction of asymmetric spaces on digital images.

[2]Regular structures known as *pattern generators* in pattern theory, are described in U. Grenander (Grenander, 1993, 1996), in building patterns from simple building blocks.

4. Antisymmetric spaces

During the 1930s, separation axioms were discovered and called *Trennungsaxiome* (*Trennung* is German for *separation*) by P. Alexandroff and H. Hopf (Alexandroff & Hopf, 1935, 58ff, §4). Hence, these axioms are named with a subscripted T as $T_n, n = 0, 1, 2, 3, 4, 5$. Often these axioms have alternate names such as Hausdorff, normal, regular, Tychonoff, and so on and there is no unanimity in the nomenclature. In this article, we consider only axioms T_0, T_1, T_2. Each of these separation axioms concern the distinctness of points.

Remark 4.1. Distinct points.
Let X be a nonempty set endowed with a proximity relation δ. Points $x, y \in X$ are *spatially distinct*, provided the closures of x and y are not near, *i.e.*, $\text{cl}\{x\} \underline{\delta} \text{ cl}\{y\}$. ■

The anti-symmetric axiom T_0 (discovered by A. Kolmogorov) is defined as follows.

> $\boldsymbol{T_0}$: (a) For every pair of distinct points, at least one of them is far from the other, or
> (b) For every pair of distinct points in a topological space X, there exists an open set containing one of the points but not the other point (*cf.* (Alexandroff & Hopf, 1935, p. 58)).

The discovery of T_0 topologies in digital images hinges on what is meant by the observation that points are descriptively distinct.

Remark 4.2. Descriptively distinct points.
Let Φ be a set of probe functions that represent features of points x in a nonempty set X. Then let X be endowed with a descriptive proximity relation δ_Φ. Points x, y are *descriptively distinct*, provided x and y are spatially distinct and the feature vectors $\Phi(x)$ and $\Phi(y)$ are not equal. For example, points x, y in a digital image X are descriptively distinct (*descriptively far*), provided x and y are spatially distinct and have different descriptions, *i.e.*, $x \underline{\delta}_\Phi y$. ■

Let Φ be a set of probe functions representing features of members of a set and let $\varepsilon > 0$. There is a descriptive form of T_0 space (denoted by T_0^Φ). Let a *descriptive open neighbourhood* $N_{\Phi(x)}$ be defined by

$$N_{\Phi(x)} = \{y \in X : \Phi(x) = \Phi(y) \text{ and } |x - y| < \varepsilon\}.$$

That is, the description of each point in $N_{\Phi(x)}$ matches the description of x. Due to the spatial restriction $|x - y| < \varepsilon$, $N_{\Phi(x)}$ is also called a *bounded descriptive neighbourhood* (Peters, 2013*d*, §1.19.3).

> $\boldsymbol{T_0^\Phi}$: For every pair x, y of descriptively distinct points in a topological space X, there exists a descriptive open neighbourhood containing one of the points but not the other point.

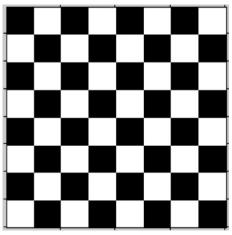

Figure 5. Sample visual space.

Example 4.1. $\boldsymbol{T_0^\Phi}$ **Visual space**.
Let X be represented by the checkerboard in Figure 5 and let x, y be black and white points in X. It is easily verified that X is a topological space. Then let $N_{\Phi(x)}$ be a descriptive open neighbourhood of x. The point y is excluded from $N_{\Phi(x)}$, since $\Phi(x) \neq \Phi(y)$. This is true for every pair of descriptively distinct points in X. Hence, X is a T_0^Φ space. ■

$\boldsymbol{T_1}$: A topological space is T_1 if, and only if, distinct points are not near.

$\boldsymbol{T_1^\Phi}$: A topological space is T_1^Φ if, and only if, descriptively distinct points are not descriptively near.

Example 4.2. Checkerboard $\boldsymbol{T_1^\Phi}$ **space**. Choose Φ to be a set of probe functions that represent greyscale and colour intensities of points in an image. Let a topological space X be represented by the checkerboard in Figure 5. X is an example of a visual T_1^Φ space. To see this, let $x, y \in X$ be points in black and white squares, respectively. The points x and y are descriptively distinct and $x \underline{\delta}_\Phi y$. In general, black and white pixels in X are descriptively distinct and not near, descriptively. Hence, the checkerboard is an example of a T_1^Φ space. ■

Lemma 4.1. *A digital image X endowed with a descriptive proximity δ_Φ such that X contains descriptively distinct points is a T_1^Φ space.*

Proof. Let X be a digital image (a set of points called pixels) endowed with a descriptive proximity δ_Φ. Choose Φ, a set of probe functions that represent features of points in X. Let points $x, y \in X$ be descriptively distinct. Then $x \underline{\delta}_\Phi y$, *i.e.*, x is descriptively not near y. Hence, X is a T_1^Φ space. □

Hausdorff observed that it is possible for a pair of distinct points to have distinct neighbourhoods and used this axiom in his work. The corresponding space with *pairs of distinct points belong to disjoint neighbourhoods* (Hausdorff, 1957*b*, §40.II) is now named after him and is called the T_2 or Hausdorff space.

T_2: A topological space is T_2, if and only if, distinct points have disjoint neighbourhoods (distinct points live in disjoint *houses*[4]).

There is a descriptive counterpart of a traditional T_2 space (denoted by T_2^Φ), introduced in (Peters, 2013*a*) (see, also, (Naimpally & Peters, 2013)). In a T_2^Φ space, one can observe that descriptively distinct points belong to disjoint descriptive neighbourhoods.

T_2^Φ: A topological space is T_2^Φ if, and only if, descriptively distinct points have disjoint descriptive neighbourhoods.

Example 4.3. A T_2^Φ Visual space. Choose Φ to be a set of probe functions that represent greyscale and colour intensities of points in an image. Let a topological space X again be represented by the checkerboard in Figure 5. X is an example of a visual T_2^Φ space. To see this, let $x, y \in X$ be points in black and white squares, respectively. Then consider a pair of descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$ of x and y, respectively. Neighbourhood $N_{\Phi(x)}$ contains only points with descriptions that match the description of x, *i.e.*, $N_{\Phi(x)}$ contains only black points. Similarly, neighbourhood $N_{\Phi(y)}$ contains only points with descriptions that match the description of y, *i.e.*, $N_{\Phi(y)}$ contains only white points. Hence, $N_{\Phi(x)}, N_{\Phi(y)}$ are disjoint. ■

Figure 6. Manitoba dragonfly.

Observe that a T_2^Φ space is also a T_1^Φ space, since, by definition, descriptively distinct points are not near. The dragonfly in Figure 6 provides an illustration of a biology-based T_2^Φ space (see Example 4.4 for details). Also observe that a T_1^Φ space is also a T_0^Φ space, since, for every pair of descriptively distinct points, one can find a descriptive open set containing of the points and not containing the other point. The penultimate example of a T_1^Φ space that is also a T_0^Φ space is a

[4]A partition is a T_2 space if, and only if, every class has no more than one point, *i.e.*, every class is single tenant ”house”.

space where descriptively distinct points belong to open descriptive neighbourhoods. From these observations, observe that $T_2^\Phi \Rightarrow T_1^\Phi \Rightarrow T_0^\Phi$.

Let $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. A *bounded descriptive neighbourhood* $N_{\Phi(x)}$ of a point x in a set X is defined by

$$N_{\Phi(x)} = \{y \in X : d(\Phi(x), \Phi(y)) = 0 \text{ and } |x - y| < \varepsilon\},$$

where d is the taxicab distance between the descriptions of x and y, *i.e.*,

$$d(\Phi(x), \Phi(y)) = \sum_{i=1}^{n} |\phi_i(x) - \phi_i(y)| : \phi_i \in \Phi.$$

Theorem 4.1. *A digital image X endowed with a descriptive proximity δ_Φ such that X contains two or more descriptively distinct points is a T_2^Φ space.*

Proof. Let X be a digital image (a set of points called pixels) endowed with a descriptive proximity δ_Φ. Choose Φ, a set of probe functions that represent features of points in X. Let points $x, y \in X$ be descriptively distinct. Let $N_{\Phi(x)}, N_{\Phi(y)}$ be descriptive neighbourhoods of x, y, respectively. If $a \in N_{\Phi(x)}$, then $d(\Phi(a), \Phi(x)) = 0$, *i.e.*, each member of $N_{\Phi(x)}$ must descriptively match x. Similarly, each $b \in N_{\Phi(y)}$ descriptively matches y. Then, $N_{\Phi(x)} \cap N_{\Phi(y)} = \emptyset$. Hence, X is a T_2^Φ space. □

Figure 7. Dragonfly edges.

Example 4.4. Dragonfly T_2^Φ Shape space.
Choose Φ to be a set of probe functions that represent the gradient orientation of the points in an image. Let a topological space X be represented by the dragonfly in Figure 6, endowed with a descriptive proximity relation δ_Φ. X is an example of a complex visual T_2^Φ shape space. To see this, let $x, y \in X$ be points along the edges of the filtered dragonfly image in Figure 7. The points x and y are descriptively distinct, since these points have different gradient orientations. In addition, points x, y are centers of disjoint descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$, respectively, in a T_2^Φ Shape Space.

Proof. We assume that $\Phi(x) \neq \Phi(y)$, *i.e.*, x and y have different gradient orientations in Figure 7. The descriptive neighbourhood $N_{\Phi(x)}$ of point x (with no spatial restriction) is defined by

$$N_{\Phi(x)} = \{a \in X : \Phi(x) = \Phi(a)\},$$

i.e., the gradient orientation of x matches the gradient orientation of each point a in $N_{\Phi(x)}$. Hence, $y \notin N_{\Phi(x)}$, since the gradient orientation of y does not match the gradient orientation of x. Similarly, observe that $x \notin N_{\Phi(y)}$. Then $N_{\Phi(x)}, N_{\Phi(y)}$ are disjoint. This is true of every pair of points in X that have unequal gradient orientations. Hence, X is an example of a descriptive T_2^{Φ} shape space. □

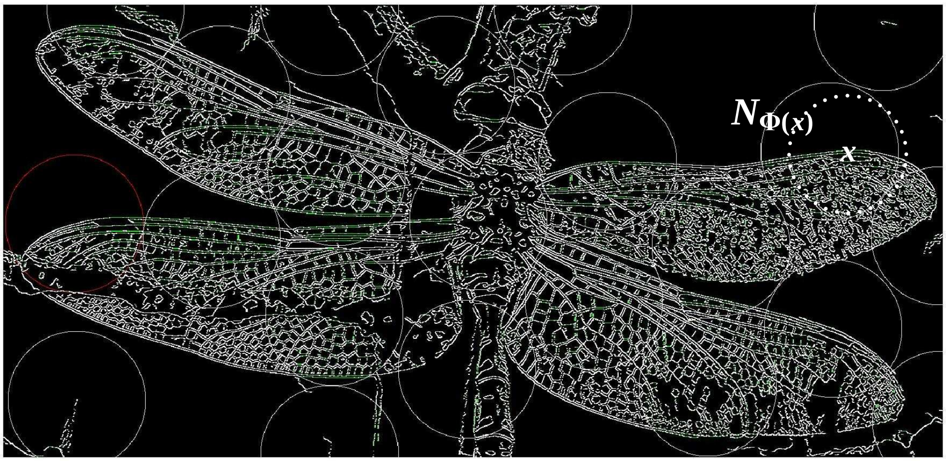

Figure 8. T_2^{Φ} Shape space.

Remark 4.3. $N_{\Phi(x)}$**,** T_2^{Φ} **Implementation details**.
A Matlab® 7.10.0 (R2010a) script written by C. Uchime has been used on the dragonfly image in Figure 6 to extract the edges shown in Figure 7. From Example 4.4, we know that the dragonfly in Figure 7 provides a basis for a T_2^{Φ} shape space. Next, bounded descriptive neighbourhoods $N_{\Phi(x)}, N_{\Phi(y)}$ of points x, y, respectively, are found by selecting x, y, radius ε, and pixel gradient orientation as the shape descriptor. For simplicity, only $N_{\Phi(x)}$ is shown in Figure 8.

Using C. Uchime's Matlab script, the selection of x, y is done manually by clicking on two points of interest on the dragonfly wings (see Figure 8). Starting with $N_{\Phi(x)}$, for example, the construction of the shape pattern $\mathfrak{P}_{\Phi}(N_{\Phi(x)})$ is carried out by using Matlab to search through the image for points (outside the motif neighbourhood) with gradient orientations that match the gradient orientation of x. For each pixel $x' \notin N_{\Phi(x)}$ such that $\Phi(x') = \Phi(x)$, a new neighbourhood is constructed. In practice, only a restricted number of neighbourhoods are found, namely, those neighbourhoods with centers that are reasonably close to the motif neighbourhood center x. ■

4.1. Descriptive nearness structures

Herrlich nearness structures are extended to descriptive nearness structures in this section. One begins the study of such structures by choosing Φ, a set of probe functions that represent features of members of a nonempty set X. Let X be endowed with a descriptive proximity relation Φ. By way of illustration, the honey bee in Figure 9 provides a basis for a shape nearness space relative to the bee image edges shown in Figure 10 (see Example 4.5 for details).

Figure 9. Bee

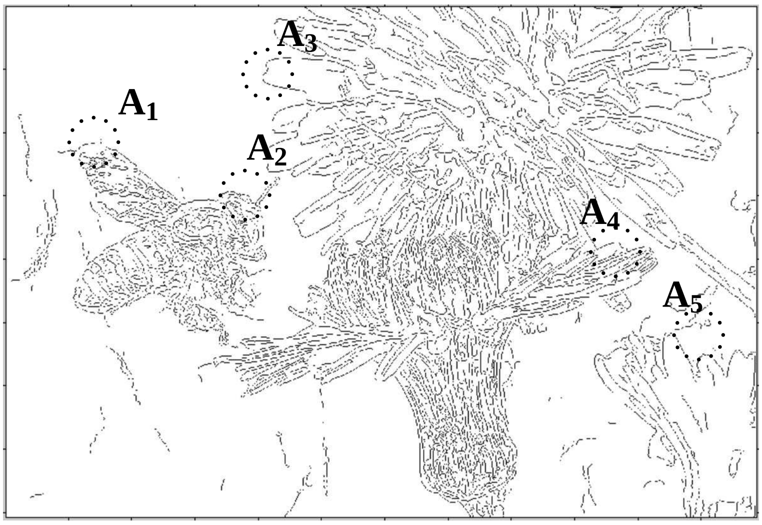

Figure 10. $\xi_{\Phi} = \{A_1, \ldots, A_5\}$

For descriptive nearness, we use the following notation.

$$\begin{aligned}
X &= \text{ nonempty set of points},\\
\Phi &= \{\text{probe functions representing features of } x \in X\},\\
\mathcal{A}, \mathcal{B} & \text{ denote collections of subsets in } X, \ i.e., \ \mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X),\\
\mathcal{Q}(\mathcal{A}) &= \{\mathcal{Q}(A) : A \in \mathcal{A}\},\\
\eta_{\Phi}\mathcal{A}, & \text{ or } \mathcal{A} \in \eta, \ i.e., \text{ members of } \mathcal{A} \text{ are descriptively near},\\
\underline{\eta}_{\Phi}\mathcal{A}, & \ i.e., \text{ members of } \mathcal{A} \text{ are not descriptively near},\\
A \ \eta_{\Phi} \ B &= \eta_{\Phi}\{A, B\} \ (A \text{ descriptively near } B),\\
\mathcal{A} \vee \mathcal{B} &= \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\},\\
cl_{\eta_{\Phi}} E &= \{x \in X : \{x, E\} \in \eta_{\Phi}\} \ (x \text{ descriptively near } E),\\
cl_{\eta_{\Phi}} \mathcal{A} &= \{cl_{\eta} A : \mathcal{Q}(A) \in \mathcal{Q}(\mathcal{A})\}.
\end{aligned}$$

A descriptive nearness structure (denoted by ξ_{Φ}) is defined by

$$\xi_{\Phi} = \left\{ \mathcal{A} \in \mathcal{P}^2(X) : \bigcap_{\Phi} \{A : A \in \mathcal{A}\} \neq \emptyset \right\}.$$

In the following axioms, let $\mathcal{A} \in \xi_{\Phi}$. It can be shown that the descriptive nearness structure ξ_{Φ} satisfies (dN.1)-(dN.5):

(dN.1) $\bigcap_{\Phi}\{A : A \in \mathcal{A}\} \neq \emptyset \Rightarrow \eta_{\Phi}\mathcal{A}$ is not empty,

(dN.2) $\underline{\eta}_{\Phi}\mathcal{A}$ and $\underline{\eta}_{\Phi}\mathcal{B} \Rightarrow \underline{\eta}(\mathcal{A} \vee \mathcal{B})$,

(dN.3) $\eta_{\Phi}\mathcal{A}$ and, for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A} : A \subset B \Rightarrow \eta_{\Phi}\mathcal{B}$,

(dN.4) $\emptyset \in \mathcal{A} \Rightarrow \underline{\eta}_{\Phi}\, \mathcal{A}$,
(dN.5) $\eta_{\Phi}(cl_{\eta}\mathcal{A}) \Rightarrow \eta_{\Phi}\mathcal{A}$ (descriptive Herrlich axiom).

Example 4.5. Descriptive Herrlich nearness.
Let the set X be represented by the set of edge pixels in Figure 10 and let Φ contain a single probe function representing pixel orientation. Each member of the collection of subsets $\mathcal{A}$ contains ridge pixels, where

$$\xi_{\Phi} = \mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\},$$

since each pair of sets in $\mathcal{A}$ contain pixels with matching orientation. Observe that there are other collections of subsets $\mathcal{B}$ in Figure 10 containing pixels with matching orientations that are not the same as the pixels orientations in the subsets in $\mathcal{A}$. Hence, ξ_{Φ} contains more collections of descriptively near subsets that are not shown in Figure 10. ■

5. Visual patterns in descriptive separation spaces

Figure 11. T_2^{Φ}.

Visual patterns arise naturally from the different forms of descriptive separation spaces. We illustrate this in terms of patterns that naturally occur in T_1^{Φ} and T_2^{Φ} spaces. Let $\mathcal{P}^2(X)$ denote the set of collections of subsets in X and let pattern $\mathfrak{P} \in \mathcal{P}^2(X)$, motif $M \in \mathcal{P}(X)$. Let Φ be a set of probe functions that represent features of members of X and let X be endowed with a descriptive proximity δ_{Φ}. For example, the 1870 Punch dancing delivery boy image in Figure 11 provides a basis for a visual pattern (see Example 5.2 for details). Further, a visual pattern $\mathfrak{P}_{\Phi}$ is a descriptive motif pattern, provided the following axioms are satisfied.

(**motif**.1) Sets in $\mathfrak{P}_{\Phi}$ are pairwise disjoint.
(**motif**.2) A is descriptively near M ($A\ \delta_{\Phi}\ M$) for each $A \in \mathfrak{P}_{\Phi}$.
(**motif**.3) If there are pairs $A, B \in \mathfrak{P}_{\Phi}$ that are copies of M, there is an isometry[5] of the plane that maps A onto B.

A descriptive motif pattern is an example of what is known as a discrete pattern in the study of patterns in tilings and weaving (see, *e.g.*, (Grünbaum & Shepard, 1987)). Observing visual patterns in an image is aided by various forms of image filtering, sharpening the features of pixel neighbourhoods, making it more possible to detect those parts of an image that are either close or remote from each other.

[5]Let A and B be sets of pixels in digital images endowed with metrics d_X and d_Y. An *isometry* is a distance-preserving map (Beckman & Quarles, 1953). For any pair pixels $x, y \in A$ with descriptions $\Phi(x), \Phi(y)$ found in B (*i.e.*, $f(\Phi(x)), f(\Phi(y)) \in B$), a map $f : A \to B$ is an isometry, provided

$$d_Y(f(\Phi(x)), f(\Phi(y))) = d_X(\Phi(x), \Phi(y)).$$

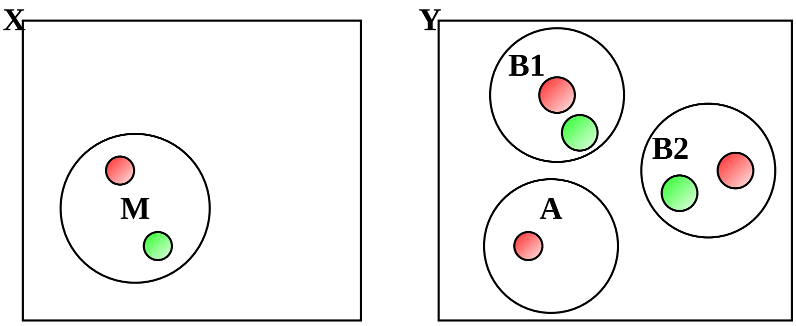

Figure 12. $\mathfrak{P}_\Phi = \{A, B1, B2\}$.

Example 5.1. Sample descriptive motif pattern.
Let sets of points X, Y endowed with a proximity relation δ be represented by Figure 12. Choose Φ to be a set of probe functions that represent greyscale and colour features of points in X, Y. The set M in Figure 12 represents a motif in a set pattern. Observe that $A, B1, B2$ are pairwise disjoint and each of the sets $A, B1, B2$ is descriptively near M. For example, M is descriptively near $B1$, since M and $B1$ contain subsets with red and green pixels. Again, for example, M is descriptively near A, since M and A contain subsets with red pixels. There is also an isometry between the descriptions of points in X and the descriptions of the points in Y. From these observation, we obtain the descriptive motif pattern $\mathfrak{P}_\Phi = \{A, B1, B2\}$. ■

5.1. Visual patterns in descriptive T1 spaces

To find visual patterns in descriptive T_1 spaces, do the following:

(1) Choose Φ, a set of probe functions representing features of points in a T_1^Φ space X.
(2) Select a pair of descriptively distinct points $x, y \in X$. By definition, $x \underline{\delta}_\Phi y$. Hence, the T_1^Φ space property is satisfied.
(3) Let M_1, M_2 denote point sets $\{x\}, \{y\}$, respectively.
(4) Determine all subsets of X containing points that descriptively match M_1 and then determine all subsets of points that descriptively match M_2.

As a result of the above steps, we can identify a pair of descriptive motif patterns $\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)$ in a T_1^Φ space X. In addition, each such a motif pattern is a member of a descriptive Herrlich topology ξ_Φ defined on X.

Let X be endowed with a proximity δ_Φ such that X is a T_1^Φ space and let $M = \{x\}$ be a motif containing a single point $x \in X$, which defines a descriptive motif pattern $\mathfrak{P}_\Phi(M)$. If $A, B \in \mathfrak{P}_\Phi(M)$, then $A \underset{\Phi}{\cap} B \neq \emptyset$. From this, we obtain the following result.

Theorem 5.1. *Let (X, δ_Φ) be a T_1^Φ space with nearness structure ξ_Φ on X and let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern determined by a motif M containing a single point x in X. Then $\mathfrak{P}_\Phi(M) \in \xi_\Phi$.*

5.2. Visual patterns in descriptive T2 spaces

To find visual patterns in descriptive T_2 spaces, do the following:

(1) Choose Φ, a set of probe functions representing features of points in a T_2^Φ space X.
(2) Select a pair of descriptively distinct points $x, y \in X$. By definition, $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$, since the description of each point in a descriptive neighbourhood matches the description of the point at the centre of the neighbourhood and $x \underline{\delta}_\Phi y$). That is, neighbourhoods $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$ are descriptively disjoint. Hence, the T_2^Φ space property is satisfied.
(3) Let M_1, M_2 denote neighbourhoods $N_{\Phi(x)} \underline{\delta}_\Phi N_{\Phi(y)}$, respectively.
(4) Determine all subsets of X that are descriptively near M_1 and then determine all subsets of X such that descriptively near M_2.

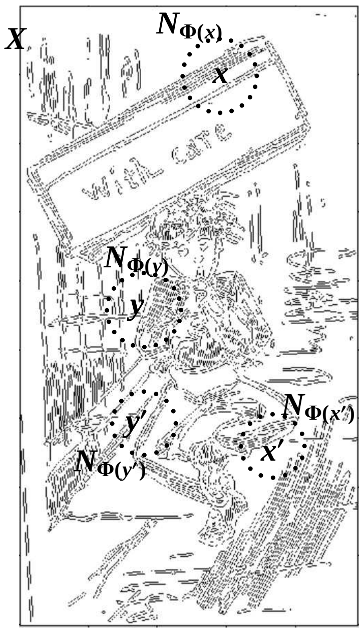

Figure 13. Sample T_2^Φ visual edge patterns.

As a result of the above steps, we can identify a pair of descriptive motif patterns $\mathfrak{P}_\Phi(M_1)$, $\mathfrak{P}_\Phi(M_2)$ in a T_2^Φ space X. In general, each such a motif pattern is not a member of the same descriptive Herrlich nearness structure ξ_Φ defined on X. To see this, consider a pair of neighbourhoods $N_{\Phi(x')}, N_{\Phi(x'')}$ that are descriptively near $N_{\Phi(x)}$. We know that $N_{\Phi(x)} \delta_\Phi N_{\Phi(x')}$ but it is possible that $N_{\Phi(x')} \underline{\delta}_\Phi N_{\Phi(x'')}$, if, for example, we compare pixel colours. $N_{\Phi(x)}$ may have a mixture of red and green colours, where $N_{\Phi(x')}$ has pixels with red colours but no green colours and $N_{\Phi(x'')}$ has pixels with green colours but no red colours. In other words, many different Herrlich nearness structures can be found in the same digital image.

Example 5.2. Edge motif pattern in T_2^Φ space.
Let X be the set of edge points in Figure 13, extracted from the 1870 Punch image in Figure 11,

using the edge function with the Canny filter[6] available in Matlab. Choose Φ to be a set of probe functions representing the orientation (gradient direction) of edge pixels in X. Observe that if pixels x, y in X have different orientations (*i.e.*, x and y are descriptively distinct), then $x \underline{\delta}_{\Phi}\ y$. Then $N_{\Phi(x)}, N_{\Phi(y)}$ are descriptively disjoint neighbourhoods. Hence, X is an example of T_2^{Φ} space.

Then let M_1, M_2 denote motif neighbourhood edge point sets $N_{\Phi(x)}, N_{\Phi(y)}$ of points x, y, respectively. An indication of the descriptive motif patterns $\mathfrak{P}_{\Phi}(M_1), \mathfrak{P}_{\Phi}(M_2)$ determined by M_1, M_2 is suggested by the edge regions containing points x', y'. In the pattern representing $\mathfrak{P}_{\Phi}(M_1)$, for example, notice that x' is the centre of bounded descriptive neighbourhood $N_{\Phi(x')}$ containing points with matching orientations. And the M_1 edge point neighbourhood is descriptively near $N_{\Phi}(x')$, since the orientation of one or more edges in $N_{\Phi(x)}$ match the orientation of one or more edges in $N_{\Phi(x')}$, *i.e.*,

$$N_{\Phi(x)}\ \delta_{\Phi}\ N_{\Phi(x')} : M_1 \doteqdot N_{\Phi(x)}.$$

Similarly, there is a descriptive neighbourhood $N_{\Phi(y')}$ in the edge pattern $\mathfrak{P}_{\Phi}(M_2)$ so that

$$N_{\Phi(y)}\ \delta_{\Phi}\ N_{\Phi(y')} : M_2 \doteqdot N_{\Phi(y)}.$$

Continuing this process, one can observe many other edge motif patterns in this particular T_2^{Φ} space. ■

Theorem 5.2. *A descriptive T_2^{Φ} space contains distinct descriptive motive patterns.*

Proof. Immediate from Lemma 4.1 and the definition of descriptive motif patterns. □

6. Stability in pattern constructions

A meaningful theory of stable pattern selection requires models of pattern-forming mechanisms that are simple enough to be understood in detail (Dee & Langer, 1983). An approach to achieving pattern selection stability in propagating patterns in either T_1 or T_2 spaces is introduced in this section. Basically in this study of descriptive patterns in a pair of digital images A, B, it is necessary to propagate a pattern in image B with some assurance that the pattern generated in B will belong to the class of images containing the image A and each new set added to a pattern does not wander or drift away from the pattern generator. That is, given a pattern generator M, each new set A added to pattern $\mathfrak{P}_{\Phi}(M)$ must be sufficiently near M, *spatially*.

P.E. Forsseén and D. Lowe observe that shape descriptors are reliable in detecting maximally stable extremal regions in digital images (Forsseén & Lowe, 2007, 1-8). In this work, descriptive motif set pattern growth is stable, provided the shape-based description of each set added to the pattern matches the shape-based description of the pattern motif. This interpretation of pattern stability is comparable to U. Grenander's notion of configuration transformation stability (Grenander, 1993, §4.1.1). To arrive at a formal definition of pattern stability, we introduce the descriptive distance between collections in terms of the Čech distance between sets.

[6]The Matlab canny filter is based on J.F. Canny's approach to edge detection introduced in his M.Sc. thesis completed in 1983 at the MIT Artificial Intelligence Laboratory (Canny, 1983). For details about this considered in the context of a topology of digital images, see (Peters, 2013*d*, §6.2).

Let $A, B \in C$ be nonempty sets in a space C and let

$$D(A, B) = \inf\{|a - b| : a \in A, b \in B\}$$

be the Čech distance between A and B. That is, a configuration transformation T on a configuration space C is stable, if, for any $\varepsilon > 0$, there exists a δ such that

$$D(A, B) \leq \delta \Rightarrow D(T(A), T(B)) \leq \varepsilon.$$

Let (X, δ_Φ) be a descriptive proximity Hausdorff space and let $A, B \in \mathcal{P}, \mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Next, consider a descriptive form of a Grenander configuration transformation, namely, T_Φ. That is, the transformation $T_\Phi \doteqdot \mathfrak{P}_\Phi : \mathcal{P}(X) \to \mathcal{P}^2(X)$ is defined by

$$\mathfrak{P}_\Phi(M) = \mathcal{A} : M\ \delta_\Phi\ B \text{ for } B \in \mathcal{A}, \text{and } D(M, B) \leq \varepsilon.$$

Definition 6.1. Pattern stability sufficiently near criterion.
Let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern constructed on a nonempty set X, $\varepsilon > 0$ and let $A \in \mathfrak{P}_\Phi(M)$. The pattern $\mathfrak{P}_\Phi(M)$ is stable, provided the distance requirement $D(M, A) < \varepsilon$ is satisfied. That is, $\mathfrak{P}_\Phi(M)$ is stable, provided A is *sufficiently near* M for each A added to $\mathfrak{P}_\Phi(M)$. ■

Let $B \ll A$ denote the fact that B is a *proximal neighbourhood* of A, provided $A \subset B$. From Def. 6.1, we obtain the following result.

Lemma 6.1. *Let $M \subset X$, a T_2^Φ space and let $\mathfrak{P}_\Phi(M)$ be a descriptive motif pattern. Let $A, B \in \mathfrak{P}_\Phi(M)$. $\mathfrak{P}_\Phi(M)$ is stable, if and only if, $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.*

From Def. 6.1 and Lemma 6.1, we obtain the following result.

Theorem 6.1. Descriptive pattern stability.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ such that Φ is a set of probe functions representing shape descriptors, let $M \in \mathcal{P}(X)$, $\varepsilon > 0$. Then the following are equivalent.
(1) $\mathfrak{P}_\Phi(M)$ is stable.
(2) $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$.
(3) $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.

Proof.
(1) ⇔ (2): $\mathfrak{P}_\Phi(M)$ is stable, if and only if, from Def. 6.1, $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$.
(1) ⇔ (3): $\mathfrak{P}_\Phi(M)$ is stable, if and only if, from Lemma 6.1, $D(M, A) < \varepsilon$ and $B \ll A$ implies $D(M, B) < \varepsilon$.
(2) ⇔ (3): $D(M, A) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M)$, if and only if, $B \in \mathfrak{P}_\Phi(M)$, provided $B \ll A$. □

Remark 6.1. Pattern stability and clustering stability.
Observe that descriptive pattern generation is a form of clustering. Recall that data clustering is a *natural grouping* of a set of patterns or points or objects (Jain, 2010). Let X be a T_2^Φ space and let $M \in \mathcal{P}(X)$. Then the use of M to generate the pattern $\mathfrak{P}_\Phi(M)$ can be considered a natural grouping

of sets in the pattern relative to the pattern generator M. That is, $A \in \mathfrak{P}_\Phi(M)$, provided $A\ \delta_\Phi\ M$. Hence, an obvious research path in the study of descriptive pattern generation is to consider the parallel between clustering stability (*e.g.*, (Ben-Hur *et al.*, 2002; Wang, 2010; Reizer, 2011)) and descriptive pattern generation stability. For example, it has been found (Ben-Hur *et al.*, 2002) that pairwise similarity between clusterings of sub-samples in a dataset provides a basis for clustering stability. A partial guarantee that descriptive pattern generation is stable, stems from the fact that $A \in \mathfrak{P}_\Phi(M)$, if and only if, A is descriptively near M. But this is only a partial guarantee of pattern generation stability, since subset similarity in a descriptive pattern does not prevent subsets from drifting or wandering away *spatially* from the pattern generator M. To achieve full descriptive pattern generation stability, we consider distance-based pattern generation in keeping with recent work on the stability of distance-based clustering (see,*e.g.*, (Wang, 2010)). In the distance-based approach to descriptive pattern stability, we introduce the *sufficiently near* criterion in Def. 6.1.
■

6.1. *Multiple pattern generation stability*

Since we are interested in constructing multiple patterns across disjoint regions of digital images that resemble each other in a T_2^Φ space, we introduce a stability criterion for the generation of multiple patterns. Again, the goal is to arrive at a view of stability of multiple patterns in a T_2^Φ space such that patterns do not wander or drift away from each other. Let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ be collections containing sets $A, B \in \mathcal{P}(X)$, respectively. To complete the definition of pattern stability, we introduce the descriptive distance D_Φ, which a descriptive form of the distance between sets introduced by E. Čech (Čech, 1966, §18.A.2). The distance D_Φ is used to define the descriptive distance $\mathbb{D}_\Phi$ between collections of sets. The descriptive distance $\mathbb{D}_\Phi : \mathcal{P}^2(X) \times \mathcal{P}^2(X) \to \mathbb{R}$ between collections $\mathcal{A}, \mathcal{B}$ is defined by

$$\mathbb{D}_\Phi(\mathcal{A}, \mathcal{B}) = \inf \{D_\Phi(A, B) : A \in \mathcal{A}, B \in \mathcal{B}\}, \text{ where,}$$
$$D_\Phi(A, B) = \inf \{d(\Phi(a), \Phi(b)) : a \in A, b \in B\}.$$

The descriptive distance $\mathbb{D}_\Phi$ can be used to measure the distance between descriptive motif set patterns, since such patterns are collections of nonempty sets that are descriptively near each other. Let $\{A\}, \{B\}$ denote collections, each containing one set. Then $\mathfrak{P}_\Phi$ is a *stable descriptive pattern*, if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mathbb{D}_\Phi(\{A\}, \{B\}) \le \delta \ \Rightarrow \mathbb{D}_\Phi(\mathfrak{P}_\Phi(A), \mathfrak{P}_\Phi(B)) < \varepsilon.$$

That is, whenever sets A and B are descriptively near, then the corresponding patterns $\mathfrak{P}_\Phi(A), \mathfrak{P}_\Phi(B)$ are descriptively near. This form of set pattern stability works well in comparing regions of pairs of digital images, where we need to guarantee that the transformation that produces the descriptive set patterns in separate image regions is stable.

Definition 6.2. Multiple pattern stability criterion.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ and let $\varepsilon > 0, \delta > 0$. Let $x, y \in X$ be distinct

points and let M_1, M_2 be disjoint neighbourhoods of x, y, respectively. Further, let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Patterns $\mathfrak{P}_\Phi(M_1) \in \mathcal{A}, \mathfrak{P}_\Phi(M_2) \in \mathcal{B}$ are stable, provided

$$\mathbb{D}_\Phi(\mathcal{A}, \mathcal{B}) \leq \delta \Rightarrow \mathbb{D}_\Phi\left(\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)\right) < \varepsilon. \qquad \blacksquare$$

To achieve stability in comparing image regions in the same digital image regions in pairs of images, it is necessary to consider pixel features that can be reliably matched, regardless of the appearance of the surroundings of a region. In this article, the focus is on constructing motif set patterns containing neighbourhoods of points defined by connected point sets that are straight edges. Neighbourhood selection is determined by the gradient orientation of the focal point of a pattern motif neighbourhood. The construction of a pattern motif (a descriptive neighbourhood of point) reduces to finding a connected set of points along an edge such that the edge points have matching gradient orientation. Hence, a gradient orientation-based motif set pattern results from finding neighbourhoods of points containing straight edges with pixel gradient orientations that match the gradient orientation of the points in the motif neighbourhood of the pattern.

Keeping in mind the underlying descriptive uniform topology in a Hausdorff T_2^Φ space X endowed with a descriptive proximity δ_Φ, an image pixel y belongs to a neighbourhood of point x, provided the gradient orientation of y matches the gradient orientation of x. Let Φ be a set of shape descriptors that includes pixel gradient orientation. In addition, let the descriptive neighbourhood $N_{\Phi(x)}$ be a pattern motif M that is a connected set of points belonging to a straight edge, *i.e.*, $y \in N_{\Phi(x)}$, provided $\Phi(y) = \Phi(x)$. Then the pattern $\mathfrak{P}_\Phi(M)$ is a collection of straight edges defined by

$$\mathfrak{P}_\Phi(M) = \left\{ N_{\Phi(y)} \in \mathcal{P}(X) : N_{\Phi(y)}\ \delta_\Phi\ M \right\}.$$

Pattern stability is achieved by guaranteeing that only matching straight edges belong to the pattern $\mathfrak{P}_\Phi(M)$. In comparing regions across pairs of digital images, stability is achieved by comparing straight edge patterns. Let $x, y \in X, Y$ be a pixels in a pair of digital images X, Y, respectively. Further, let $\mathfrak{P}_\Phi(M_1), \mathfrak{P}_\Phi(M_2)$ be straight edge shape patterns in images X, Y, respectively, such that $M_1 = N_{\Phi(x)}, M_2 = N_{\Phi(y)}$. Pattern $\mathfrak{P}_\Phi(M_x)$ is close to pattern $\mathfrak{P}_\Phi(M_y)$, provided the straight edges represented by neighbourhoods in the patterns have matching edge-neighbourhood motifs, *i.e.*,

$$\begin{aligned} &\mathfrak{P}_\Phi(M_1)\ \delta_\Phi\ \mathfrak{P}_\Phi(M_2), \text{ if and only if,} \\ &N_{\Phi(x)}\ \delta_\Phi\ N_{\Phi(y)}, \text{ if and only if,} \\ &\Phi(x) = \Phi(y). \end{aligned} \qquad \blacksquare$$

From Def. 6.2 and Theorem 6.1, we obtain the following result.

Theorem 6.2. Multiple pattern generation stability.
Let $\mathfrak{P}_\Phi$ be a pattern configuration transformation used to construct collections of patterns on X, a T_2^Φ space endowed with a descriptive proximity δ_Φ such that Φ is a set of probe functions representing shape descriptors, let $M_1, M_2 \in \mathcal{P}(X)$, and let $\varepsilon > 0$. Further, let $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$. Then the following are equivalent.
(1) *$\mathfrak{P}_\Phi(M_1) \in \mathcal{A}, \mathfrak{P}_\Phi(M_2) \in \mathcal{B}$ are stable.*
(2) *$D(M_1, A) < \varepsilon, D(M_2, B) < \varepsilon$ for each $A \in \mathfrak{P}_\Phi(M_1)$ and for each $B \in \mathfrak{P}_\Phi(M_2)$.*

6.2. Comparison with existing clustering stability analysis

One of the most widely used clustering techniques is k-means clustering. This is a non-hierarchical clustering approach, which aims to partition n p-dimensional observations into k clusters ($k \leq n$) by minimizing a measure of dispersion within the clusters. In k-means clustering, the selection of the number of clusters affects the clustering stability significantly (Ben-Hur *et al.*, 2002). Let k be the true number of clusters in an image. If the number of clusters is greater than k, then some of the true clusters will be split into smaller clusters during clustering. On the other hand, if the number of clusters is less than k, then some of the true clusters will be merged into bigger clusters during clustering. Both cases will lead to unstable clusterings. Hence, clustering stability can be used as a quality measure of the clustering algorithm.

Ben-Hur, Elisseeff and Guyon propose distribution of pairwise similarity between clusterings of sub-samples of a dataset as a stability measure of a partition (Ben-Hur *et al.*, 2002). Another notion of stability as proposed in (Lange *et al.*, 2004) is based on the average dissimilarity of solutions computed on two different data sets. While the aforementioned approaches focus on maximizing the within-cluster similarity and within-cluster dissimilarity, Wang proposes a new measure of the quality of clusterings based on the clustering instability from sample to sample (Wang, 2010). On the other hand, Reizer proposes to measure the quality of clustering through stability from sample to sample (Reizer, 2011).

In contrast to the traditional clustering methods, the descriptive-based pattern generation method proposed in this article does not require the number of clusters to be pre-determined. The pattern $\mathfrak{P}_\Phi(M)$ may grow as long as it satisfies the condition that each new set A added to pattern $\mathfrak{P}_\Phi(M)$ is sufficiently near M, both spatially and descriptively. However, similar to clustering stability, we may say that the pattern generation is stable, provided it produces similar patterns on data originating from the same source. Based on this argument, a definition for pattern stability can be derived from the clustering stability model given in (Reizer, 2011).

Since we are interested in determining when a generated pattern in a sample digital image Y serves as an indicator that Y belongs to the class of digital images represented by a pattern generated in a query image X, we define pattern stability in terms of the expected descriptive distance between $\mathfrak{P}_\Phi(M, X)$ (pattern generated in X) and $\mathfrak{P}_\Phi(M, Y)$ (pattern generated in Y).

Definition 6.3. Pattern Stability.
Let $th > 0$ denote an expectation threshold and let $E[\cdot]$ denote the expected value of $\cdot$. Further, let $\mathfrak{P}_\Phi(M, X)$ be a pattern generated by M in X and $\mathfrak{P}_\Phi(M, Y)$, pattern generated by M in Y. The *stability of any description-based pattern* $\mathfrak{P}_\Phi(M)$ (denoted by $Stab(\mathfrak{P}_\Phi(M))$) is defined by

$$Stab(\mathfrak{P}_\Phi(M)) = \begin{cases} 1, & \text{if } E\left[\mathbb{D}_\Phi(\mathfrak{P}_\Phi(M, X), \mathfrak{P}_\Phi(M, Y))\right] \leq th, \\ 0, & \text{otherwise } \mathfrak{P}_\Phi(M) \text{ is unstable.} \end{cases}$$

where X and Y are two independent samples from some unknown distribution. Pattern $\mathfrak{P}_\Phi(M)$ is stable, provided $Stab(\mathfrak{P}_\Phi(M)) = 1$. ■

Furthermore, given two patterns $\mathfrak{P}_\Phi(M_1)$ and $\mathfrak{P}_\Phi(M_2)$, pattern generation will be stable, provided $M_1 \underline{\delta}_\phi M_2$ *and* $Stab(\mathfrak{P}_\Phi(M_1)) = Stab(\mathfrak{P}_\Phi(M_2)) = 1$. In addition, for any set A, $A \, \delta_\Phi \, M_1$ and $A \, \underline{\delta}_\phi \, M_2$ will ensure that set A will always be added to pattern $\mathfrak{P}_\Phi(M_1)$. This is advantageous in achieving pattern stability for the method proposed in this article compared to the traditional clustering methods such as k-means clustering, since pattern stability, in our case, derives its strength from the fact that each set A added to a pattern has *descriptive proximity* to the pattern generator M in a descriptive proximity space.

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Analyzing Trends for Maintenance Request Process Assessment: Empirical Investigation in a Very Small Software Company

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Abstract

Assessment and improvement of software maintenance processes in small software companies is very important because of large costs of maintenance and constraints of small software companies. This study presents an approach to assessment of software maintenance requests' processing in a very small local software company. The approach is context dependent and uses trend analysis and feedback sessions for assessing the current state of maintenance request processing. The analysis is based on various sources of data such as: internal repository of maintenance requests, company documents, transcribed records of interviews with company employees, and transcribed records of feedback sessions. Monthly trends for maintenance requests, working hours and types of maintenance tasks, by considering clients and software products are presented in the article. Identified trends were discussed during feedback sessions in the company. Participants in feedback sessions were company employees and researchers. During discussions of trends, some directions for further improvement of maintenance requests' processing were proposed. The article concludes with implications for practitioners from industry and researchers, as well as further research directions.

Keywords: software maintenance, process assessment, trend analysis, feedback session, very small software company.
ACM CCS: D.2.7 Distribution, Maintenance, and Enhancement, D.2.9 Management—Life cycle, K.6.3 Software Management—Software maintenance.

1. Introduction

Software maintenance includes all activities related to the preservation of consistency and efficiency of complex software systems. Maintenance consumes most of the costs of software systems (between 40% and 90% of the total costs) in software life-cycle (Lientz *et al.*, 1978; Kajko-Mattsson *et al.*, 2001; Abran *et al.*, 2004). Maintenance costs for systems that are in use for a very long time usually greatly exceed the costs of development. Despite that fact, software maintenance attracts less attention in scientific literature comparing to software development.

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Software Maintenance is in literature recognized as the last phase in software life-cycle, which does not attract enough attention when compared with software development. Developers and managers consider maintenance requests as short-term jobs that should be done as quickly as it is possible (Junio *et al.*, 2011). Research on the maintenance process conducted with people involved in the process indicates that only 2.7% thought that the maintenance process is effective, while 70.2% of them believe that the maintenance process is ineffective (Sousa, 1998).

Small companies are dominant in economies across the globe (Richardson & von Wangenheim, 2007). U.S. Census Bureaus "1995 County Business Patterns" pointed that the vast majority of software and data processing companies are small, and that those with more than 50 employees comprise only a few percent of the total number (Fayad *et al.*, 2000). Laporte *et al.* (2006) reported that in Europe, 85% of IT sector companies have between 1 and 10 employees.

Small software companies are typically characterized as economically vulnerable with low budget to perform corrective post delivery maintenance, as well as with limited resources and the lack of knowledge and capacities to implement software process assessment and improvement activities (Laporte *et al.*, 2008). According to Vasilev (2012), rationalization of processes indirectly affects company business and reduces managerial costs. Small software companies have not adopted assessment directives proposed by software process improvement (SPI) models (Capability Maturity Model (CMM) and more recently CMMI) or intenational process-related standards (ISO 15504 and ISO 9001) (Fayad & Laitinen, 1997). Because of limitations in scale and resources, small software companies find software process improvement a major challenge that should be supported with short, light and tailored assessment methods (Mc Caffery *et al.*, 2007). Qualitative empirical study about the maintenance practice in local small software companies (companies from Timisoara and Zrenjanin), with the focus on collecting and processing client requests, revealed that they face many problems, both technical and organizational (Stojanov, 2011; Stojanov *et al.*, 2011; Stojanov, 2012*b*). Therefore, software maintenance practice assessment and improvement in these companies require more attention.

This paper presents an approach to maintenance assessment in a very small software company based on analyzing trends in available maintenance data. Practice assessment is based on a tailored lightweight approach with frequent feedback, with the following phases: initialization, planning, execution, and final reporting on assessment. Since the aim of this paper is to present the use of trend analysis as a valuable tool in process assessment, assessment phases will not be discussed in more details.

The research was conducted in a very small software company with seven employees (classified as micro enterprise according to (Commission, 2005)). This study is a part of a large project (from 2011 to 2014) with the aim to assess and improve maintenance practice in the selected software company. Data collected in the company through practice observation, interviews with programmers, and analysis of company documents and maintenance repository provide the basis for assessment of the maintenance practice. The objective of the paper is to present a light assessment approach of software maintenance practice based on trend analysis.

The paper is organized as follows. Section 2 contains related work that presents the use of trend analysis in software engineering. After that are described the context of the research in section 3, and analysis of maintenance trends in section 4. Discussion of results follows in section 5, while discussion of treats to validity is in section 6. The last section of the paper contains concluding remarks with implications for research community and practitioners from industry, and further research directions.

2. Related work

Software maintenance trend analysis requires systematic data collection over an appropriate period. This is very important since maintenance requests occur randomly, and cannot be planned neither technically nor in budget. It is also important to note that maintenance workload cannot be managed using project management techniques (April, 2010).

Trend analysis can help in analyzing and controlling the activities and processes, and in assessing the efficiency of observed processes. Trend analysis is based on real empirical data and provides information of prime importance for organization (Buglear, 2001). A trend can be seen as an underlying longer-term movement in the observed data series. In addition, trend analysis is also important for providing evidence on deviations from trends. Trends are generally related to long-term observation and data collection, although the term ”*long term*” is based on the subjective assessment (Chatfield, 1996). Kanoun & Laprie (1996) argued that trend analyses are usually applied intuitively and empirically rather than in quantified and well-defined manner. Results of trend analyses provide valuable information for assessing maintenance activities and workload of maintenance personnel.

In the paper (Kenmei *et al.*, 2008) is proposed a trend analysis approach of change requests based on time series analysis of data extracted from version control and bug tracking systems. The empirical study is based on data from three large-scale open source software projects (Eclipse, Mozilla and JBoss). The study proved that time series are efficient tools for modeling change request density and further trends in receiving new change requests.

The study (Ahmed & Gokhale, 2009) presented an approach to modelling the behaviour of bugs inside Linux kernels. The study included the analysis of bug distribution, lifetime, and clustering inside the kernel modules, as well as a deeper analysis of the statistical trends in the bug data from an architectural perspective. The aim of the study was to gain insight into the factors that impact system reliability. The analysis was related to: trends across the three releases of the kernel, the manner in which bugs were resolved, and on understanding the impact of bug severities on the resolution time of the bugs. From the architectural perspective of the kernel, the results of the study based on the statistical trends suggested that the module dependencies and interactions have higher impact on the bugs than the individual modules themselves.

April (2010) presented trend analysis of software maintenance services. Analysis includes supply and demand of software maintenance services. The research was conducted as a part of a process improvement activity in Integratik, an ERP development firm in Canada. The improvement aim is the implementation of maintenance request tracking process and information system. This process should ensure that each request would be recorded, dispatched and tracked formally, as well as time recording of maintenance personnel effort. In addition, this improvement ensured that the maintenance demands would be properly measured and analyzed. The author investigated trends related to distribution of requests per months, maintenance personnel effort per months, distribution of requests and work effort for particular software applications.

Zhu *et al.* (2011) proposed an approach for quality evolution monitoring based on the analysis of deviation trends of different modularity views of software. The approach includes monitoring the following views: package view that prescribes how developers intentionally group related source files as modules (packages), structural cluster view that reflects the nature of inter-file dependencies and method invocation relations, and semantic cluster view that reflects the nature of vocabulary used and topics involved in different source files and their correlations. The approach is based on an assumption that the deviation between different modularity views can influence quality evolution. If the views are properly aligned, the developers will be able to easily locate concepts and implement modifications. Deviation between different trends was measured with SiMo (Similarity between the Modularity views) metrics. Deviations for individual versions were computed and analyzed, and after that deviation trends in a sequence of versions were analyzed. The main activities in the approach are: construction of modularity views, computation of similarity metrics and analysis of deviation trends. Deviation trend of different modularity views is a combination of three change trends (i.e. rise, drop, hold) of three SiMo metrics, which is totally 27 patterns of deviation trends. Empirical study was conducted on three open-source Java systems, JFreeChart, JHotDraw and Jedit, that are available at SourceForge.net Subversion (SVN) repositories. Presented empirical study confirms that continuous monitoring of deviation trends provides useful feedback.

3. Context

Proper understanding of the assessment approach requires more detailed insight into the organizational context where the study is conducted. The approach is tailored to a selected small software company and therefore it is necessary to outline basic facts about the company.

This research was realized in a very small software company with seven employees (six programmers and one technical secretary). Software development and maintenance activities are organized in the way that one or more programmers are assigned to each software application. When a maintenance request (MR) is received from a client, it is forwarded to a programmer from a set of assigned programmers. Programmers' assignments to software applications are documented and available to all employees.

The company maintains over 30 business software applications used by local clients in Serbia. Clients are classified in two groups: clients that have signed Maintenance Service Agreement (MSA) and pay for maintenance services on the monthly basis, and clients that do not have signed MSA and pay for each maintenance service after its completion.

3.1. Maintenance request processing

Analysis of trends for a long period of time requires the existence of systematically collected data, and the process that is implemented and followed by all stakeholders. All requests are recorded in the internal software application with the repository for issues tracking (requests, tasks, work orders). Although a process is usually tailored for the current request and a user, a general process is defined and implemented in the company. The process includes the following steps: receiving and recording a request in the internal repository, sending a notification email with the request info to an appropriate programmer, checking a client's status in order to define the priority of the request, assigning a programmer to a request, collecting additional information that is necessary for understanding and solving a request (if necessary), preparing a bid for work that is supposed to be done for clients that do not have signed MSA, and solving the request.

The request processing is completed when a client confirms correctness of finished work. In the repository is recorded who confirms the correctness, the date and the way of confirming. After that a working order is printed and sent to a client, and a request is labeled as closed.

3.2. Internal repository

The repository provides the efficient platform for storing and tracking tasks and maintenance requests. Practically, maintenance requests include all types of requests for maintenance, not only requests related to modification of software applications. In order to provide support for complete set of activities related to tracking requests, in the repository are also stored data about clients, software applications and work orders that are associated to requests. The repository is managed through an internal Web based application.

Issue tracking system does not contain only records for clients maintenance requests, but also records for all other, non-maintenance, tasks. However, analysis of all records for the period from May 2010 to November 2011 provides the evidence that 1896 tasks of totally 2252 tasks are related to software maintenance (84%), while 356 tasks are related to other activities (16%). A period of 19 months that begins two years after introducing the issue tracking system in the company is selected for the analysis. Discussions with programmers in the company confirmed that they are accustomed in using the system, which ensures extraction of more reliable data from the repository.

3.3. Programmers' working hours

Working hours spent on solving clients' requests provide the real basis for charging maintenance services. These working hours are hours that a programmer spends on a specific task associated to a request. In addition, these working hours are a part of a programmer's daily activities. Repository of MRs contains recorded working hours for each request. Three types of working hours exist in the repository: working hours spent in the company, working hours spent on Internet (activities that require Internet access to clients' information system), working hours spent at client side (in client's company). The total number of working hours can be calculated as a sum of these three types of working hours.

4. Maintenance trends analysis

Two sources of data were used for the trend analysis: company documents containing description of organizational structure of the company, and data extracted from the internal repository by using SQL script. Data about programmers' assignments to software applications, and the list of clients with MSA were extracted from company documents. Internal repository contains data about users' requests and other entities that are necessary to track all activities associated to each request. Data was extracted from tables `UserRequest`, `Worker` (programmers), `SoftwareApplication`, `User` (clients) and `WorkOrder`. In table 1 is presented the monthly distribution of solved (completed) maintenance requests used for the analysis in this paper, for clients

Table 1. Monthly distribution of solved maintenance requests

Month	Clients with MSA	Clients without MSA
5.2010.	43	17
6.2010.	37	29
7.2010.	52	24
8.2010.	57	16
9.2010.	42	18
10.2010.	80	18
11.2010.	94	28
12.2010.	88	41
1.2011.	73	49
2.2011.	85	33
3.2011.	85	44
4.2011.	88	30
5.2011.	57	28
6.2011.	60	35
7.2011.	64	30
8.2011.	83	39
9.2011.	49	30
10.2011.	78	54
11.2011.	70	48
Total	**1285**	**611**

with MSA and clients without MSA. The following trends can be drawn: (1) Clients with MSA submit more requests, which is expected since the costs of their requests usually fit the contracted amount in MSA, and (2) The average number of requests per month is 99.79, which practically means that all clients submit approximately four request per working day.

These trends do not provide enough information on maintenance requests' processing. Trends are too general, and therefore not suitable for more detailed analysis. However, these trends prove the high demand for maintenance services. In addition, these trends show that clients with MSA require more maintenance services than clients without MSA. In order to get deeper insight into maintenance trends it is necessary to include details about particular software applications that are maintained, about clients, and about types of maintenance tasks. This analysis enables detection of trends in demands for maintenance by various clients, discovery of distribution of requests per applications, and detection of trends for types of maintenance tasks.

4.1. Monthly trends for maintenance requests per client

Previous analysis shows that clients with MSA submit two times more requests. Therefore, it would be beneficial to find out the distribution of requests per clients, to find out clients with the highest demand for maintenance and based on that to suggest improved versions of MSA that will be tailored to each client or a group of clients. Detailed monthly trends with the number of requests for clients that submit more than five requests per month in average is presented in figure 1. Names of clients' companies are coded with letters A,B,C,D and E in order to preserve their anonymity according to guidelines for ethical issues in empirical studies in software engineering (Singer &

Table 2. Total and average number of MRs for clients with MSA

	Client A	Client B	Client C	Client D	Client E
Total number	171	170	95	184	142
Average	9.00	8.95	5.00	10.82	9.47

Vinson, 2002). It should be noted that clients D and E have zero requests in the beginning of observed interval because client D started to use software applications in July 2010 and client E in September 2010.

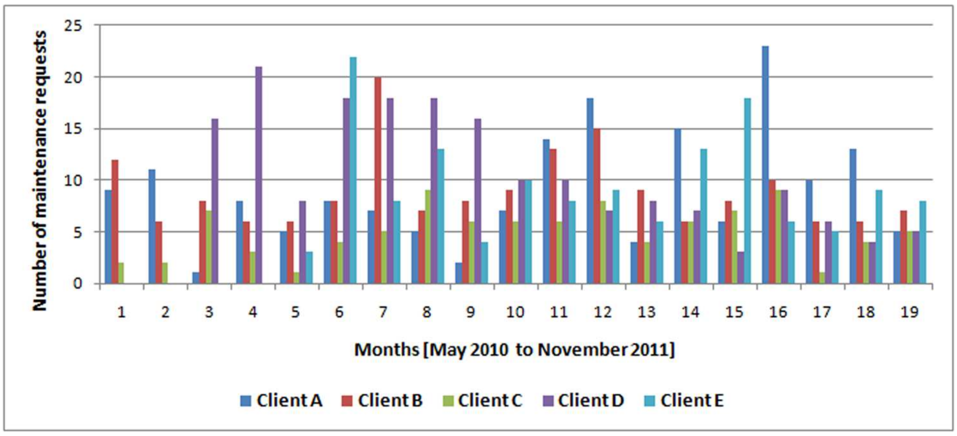

Figure 1. Monthly distribution of MRs for clients that submit more than five requests per month in average

Data presented in figure 1 are related to requests submitted by clients with MSA. Total number of requests, and the average number of requests per month for clients *A*,*B*,*C*,*D* and *E* are shown in table 2. Other clients with MSA submit smaller number of requests, but they submit them in each month.

Clients without MSA submit smaller number of requests than clients with MSA. In addition, they do not submit requests regularly. This means that there are longer periods of time without requests from them. Typical trends for requests submitted by clients (*K*,*M* and *N*) without MSA are presented in figure 2.

Trend analysis of the number of MRs for particular clients can be used for the proactive management of software maintenance activities. In addition, these data can be used also for improvement of policies in MSAs. Since trends for clients with MSA are regular, they could be also used as parameters for estimating future maintenance activities and workload. For clients without MSA it is very hard to draw any conclusion because of irregularity in trends.

Analysis of the number of working hours spent for each client shows the real state of the maintenance workload per client. Figure 3 shows the monthly distribution of working hours for selected clients with MSA.

The average number of working hours for clients *A*,*B*,*C*,*D* and *E* that have MSA per month, and the average number of working hours for clients *K*,*M* and *N* that do not have MSA are shown

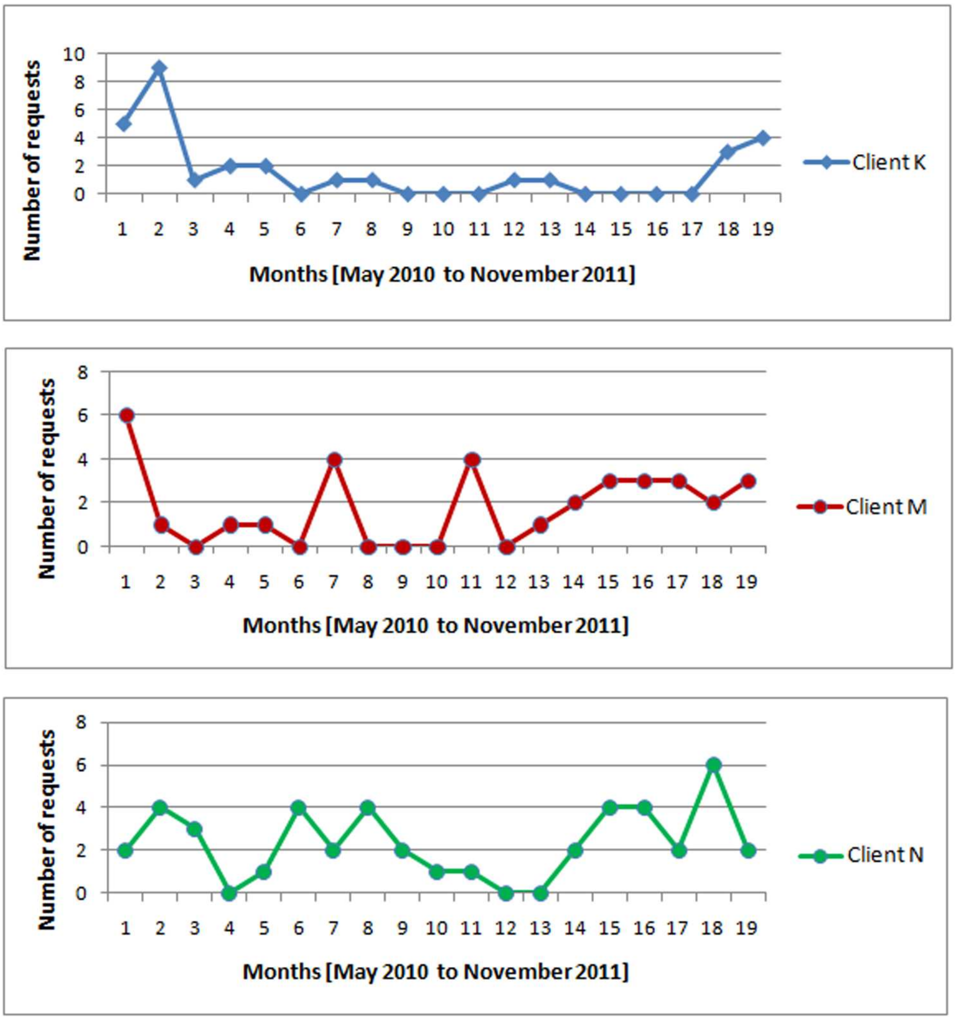

Figure 2. Typical MRs monthly trends for clients without MSA

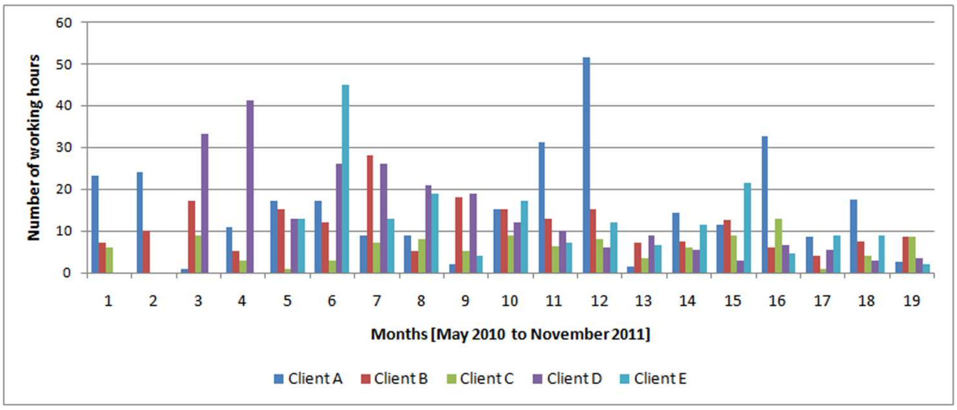

Figure 3. Monthly distribution of working hours for clients with MSA that submit more than five requests per month in average

Table 3. Average number of of working hours per month and per request for clients with MSA that submit more than five requests per month in average

	Client A	Client B	Client C	Client D	Client E
Average number of working hours per month	15.72	11.21	5.81	14.29	12.93
Average number of working hours per request	1.65	1.30	1.16	1¿,17	1.42

Table 4. Average number of of working hours per month and per request for clients without MSA

	Client K	Client M	Client N
Average number of working hours per month	1.13	2.61	1.76
Average number of working hours per request	0.44	0.92	0.64

in tables 3 and 4 respectively.

The first insight into the data related to clients and their requests suggests that clients with MSA consume more time and resources than clients without MSA. This is somehow expected, but directs the thinking towards tailoring appropriate service agreements for particular clients that have not signed agreements yet.

4.2. Monthly trends for maintenance requests per software application

Application portfolio consists of over 30 software applications used by local clients. Organization of maintenance activities is based on assignments of programmers to software applications, which is documented in the company. This means that when somebody receives a request, he/she knows who are potential programmers that should solve it. It is very important to know the distribution of MRs and working hours per software applications in order to improve the maintenance practice.

For that purpose was conducted trend analysis that shows the distribution of maintenance requests per applications, and the distribution of working hours per applications. Analysis disclosed that 75.84 percent of all requests are related to five software applications (named *app1*, *app2*, *app3*, *app4* and *app5*), while 87.39 percent of all requests are distributed to totally nine software applications (see figure 4).

Table 5 shows the average number of working hours per month for five most frequently used software applications, and the average number of working hours per request for these five applications.

4.3. Trends for types of maintenance tasks associated to requests

Classification of maintenance tasks, or maintenance types in the practice is subjected of several studies. From the first typology of software maintenance proposed by Swanson (1976), many authors have proposed different typologies. General agreement among the researchers and practitioners is that maintenance types are: corrective, perfective, adaptive and preventive. However, in practice, software organizations define their typologies according to their needs (Stojanov, 2012*a*).

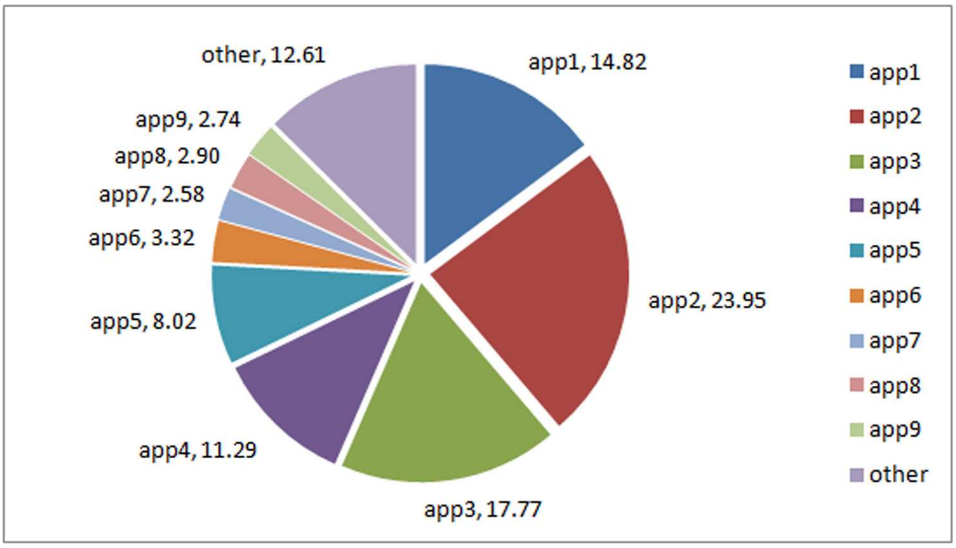

Figure 4. Monthly distribution of MRs per sotware applications

Table 5. Average number of working hours per month and per request for five most frequently used applications

	app1	app2	app3	app4	app5
Average number of working hours per month	19.05	33.07	20.91	19.99	9.51
Average number of working hours per request	1.20	1.41	1.18	1.78	1.10

Jones (2010) proposed the list of 23 types of maintenance tasks based on the best practice in industry.

In the selected small company, all maintenance tasks have been recorded in the internal repository. Despite of the large experience in industry, leading experts in the company have not proposed any classification of maintenance tasks based on proposals in available literature, but rather on their own experience and needs. In the repository are defined the following types of tasks: change (any type of change on software products), training, mandatory change (changes proposed by regulative and law), and all other tasks (updates, adaptations). However, more helpful analysis requires more detailed classification of maintenance tasks. For that purpose, general change tasks were manually classified by the company leading programmer in two groups: corrections tasks related to fixing detected faults, and enhancements tasks related to adding new features and other changes not related to faults. Classification was based on detailed description of tasks provided by clients and programmers.

The new classification schema for maintenance tasks is: corrections, enhancements, training, mandatory changes and other. Figure 5 presents trends for types of maintenance tasks in the company. The most of maintenance work is related to enhancing software product capabilities (60.18%), while corrections are related to 23.32% of all maintenance works. All other maintenance tasks contribute with about 10% .

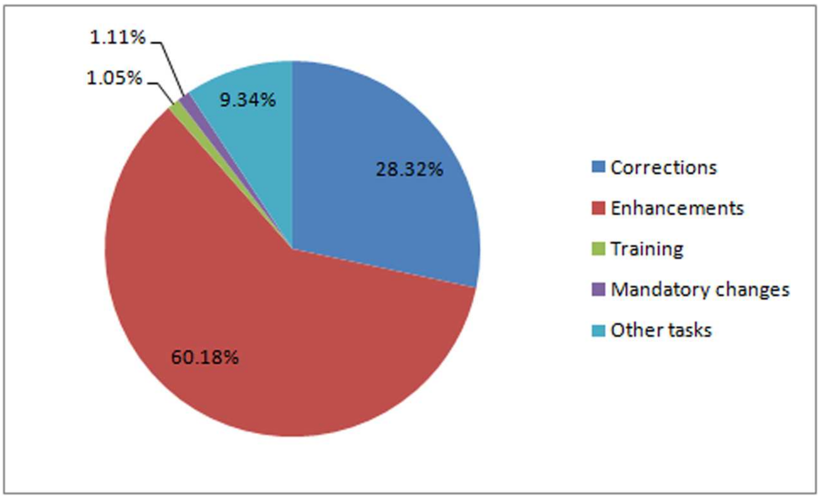

Figure 5. Trends for maintenance tasks

5. Discussion of results in the company

According to assessment plan, feedback meetings (sessions) were regularly organized in the company. Feedback meeting is an effective tool that helps in discussing the state of the assessment process, current findings, and further steps (Dyba *et al.*, 2004; Oktaba & Piattini, 2008). Hattie & Timperley (2007) argued that feedback provides directions for the current practice assessment, learning based on the experience, and further performance improvement.

Company manager and leading programmers participated in the feedback sessions. All sessions were prepared in advance in order to reflect the current state, discussions were type-recorded, and records were later transcribed and analyzed. Session lasted between 30 and 60 minutes. Practically, sessions were semi-structured, which means that a session plan and initial discussion had been prepared in advance, but discussions during the sessions included many issues that had not been planed.

Discussions related to the analysis of the average number of working hours per clients revealed the following trends:

- Programmers spend approximately six times more time on average for the realization of requests submitted by clients with MSA. According to MSA, clients pay for contracted number of working hours on the monthly basis, and that makes them to feel more comfortable in submitting new requests.
- On the other hand, clients without MSA submits less requests. In addition, their requests require less time in average on the monthly basis.
- The next issues that is obvious is that MRs submitted by clients with MSA consume more time comparing to requests originated from clients without MSA. In addition, clients without MSA are less interested in the improvement of the software applications they use.

Discussions related to the analysis of the average number of working hours per software applications revealed the following trends:

- Software applications labeled with *app1* to *app5* are usually installed as comprehensive business solution for accounting and management of resources in organizations. This explains their dominance regarding the number of requests and consumed working hours. Other applications are not so frequently used and usually are sold as independent software solutions. This implies that the package of these applications should be considered as a candidate for tailoring a special type of MSA for clients that regularly use them, and to offer this option also to other clients.
- For software applications that consume less working hours, solutions that will increase their usability to clients should be identified, which will lead to increase of associated maintenance activities and, therefore, to increased profit to the company. There are few possible directions for further activities that will help in increasing the profit from software applications that are not regularly used: include them in integrated business software solutions, or retire some of them and introduce substitutions that are more attractive to clients.

Discussions related to the analysis of trends in maintenance tasks revealed that the most of the work is related to enhancements and corrections. However, data available in the repository are not suitable for detailed analysis of trends because maintenance tasks have not been properly differentiated.

5.1. Improvement directions

Software process assessment is usually considered in literature as the initial phase of a process improvement (Gray & Smith, 1998). Assessment leads to the identification of key process (practice) elements that need improvements, or towards identification of the strengths and weaknesses that should be considered during improvements' planning (von Wangenheim *et al.*, 2006). Based on the presented trend analysis, the following improvement directions are identified:

- Development of effort estimation models that will be useful in planning programmers' workload. These models will consider monthly distributions of maintenance requests per software applications and per clients and distribution of responsibilities in the company.
- Improvement of planning activities in the company related to distribution of workloads among programmers in order to achieve more efficient and faster processing of maintenance requests.
- Improvements of service agreements for clients. This direction includes proposing different types of MSA that will include various types of software applications. This will lead to portfolio of MSAs that are tailored for special clients' needs.
- Improvements of software applications portfolio management that will consider software applications that are irregularly used, and have very small number of maintenance requests.This direction includes planning the retirement of unsuccessful software and introduction of appropriate substitutions.
- Development of more detailed typology of maintenance tasks that will enable derivation of trends that will cross data about software applications, clients, programmers workload, and maintenance tasks.

6. Limitations and threats to validity

Discussion about internal end external validity, and any other possible limitation is mandatory for empirical studies (Kitchenham *et al.*, 2002).

Internal validity relates to the design of the research, consistency of analysis, and the influence of unexpected sources of bias. Analysis is based on trend analysis techniques that are easy to implement, but requires deeper understanding of the context and full engagement of both researchers (assessors) and company employees that perform the process. This is accomplished by joint work on proposing general improvement and assessment goals, selection of appropriate techniques and methods, and joint analysis of all findings during feedback meetings in the company. The problem with the bias is not addressed since the general goal of the study is practice assessment and improvement and we assume that company employees will provide the full assistance in order to achieve the best results for them. In addition, using rigorous data analysis methods based on traceable data minimized researchers' bias in the research.

The threat to the external validity primarily is related to applicability of this approach in other industrial settings. The approach assumes deeper understanding of the context and involvement of all company employees in all phases of the research from planning, through collecting and analysing data, to discussing and presenting research findings. Since very small software companies have the similar problems in their business, the approach could be adapted to other small software companies, by considering specificity of their internal organization. The analysis process presented in this study could be also adapted to other, preferably small software companies or small teams. Subsequent applications of this approach would provide evidence about its validity and usefulness.

7. Conclusions

In this paper is presented an approach to software maintenance assessment based on trend analysis and feedback sessions. The study was conducted in a very small software company in Serbia, which is oriented towards local clients. Trend analysis includes analysis of maintenance requests' processing trends with the focus on the number of requests and working hours spent per clients and software applications, as well as simple analysis of maintenence tasks' trends.

The observations and conclusions from trend analysis will be used as directions for process improvement activities in the company. Both technical and organizational issues in the company are subject of improvement based on the results of trend analysis. For example, improvement of client service agreements is the obvious directions for practice improvement related to organizational issues. Improvement of the practice can be achieved also by proposing effort estimation models based on the current trends, such as the model presented by Stojanov *et al.* (2013). The next direction for practice improvement is related to development of more detailed typology of maintenance tasks that will enable analysis of trends for various types of tasks regarding software applications and clients.

The main contribution of the presented approach is related to implementation of assessment method based on trend analysis tailored to a very small software company. The method is based on collecting field data from company maintenance repository, analyzing data by using trend analysis, and identification of relevant conclusions and directions for further improvements of the maintenance practice. The next contribution is detailed presentation of trend analysis as a part of assessment method tailored to specific context, which will be helpful for other small software companies.

The approach is designed for small software companies or teams, and can be tailored to other similar settings. Findings of this research contain lessons that can be used by software practitioners in small software companies in order to assess and improve their decision-making and maintenance requests' processing. On the other hand, researchers could find some useful guidelines how to conduct *light* maintenance assessment based on trend analysis by considering given context.

Further work includes developing a formal model of light assessment approach for software maintenance in very small software companies, and adaptation and implementation of the approach in other similar settings. This will provide the opportunity to replicate the research in order to validate usability of presented approach. The next promising research direction is related to adapting this assessment approach to other processes in small software companies, or to companies that are mostly oriented towards outsourcing of products and services.

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Steady Non Isothermal Two-Dimensional Flow of Newtonian Fluid in a Stenosed Channel

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Abstract

In this paper, the steady two-dimensional motion of an incompressible Newtonian fluid between two parallel plates with heat transfer in the presence of a cosine shaped stenosis is studied. The governing equations are transformed into a compatibility and energy equations, which is solved analytically with the help of the regular perturbation technique. The solutions obtained from the present analysis are given in terms of streamlines, wall shear stress, separation and reattachment points, pressure and temperature distributions through the stenosed channel. The accuracy of the results are verified from available literature. It is found that the wall shear stress, pressure gradient and temperature increase with the development of the stenosis, causing separation and reattachment points in the region. It is also observed that even at low velocity, separation occurs if the thickness of the stenosis is increased. We present the results in graphical form.

Keywords: Newtonian fluid, stenosis, heat transfer.
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1. Introduction

The motivation of this study comes from the investigation of abnormal blood flow in a stenosed artery, which may be due to atherosclerotic plaques developed at various locations in the artery. Its effect on the flow of blood is discussed by many authors theoretically, experimentally as well as numerically. Forrester and Young (Forrester & Young, 1970) presented the theoretical as well as experimental results of an axisymmetric, steady flow through a converging and diverging tube with mild stenosis. It is observed that there is an abundant amount of evidence to support the conclusion that the abnormal flow conditions developed in a stenotic obstruction can be an important

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factor in the development and progression of arterial disease. Further indicate that even a mild collarlike stenosis in a small artery can create significant abnormalities in the flow. Morgan and Young (Morgan & Young, 1974) provided the approximate analytical solution of axisymmetric, steady flow of incompressible Newtonian fluid both for mild and severe stenosis by using an integral method; basically they presented the extension of Forrester (Forrester & Young, 1970). It is observed that even a mild stenosis can cause a radical alteration in flow characteristics and that the effect in general becomes more drastic as the stenosis becomes more severe and the Reynolds number increases and also wall shearing stress is especially affected. Analysis of blood flow using an incompressible Newtonian fluid through an axisymmetric stenosed artery of cosine shape has been done by K. Haldar (Haldar, 1991). It is shown that for any given Reynolds number or tube constriction the separation point moves towards the throat of the tube and the reattachment point moves downstream with the enlargement of the region of separation which is physiologically unfavorable. Layek and Midya (Layek & Midya, 2007) presented the numerical solution of a time dependent incompressible Newtonian fluid for symmetric stenosis in a two dimensional channel. It is noticed that the maximum stress and the length of the recirculating region associated with two shear layers of the constriction increase with the increase of the area reduction of the constriction. It is observed that the critical values for three constriction heights $h = 0.25, 0.3, 0, 35$ are $600, 300, 210$ respectively. Chow et al. (Chow *et al.*, 1971) analyzed the steady laminar flow of an incompressible Newtonian fluid for different physical parameters by considering a sinusoidal boundary. It is observed that by increasing either Re or ϵ, the separation point would move down towards the throat in the divergent part of the channel with subsequent enlargement of the region of separation. Lee and Fung (Lee & Fung, 1970) solved the flow model of the Newtonian fluid numerically through locally constricted tube for the low Reynolds number. The constraints in their numerical procedure restricted the shape of the tube to be fixed and the Reynolds number to be moderate. Haldar (Haldar, 1985) discussed the effect of the shape of constriction on the resistance of blood flow through an artery with mild local narrowing. It is shown that the resistance to flow decreases as the shape of the stenosis changes and maximum resistance is attained for symmetric stenosis. S. Chakravarty and A. Ghosh Chakravarty (Chakravarty & Chakravarty, 1988) presented analytical solutions by considering an anisotropically elastic cylindrical tube filled with viscous incompressible fluid representing blood having stenosis. The analysis is carried out for an artery with mild local narrowing in its lumen forming a stenosis. K. Haldar (Kruszewski *et al.*, 2008) studied the oscillatory flow of blood which behaves as a Newtonian fluid having surface roughness of cosine shape. It is observed that the resistive impedance and wall shear stress increases as the phase lag increases for a particular value of stenosis height. It is also observed that impedance and wall shear stress increases with the increase in the stenosis height. Newman et al. (Newman *et al.*, 1979) investigated the oscillatory flow numerically in a rigid tube with stenosis. The predictions of the numerical results agreed well with the experimental works. This paper deals with the problem of oscillatory blood flow through a rigid tube with a mild constriction under a simple-harmonic pressure gradient examines the effect of stenosis on the flow field by considering blood as a Newtonian fluid. Mehrotra et al. (Mehrotra *et al.*, 1985) presented analysis by considering the flow in a stenotic tube where the cross-section is elliptic. It is observed that the theoretical study of pulsatile flow in a stenotic tube confirms the view that the fluid dynamics characteristics of the flow are affected by the percentage of stenosis as well as the geometry of the stenosis. The frequency of oscillation also influences the shearing stress. Srivastava and Rastogi (Srivastava & Rastogi, 2010) investigated the blood flow through narrow catheterized artery with an axially nonsymmetrical stenosis. It is found that the flow resistance increases with the catheter size, the hematocrit and the stenosis size but decreases with the shape parameter. A significant increase in the magnitude of the impedance and the wall shear stress occurs even for a small increase in the catheter size. The shear stress at the stenosis throat decreases with the increasing catheter size. The abnormal flow conditions developed due to stenosis can be an important factor in the development and progression of arterial diseases. Some of the further major complications developed through these stenosis are the growth of tissues into arteries, development of an intravascular clot and post-stenotic dilatation. This type of flow also has applications in various fields like physiological flows and polymer science.

In the present paper, the effect of stenosis height and Reynolds number on flow characteristics, wall shear stress, pressure gradient, separation and reattachment points and heat transfer are analyzed. The study of the Peclet number and Brinkman number on the temperature distribution is also presented. It is observed that the general pattern of flow is similar to the results given in (Haldar, 1991) - (Chow *et al.*, 1971). The results of the present investigation indicate that even a mild collar like stenosis in a small artery can create significant abnormalities in the flow including the phenomenon of separation. This study presents the steady, two-dimensional motion of an incompressible Newtonian fluid in a cosine shaped stenosed channel with heat transfer. In this analysis blood is assumed as Newtonian fluid and the geometry of the artery is approximated by a channel. The layout of the paper is as follows: The basic equations governing the flow, in the Cartesian coordinate, are given in section 2. Problem formulation is presented in Section 3. In Section 4 the method is discussed and section 5 is dedicated the solution for different parameters. Section 6 provides a graphical discussion. A summary is given in section 7.

2. Governing equations

The basic governing equations for steady two dimensional flow of a non-isothermal, incompressible linearly viscous fluid in the absence of body forces are

$$\widetilde{\nabla} \cdot \widetilde{\mathbf{V}} = 0, \tag{2.1}$$

$$\rho \frac{d\widetilde{\mathbf{V}}}{dt} = -\widetilde{\nabla}\widetilde{p} + \widetilde{\nabla}\widetilde{\tau}, \tag{2.2}$$

$$\rho c_p \frac{d\widetilde{T}}{dt} = \kappa \widetilde{\nabla}^2 \widetilde{T} + \phi, \tag{2.3}$$

where $\widetilde{\mathbf{V}}$, $\widetilde{T}$ and ρ are the velocity vector, temperature and constant density of the fluid respectively, $\widetilde{p}$ is the dynamic pressure, c_p and κ are the specific heat and thermal conductivity parameters respectively, $\widetilde{\nabla}^2$ is the Laplacian, ϕ the viscous dissipation function defined as $\phi = \widetilde{\tau} \cdot \widetilde{\nabla}\widetilde{\mathbf{V}}$ and d/dt the material time derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \widetilde{u}\frac{\partial}{\partial \widetilde{x}} + \widetilde{v}\frac{\partial}{\partial \widetilde{y}}, \tag{2.4}$$

where $\widetilde{u}$ and $\widetilde{v}$ are the velocity components in $\widetilde{x}$ and $\widetilde{y}$ directions, respectively and $\widetilde{\tau}$ is the extra stress tensor defined as follows

$$\widetilde{\tau} = \mu \widetilde{\mathbf{A}}_1, \tag{2.5}$$

where μ is the dynamic viscosity and $\mathbf{A}_1$ the first Rivlin-Ericksen tensor defined as

$$\widetilde{\mathbf{A}}_1 = \widetilde{\nabla}\widetilde{\mathbf{V}} + (\widetilde{\nabla}\widetilde{\mathbf{V}})^{\top}, \tag{2.6}$$

here $\top$ indicates the transpose.

3. Problem formulation

Consider the non-isothermal Newtonian flow through the channel of infinite length with heat transfer having stenosis of length $l_o/2$. The coordinate system is chosen in such a way that the arterial system lies in the $\widetilde{x}\widetilde{y}-plane$, such that $\widetilde{x}-axis$ coincide with the center line in the direction of flow and $\widetilde{y} - axis$ perpendicular to $\widetilde{x} - axis$.

Consider the boundary of the stenosed region of the form (Haldar, 1991)

$$\begin{aligned} h\left(\widetilde{x}\right) &= h_o - \frac{\lambda}{2}\left(1 + \cos\left(\frac{4\pi\widetilde{x}}{l_o}\right)\right) \quad -\frac{l_o}{4} < \widetilde{x} < \frac{l_o}{4}, \\ &= h_o \quad \text{otherwise}, \end{aligned} \tag{3.1}$$

where $h(\widetilde{x})$ is variable gap between the stenosis, $2h_o$ the width of unobstructed channel and λ the maximum height of stenosis.

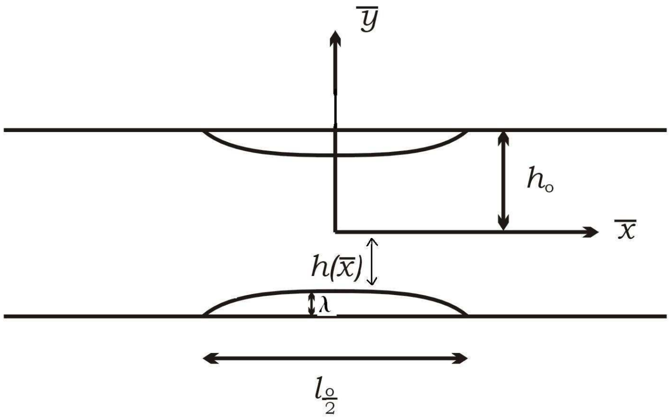


Figure 1. Geometry of the problem.

Boundary conditions for the present problem are

$$\begin{aligned} &\widetilde{u} = \widetilde{v} = 0, \quad \widetilde{T} = T_1 \quad at \quad \widetilde{y} = h(\widetilde{x}), \\ &\frac{\partial \widetilde{u}}{\partial \widetilde{y}} = 0, \quad \frac{\partial \widetilde{T}}{\partial \widetilde{y}} = 0 \quad at \quad \widetilde{y} = 0, \\ &\widetilde{Q} = 2\int_0^{h(\widetilde{x})} \widetilde{u} d\widetilde{y} = -u_o h_o, \end{aligned} \tag{3.2}$$

where u_o is the average velocity and $\widetilde{Q}$ the volume flow rate. Assume that the blood behaves like Newtonian fluid and for steady, homogeneous, incompressible two dimensional flow of blood velocity field is assumed as

$$\widetilde{\mathbf{V}} = \left(\widetilde{u}\left(\widetilde{x},\widetilde{y}\right), \widetilde{v}\left(\widetilde{x},\widetilde{y}\right), 0\right). \tag{3.3}$$

Introducing the dimensionless parameters as follows

$$u = \frac{\widetilde{u}}{u_o}, \quad v = \frac{\widetilde{v}}{u_o}, \quad x = \frac{\widetilde{x}}{l_o}, \quad y = \frac{\widetilde{y}}{h_o}, \quad p = \frac{h_o^2}{\mu u_o l_o}\widetilde{p}, \quad \theta = \frac{\widetilde{T} - T_o}{T_1 - T_o}, \tag{3.4}$$

where T_1 and T_o are temperatures on the boundary of stenosis and fluid respectively.

Substituting equations (2.4)-(2.6) in equations (2.1) - (2.3) and making use of (3.3) and (3.4), nondimensional form of equations becomes

$$\delta\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.5}$$

$$Re\left(\delta u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \nabla^2 u, \tag{3.6}$$

$$Re\delta\left(\delta u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \delta\nabla^2 v, \tag{3.7}$$

$$Pe\left(\delta u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\theta = \nabla^2\theta + Br\left(4\delta^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \delta\frac{\partial v}{\partial x}\right)^2\right), \tag{3.8}$$

where

$$\delta = \frac{h_o}{l_o}, \quad Re = \frac{u_o h_o}{\nu}, \quad Br = \frac{\mu u_o^2}{\kappa(T_1 - T_o)}, \quad Pe = \frac{\rho c_p h_o u_o}{\kappa}, \tag{3.9}$$

in which Re is the Reynolds number, Br the Brinkman number, Pe the Peclet number.

Now to convert these equations in single variable, introducing the stream function defined as

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\delta\frac{\partial\psi}{\partial x}, \tag{3.10}$$

which satisfy the continuity equation (3.5) identically. After eliminating pressure gradient term from momentum equations (3.6)-(3.7) and making use of (3.10), compatibility equation is obtained of the form

$$Re\delta\frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial\left(y, x\right)} = \nabla^4\psi, \tag{3.11}$$

where $\nabla^2 = \delta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, is the dimensionless form of the Laplacian. Energy equation (3.8) in terms of stream function becomes

$$Pe\delta\frac{\partial(\psi, \theta)}{\partial(y, x)} = \nabla^2\theta + Br\left(4\delta^2\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial y^2} - \delta^2\frac{\partial^2\psi}{\partial x^2}\right)^2\right). \tag{3.12}$$

The dimensionless stenosis profile (3.1) takes the form

$$f(x) = 1 - \frac{\epsilon}{2}(1 + \cos 4\pi x) \quad -\frac{1}{4} < x < \frac{1}{4}, \\ = 1 \quad \text{otherwise}, \tag{3.13}$$

where $f = \frac{h(\bar{x})}{h_o}$ and $\epsilon = \frac{\lambda}{h_o}$.
Boundary conditions in terms of stream function becomes

$$\frac{\partial \psi}{\partial y} = 0, \quad \psi = -\frac{1}{2}, \quad \theta = 1 \quad at \quad y = f, \\ \frac{\partial^2 \psi}{\partial y^2} = 0, \quad \psi = 0, \quad \frac{\partial \theta}{\partial y} = 0 \quad at \quad y = 0. \tag{3.14}$$

Due to non-linearity of (3.11) and (3.12), the regular perturbation technique is applied to find the analytical solution along with the boundary conditions defined in (3.14).

4. Perturbation method

In this section we shall discuss the perturbation method by considering a linear or nonlinear differential equation

$$L(\psi, \delta) = 0, \tag{4.1}$$

that depends on the small positive parameter δ. The boundary or initial conditions may depend on δ. The reduced or unperturbed problem associated with the problem is obtained by setting $\delta = 0$ along with its boundary or initial conditions. We expand the solution ψ in the perturbation series

$$\psi = \sum_{n=0}^{\infty} \psi_n \delta^n, \tag{4.2}$$

the difference between ψ and ψ_o is refereed to as a perturbation on the solution ψ_o of the reduced problem. Inserting this equation into (4.1) gives

$$L(\psi, \delta) = L(\sum_{n=0}^{\infty} \psi_n \delta^n, \delta) = 0. \tag{4.3}$$

We assume that $L(\psi, \delta)$ can be expanded in a power series in ψ and δ. As a result above equation (4.3) can be expanded in the form of the series

$$L(\psi, \delta) = \sum_{n=0}^{\infty} L_n(\psi_n, \psi_{n-1}, ...\psi_1, \psi_o)\delta^n = 0, \tag{4.4}$$

where L_n represents differential operator, which may be linear or nonlinear. The series (4.2) is also inserted into the given initial and boundary conditions for the problem.

To solve the given problem by means of the perturbation method, we put the coefficient of δ^n in (4.4) equal to zero and obtain

$$L_n(\psi_n, \psi_{n-1}, \cdots \psi_1, \psi_o) = 0, n = 0, 1, 2, \cdots. \tag{4.5}$$

Similarly we equate coefficient of like powers of δ in the initial or boundary conditions. This yields the system of equations (4.5) with appropriate boundary conditions that we solve recursively.
We first solve the reduced equation

$$L_o(\psi_o) = 0, \tag{4.6}$$

with relevant boundary conditions. Once ψ_o is found, then equation for ψ_1 with boundary conditions is

$$L_1(\psi_1, \psi_o) = 0, \tag{4.7}$$

is solved and then the equations for $\psi_2, \psi_3, \cdots$ with relevant boundary or initial conditions are solved successively.

5. Solution

To solve the compatibility equation and energy equation along with boundary conditions (3.14), the flow variables ψ and θ are perturbed as

$$\begin{aligned} \psi &= \psi_o + \delta\psi_1 + \delta^2\psi_2 + \cdots, \\ \theta &= \theta_o + \delta\theta_1 + \delta^2\theta_2 + \cdots. \end{aligned} \tag{5.1}$$

where δ is a small parameter.

5.1. Zeroth order problem and its solution

Zeroth order system of equations is obtained by substituting (5.1) in equations (3.11)-(3.12), (3.14) and equating the coefficients of δ^0 as

$$\frac{\partial^4\psi_o}{\partial y^4} = 0, \tag{5.2}$$

$$\frac{\partial^2\theta_o}{\partial y^2} = -Br\left(\frac{\partial^2\psi_o}{\partial y^2}\right)^2, \tag{5.3}$$

and corresponding boundary conditions

$$\begin{aligned} &\frac{\partial\psi_o}{\partial y} = 0, \quad \psi_o = -\frac{1}{2}, \quad \theta_o = 1 \quad at \quad y = f, \\ &\frac{\partial^2\psi_o}{\partial y^2} = 0, \quad \psi_o = 0, \quad \frac{\partial\theta_o}{\partial y} = 0 \quad at \quad y = 0. \end{aligned} \tag{5.4}$$

The solution of equation (5.2) along with boundary conditions (5.4) is given of the form

$$\psi_o = \frac{\eta}{4}\left(\eta^2 - 3\right), \qquad where \quad \eta = \frac{y}{f}. \tag{5.5}$$

After substitution of (5.5) in (5.3) subject to (5.4), zeroth order temperature is obtained as

$$\theta_o = 1 - \frac{3Br}{16f^2}\left(\eta^4 - 1\right), \tag{5.6}$$

which indicates that the temperature depends upon the ratio of heat production by viscous dissipation to heat transport by conduction.

5.2. *First order problem and its solution*

For the first order system comparing the coefficients of δ, we get

$$\frac{\partial^4\psi_1}{\partial y^4} = Re\frac{\partial\left(\psi_o, \frac{\partial^2\psi_o}{\partial y^2}\right)}{\partial\,(y,x)}, \tag{5.7}$$

$$\frac{\partial^2\theta_1}{\partial y^2} = Pe\frac{\partial\,(\psi_o, \theta_o)}{\partial(y,x)} - 2Br\left(\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2}\right), \tag{5.8}$$

and boundary conditions

$$\begin{aligned} &\frac{\partial\psi_1}{\partial y} = 0, \quad \psi_1 = 0 \quad \theta_1 = 0 \quad at \quad y = f, \\ &\frac{\partial^2\psi_1}{\partial y^2} = 0, \quad \psi_1 = 0, \quad \frac{\partial\theta_1}{\partial y} = 0 \quad at \quad y = 0. \end{aligned} \tag{5.9}$$

The solution of equation (5.7) by making use (5.5) and (5.9) becomes

$$\psi_1 = -\frac{3Ref'\eta}{1120}\left(\eta^6 - 7\eta^4 + 11\eta^2 - 5\right). \tag{5.10}$$

By substitution of (5.5) and (5.10) in equation (5.8) and making use of (5.9), the first order temperature profile is obtained of the form

$$\theta_1 = \frac{3Brf'\left(\eta^2 - 1\right)}{8960f^2}\left\{2Re\left(9\eta^6 - 47\eta^4 + 19\eta^2 + 19\right) + Pe\left(15\eta^6 - 13\eta^4 - 83\eta^2 + 337\right)\right\}. \tag{5.11}$$

It is observed that the first order temperature depends upon the ratio of heat production by viscous dissipation and heat transport by convection to heat transport by conduction.

5.3. *Second order problem and its solution*

Comparing the coefficients of δ^2 to get the second order system as

$$\frac{\partial^4\psi_2}{\partial y^4} = Re\left[\frac{\partial\left(\psi_o, \frac{\partial^2\psi_1}{\partial y^2}\right)}{\partial\,(y,x)} + \frac{\partial\left(\psi_1, \frac{\partial^2\psi_o}{\partial y^2}\right)}{\partial\,(y,x)}\right] - 2\frac{\partial^4\psi_o}{\partial x^2\partial y^2}, \tag{5.12}$$

$$\frac{\partial^2\theta_2}{\partial y^2} = Pe\left[\frac{\partial(\psi_o,\theta_1)}{\partial(y,x)} + \frac{\partial(\psi_1,\theta_o)}{\partial(y,x)}\right] - \frac{\partial^2\theta_o}{\partial x^2} - Br\left[4\left(\frac{\partial^2\psi_o}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi_1}{\partial y^2}\right)^2 + 2\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_2}{\partial y^2} - 2\frac{\partial^2\psi_o}{\partial y^2}\frac{\partial^2\psi_o}{\partial x^2}\right], \tag{5.13}$$

boundary conditions for second order system are

$$\begin{aligned} &\frac{\partial\psi_2}{\partial y} = 0, \quad \psi_2 = 0, \quad \theta_2 = 0 \quad at \quad y = f, \\ &\frac{\partial^2\psi_2}{\partial y^2} = 0, \quad \psi_2 = 0, \quad \frac{\partial\theta_2}{\partial y} = 0 \quad at \quad y = 0. \end{aligned} \tag{5.14}$$

Using (5.5) and (5.10) in equation (5.12), the solution is obtained by successive integration along with the boundary conditions defined in (5.14) as follows

$$\psi_2 = CRe^2\eta\left[f'^2\left(98\eta^{10} - 1155\eta^8 + 4488\eta^6 - 8778\eta^4 + 8222\eta^2 - 2875\right) - ff''\left(35\eta^{10} - 385\eta^8 + 1518\eta^6 - 3234\eta^4 + 3279\eta^2 - 1213\right)\right] - \frac{3\eta(4f'^2 - ff'')}{40}(\eta^4 - 2\eta^2 = 1), \tag{5.15}$$

which is second order solution for stream lines. To find the second order temperature, using (5.5), (5.10) and (5.15) in equation (5.13), with the help of MATHEMATICA, we get

$$\begin{aligned} \theta_2 = &\frac{C_1(\eta^2-1)}{f^2}\Big[-4f'^2\Big\{7Pe^2\left(225\eta^{10} - 721\eta^8 - 220\eta^6 + 30134\eta^4 - 94771\eta^2 + 238859\right) \\ &+2PeRe\left(840\eta^{10} - 6860\eta^8 + 11455\eta^6 + 3139\eta^4 - 26891\eta^2 + 56269\right) + Re^2\left(2303\eta^{10}\right. \\ &\left.-21721\eta^8 + 63122\eta^6 - 68086\eta^4 + 17183\eta^2 + 17183\right) - 517440(13\eta^4 + 9\eta^2 - 36)\Big\} \\ &+ff''\Big\{3Br\Big(7Pe^2\left(225\eta^{10} - 721\eta^8 - 2206\eta^6 + 30134\eta^4 - 94771\eta^2 + 238859\right) \\ &+2PeRe\left(525\eta^{10} - 5173\eta^8 + 14132\eta^6 - 7120\eta^4 - 29065\eta^2 + 102605\right) + 8Re^2\left(175\eta^{10}\right. \\ &\left.-1673\eta^8 + 5158\eta^6 - 7778\eta^4 + 2059\eta^2 + 2059\right) - 2069760\left(7\eta^4 - 4\eta^2 - 9\right)\Big)\Big\}\Big]. \end{aligned} \tag{5.16}$$

5.4. Velocity and temperature fields

The dimensionless velocity components in x and y directions are obtained from (3.10), we arrive at the axial component of velocity as

$$\begin{aligned} u = &\frac{(\eta^2-1)}{f}\Big[\frac{3}{4} - \frac{3Re\delta f'}{1120}\left(7\eta^4 - 28\eta^2 + 5\right) + \delta^2\Big\{\frac{3}{40}\left(ff'' - 4f'^2\right)\left(5\eta^2 - 1\right) \\ &+CRe^2\Big\{f'^2\left(1078\eta^8 - 9317\eta^6 + 22099\eta^4 - 21791\eta^2 + 2875\right) \\ &-Cff''\left(385\eta^8 - 3080\eta^6 + 7546\eta^4 - 8624\eta^2 + 1213\right)\Big\}\Big\}\Big], \end{aligned} \tag{5.17}$$

and the normal component of velocity is

$$v = \frac{\delta\eta\left(\eta^2-1\right)}{f}\left[\frac{3}{4}f' + \frac{3Re\delta}{20}\left(ff''\left(\eta^2-1\right)\left(\eta^2-5\right) - f'^2\left(7\eta^4-28\eta^2+5\right)\right)\right.$$
$$+\delta^2\left\{\frac{3ff'f''}{10}\left(3\eta^2-1\right) - \frac{3f^2f''}{40}\left(\eta^2-1\right) - \frac{3f'^3}{10}\left(5\eta^2-1\right) + Re^2\left\{Cf'^3\left(1078\eta^8\right.\right.\right. \tag{5.18}$$
$$-93172\eta^6+22099\eta^4-21791\eta^2+28875\Big) - C_2ff'f''\left(273\eta^8-2422\eta^6+6620\eta^4\right.$$
$$\left.\left.\left.\left.-8626\eta^2+2875\right) + Cf^2f''\left(\eta^2-1\right)\left(35\eta^6-315\eta^4+853\eta^2-1213\right)\right\}\right\}\right].$$

The temperature distribution up to second order is obtained from (5.1), we arrive

$$\theta = 1 - \frac{Br}{f^2}\left[\frac{3}{16}(\eta^4-1) - \delta\left\{\frac{3f'(\eta^2-1)}{8960}\left(2Re\left(9\eta^6-47\eta^4+19\eta^2+19\right) + Pe\left(15\eta^6-13\eta^4\right.\right.\right.\right.$$
$$\left.\left.\left.-83\eta^2+337\right)\right)\right\} - \delta^2\left[C_1(\eta^2-1)\left\{-2f'^2\left\{7Pe^2\left(225\eta^{10}-721\eta^8-2206\eta^6+30134\eta^4\right.\right.\right.\right.$$
$$\left.-94771\eta^2+238859\right) + 4PeRe\left(840\eta^{10}-6860\eta^8+11455\eta^6+3139\eta^4-26891\eta^2+56269\right)$$
$$+2Re^2\left(2303\eta^{10}-21721\eta^8+63122\eta^6-68086\eta^4+17183\eta^2+17183\right) - 1034880\left(13\eta^4\right.$$
$$\left.\left.\left.+9\eta^2-36\right)\right\}\right\} + ff''\left\{3Br\left\{7Pe^2\left(225\eta^{10}-721\eta^8-2206\eta^6+30134\eta^4-94771\eta^2+238859\right)\right.\right.$$
$$+2PeRe\left(525\eta^{10}-5173\eta^8+14132\eta^6-7120\eta^4-29065\eta^2+102605\right) + 8Re^2\left(175\eta^{10}-1673\eta^8\right.$$
$$\left.\left.\left.\left.\left.+5158\eta^6-7778\eta^4+2059\eta^2+2059\right) - 2069760\left(7\eta^4-4\eta^2-9\right)\right\}\right\}\right]\right], \tag{5.19}$$

where $C = \frac{1}{3449600}$, $C_1 = \frac{1}{55193600}$, $C_2 = \frac{1}{1724800}$. Dimensionless wall shear stress for viscous fluid up to second order is given by

$$\begin{aligned}\tau_\omega &= \left(\frac{\partial u}{\partial y} + \delta\frac{\partial v}{\partial x}\right)_{y=f} \\ &= \frac{3}{f^2}\left[\frac{1}{2} + \frac{Ref'}{35}\delta + \frac{\delta^2}{10}\left\{\frac{Re^2}{8085}\left(40ff''-79f'^2\right) + \left(2ff''-13f'^2\right)\right\}\right].\end{aligned} \tag{5.20}$$

The points of separation and reattachment are defined as the back flow at wall, where the wall shear stress is zero, i.e.$\tau_\omega = 0$, then above equation reduces as

$$40425 + 2310Ref'\delta + \delta^2\left\{10ff''\left(40Re^2+16170\right) - f'^2\left(79Re^2+105105\right)\right\} = 0. \tag{5.21}$$

The solution of (5.21) in terms of Reynolds number Re is

$$Re = \frac{7}{\delta\left(40ff''-79f'^2\right)}\left\{-165f' \pm \sqrt{165\left\{165f'^2 - (40ff''-79f'^2)(5-13\delta^2f'^2+2\delta^2ff'')\right\}}\right\}. \tag{5.22}$$

By using equation (5.22), our aim is to find graphically the critical Reynolds number at which the back flow occur.

5.5. Pressure distribution

To find the pressure distribution along x-axis within the channel, the equations (3.6) and (3.7) are converted in terms of stream function and then perturb these equation by using (5.1) and

$$p = p_o + \delta p_1 + \delta^2 p_2 + \cdots, \tag{5.23}$$

system of equations is obtained as follows.

5.5.1. Zeroth order pressure and solution

Comparing the coefficients of δ^0 , we get

$$\frac{\partial p_o}{\partial x} = \frac{\partial^3 \psi_o}{\partial y^3}, \tag{5.24}$$

$$\frac{\partial p_o}{\partial y} = 0. \tag{5.25}$$

By integrating the above two equations and making use of (5.5), the zeroth order pressure is obtained of the form

$$p_o = \frac{3}{32\pi(\epsilon-1)^2}\left[\frac{1}{\sqrt{1-\epsilon}}\left(3\epsilon^2 - 8\epsilon + 8\right)\tan^{-}1\left(\frac{\tan 2\pi x}{\sqrt{1-\epsilon}}\right) - \frac{f'}{8\pi f^2}\left\{16(\epsilon-1) - 3\epsilon^2 \right.\right.$$
$$\left.\left. -3\epsilon(\epsilon-2)\cos(4\pi x)\right\}\right], \tag{5.26}$$

which involves the trigonometric and inverse trigonometric function.

5.5.2. First order pressure and solution

Equating the coefficients of δ , we obtain

$$\frac{\partial p_1}{\partial x} = \frac{\partial^3 \psi_1}{\partial y^3} - Re\frac{\partial\left(\psi_o, \frac{\partial \psi_o}{\partial y}\right)}{\partial\left(y, x\right)}, \tag{5.27}$$

$$\frac{\partial p_1}{\partial y} = 0, \tag{5.28}$$

by making use of equations (5.5), (5.10) and solving (5.27)-(5.28), the first order solution for pressure is obtained by applying

$$p_1 = \int_0^x \frac{\partial p_1}{\partial x}dx + \int_0^y \frac{\partial p_1}{\partial y}dy, \tag{5.29}$$

of the form

$$p_1 = \frac{27Re}{140 f^2}\left(\frac{1}{(1-\epsilon)^2} - \frac{1}{f^2}\right). \tag{5.30}$$

5.5.3. Second order pressure and solution

Comparing the coefficients of δ^2 , we arrive at

$$\frac{\partial p_2}{\partial x} = \frac{\partial^3 \psi_2}{\partial y^3} - Re\left[\frac{\partial\left(\psi_o, \frac{\partial \psi_1}{\partial y}\right)}{\partial(y,x)} + \frac{\partial\left(\psi_1, \frac{\partial \psi_o}{\partial y}\right)}{\partial(y,x)}\right], \tag{5.31}$$

$$\frac{\partial p_2}{\partial y} = -\frac{\partial^3 \psi_o}{\partial x \partial y^2}, \tag{5.32}$$

by integrating the equations (5.31)-(5.32) and making use of (5.5), (5.10) and (5.15) ,we arrive at the second order pressure as follows

$$\begin{aligned} p_2 = & \frac{3}{13475}\left[\frac{\pi\epsilon^2}{(1-\epsilon)^{\frac{3}{2}}}\left(13Re^2 + 8085\right)\tan^-1\left(\frac{\tan 2\pi x}{\sqrt{1-\epsilon}}\right)\right. \\ & \left. + f'\left\{\frac{40425\eta^2}{2f^2} + \frac{52Re^2 - 18865}{4f^2} + \frac{(\epsilon-2)(13Re^2 + 8085)}{4f(\epsilon-1)}\right\}\right]. \end{aligned} \tag{5.33}$$

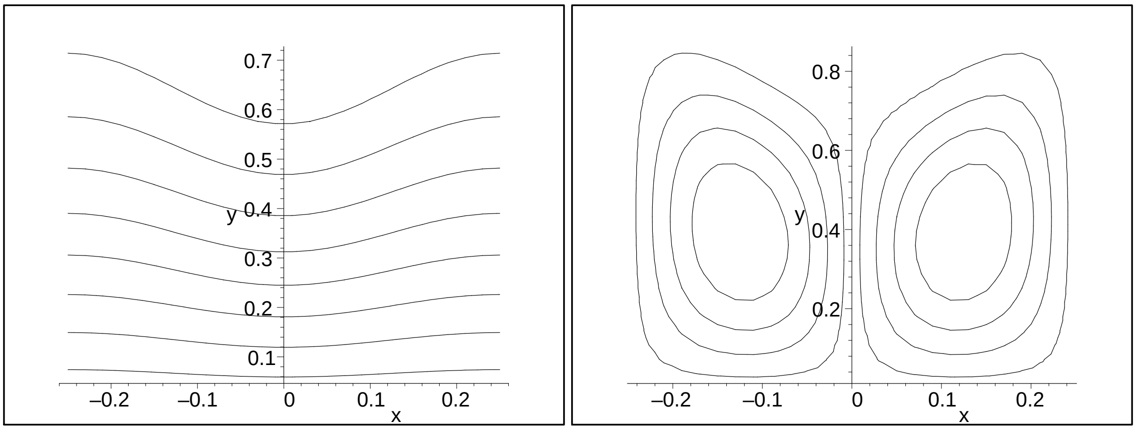
Now one can easily find the pressure up to second order by using equations (5.26), (5.30) and (5.33).

6. Graphical discussion

In this section the effect of different pertinent parameters on stream lines, wall shear stress, pressure distribution, separation and reattachment points and analysis for heat transfer are presented graphically. The geometry of the proposed model for the study of the stenosed artery is depicted in Figure 1. The radii of obstructed and unobstructed regions are $h(x)$ and h_o. The point of separation lies near the throat of the stenosed region in the converging section. Separation point means the point where reverse flow occurs. Figure 2,3 presents the behavior of stream lines for zeroth order in 2(a), first order in 2(b), second order in 3(a) and up to second order in 3(b) respectively, for the fixed values of $Re = 12, \epsilon = 0.2, \delta = 0.1, \alpha = 0.04$. In these figures $x - axis$ lies in the horizontal direction and $y - axis$ perpendicular to it. The zeroth order solution corresponds to the flow with vanishing wall slopes and reduces to the flow between parallel plates for $\epsilon = 0$. The stream lines are relatively straight in the center of the channel. The first order solution induces the clockwise and counterclockwise rotational motion in the converging and diverging regions, which indicates the separation point in the converging region and reattachment point in the diverging region. Figure 3(a) shows the stream lines for second order solution reinforce the first order solution and observe the rotational motion which predicts the separation and reattachment points. Figure 3(b) presents the stream lines up to second order. It is observed that the stream lines becomes relatively straight in the center of the channel as compares to the walls of the channel and similar to (Chow *et al.*, 1971).

The distribution of wall shear stress across the stenosis has been described for the variation of Re in figure 4(a) for fixed $\epsilon = 0.2, \delta = 0.1$. An increase in Re, wall shear stress increases near the throat of stenosed region and becomes negative in the converging and diverging section of channel due to back flow. The adverse shearing in converging and diverging sections of channel indicates that there is point of separation in the upstream region and reattachment point in the downstream region of channel. It is observed that wall shear stress holds for both small and large *Re*.

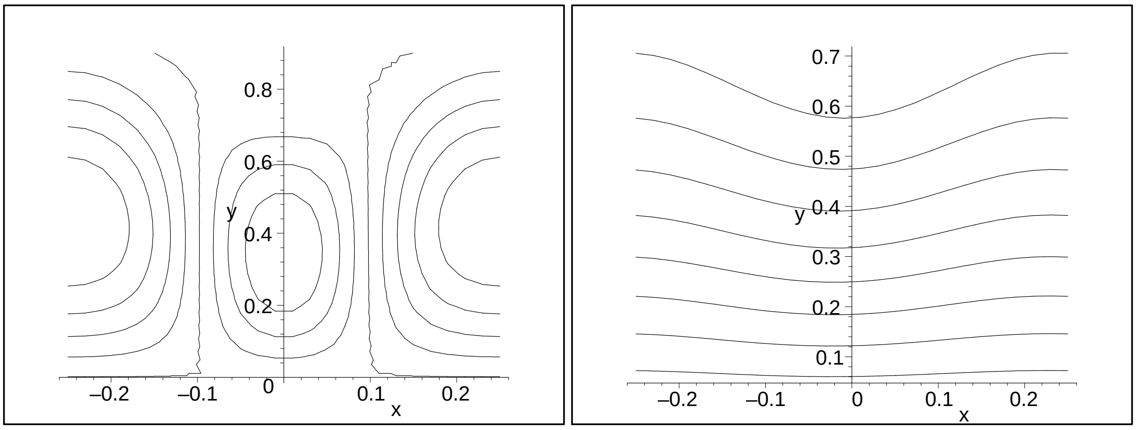
In figure 4(b) effect of ϵ on wall shear stress is presented. The straight line indicates that there is no stenosis and the flow is Poiseuille flow. By the increase in ϵ wall shear stress increases over the stenosed region and becomes negative in the converging section of channel due to adverse flow, which is prediction for the point of separation. The separation point was considered to be the point nearest the throat where reversed flow along the wall of channel could be observed. The point farthest down stream from the throat where back flow occur is defined as reattachment point. It is expected that the wall shear stress plays an important role in the formation of the stenosis and its further growth. Because the deposit of cholesterol and proliferation of connective tissue may be responsible for the abnormal growth in lumen of artery. Its actual cause may not be known exactly but its effect on the cardiovascular system can easily be understood by studying the blood flow in its vicinity. One of the practical applications of blood flow through a membrane oxygenator is the flow with an irregular wall surface.



(a) (b)

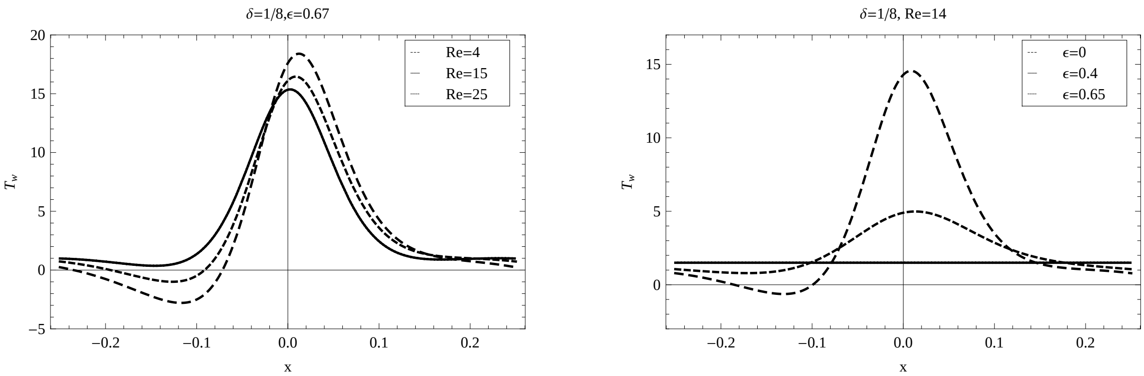
Figure 2. The zeroth order stream lines for $\epsilon = 0.20, Re = 12, \delta = 0.1$ are shown in (a) and the first order stream lines are shown in (b).

Figure 5(a) depict the distribution for the point of separation in converging section of channel for different ϵ along with fixed δ. The separation point lie to the right of minimum point, actually the purpose for zero wall shear stress is to find the critical Reynolds number where separation occur. The critical value of *Re* in the converging region for $\epsilon = 0.6$ is 70. The theory that the critical *Re* decreases with the increase in ϵ is verified. In figure 5(b) zero wall shear stress is plotted for ϵ having fixed value of δ in diverging section of channel. The aim of investigation is to determine the critical value of *Re* at which reattachment occurred in the diverging region of the channel. As the critical *Re* reached the reattachment occur in the diverging region of channel and separation point occur in the upstream region of channel. It is observed that the critical value of *Re* for $\epsilon = 0.6$ is 380. It is also observed form figure 6 that as ϵ increases critical value of *Re* decreases.



(a) (b)

Figure 3. The second order stream lines are shown in (a) and the streamlines correct up to the second order in δ are shown in (b).



(a) (b)

Figure 4. The effect of *Re* on wall shear stress is shown in (a) and the effect of ϵ on wall shear stress is shown in (b).

Figure 6(a) presents the effect for the various values of *Re* on pressure distribution. It may be noted that with the increase in *Re* leads to increase the pressure gradient over the stenosed region and becomes negative in the converging and diverging regions, due to the dependence of *Re* on average velocity. The adverse pressure gradient in these regions causes back flows as observed earlier. These back flows predicts the separation point in converging region and reattachment point in diverging region of the channel. It is observed that the magnitude of adverse pressure gradient in the diverging region is smaller as compared to that in the converging region.

Effect of ϵ on pressure gradient is studied in figure 6(b). It is observed that with the increase in ϵ, pressure gradient increases over the region having stenosis and becomes negative in the upstream and downstream regions of channel due to back flow. The adverse pressure gradient in the converging part of stenosis describing the flow separation and reattachment in the diverging

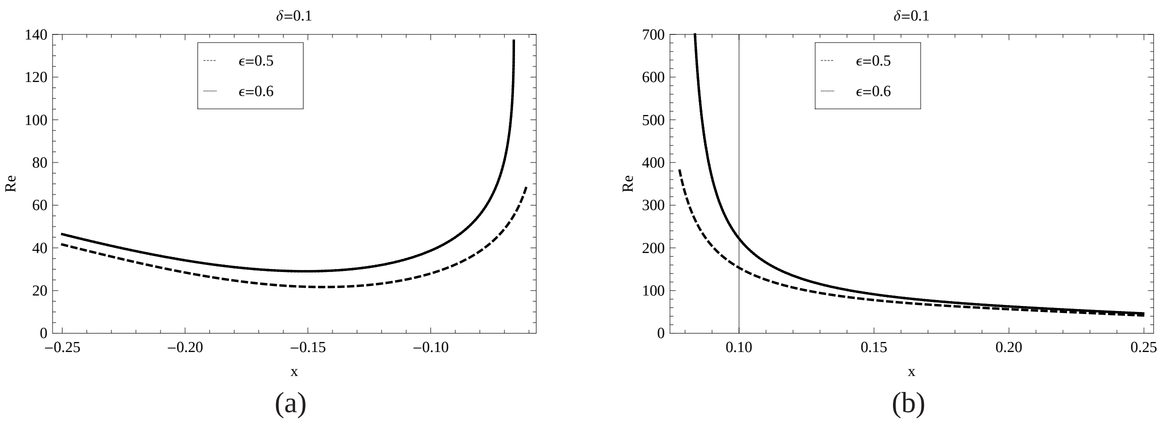


Figure 5. The separation point in converging region are shown in (a) and the reattachment point in diverging region are shown in (b).

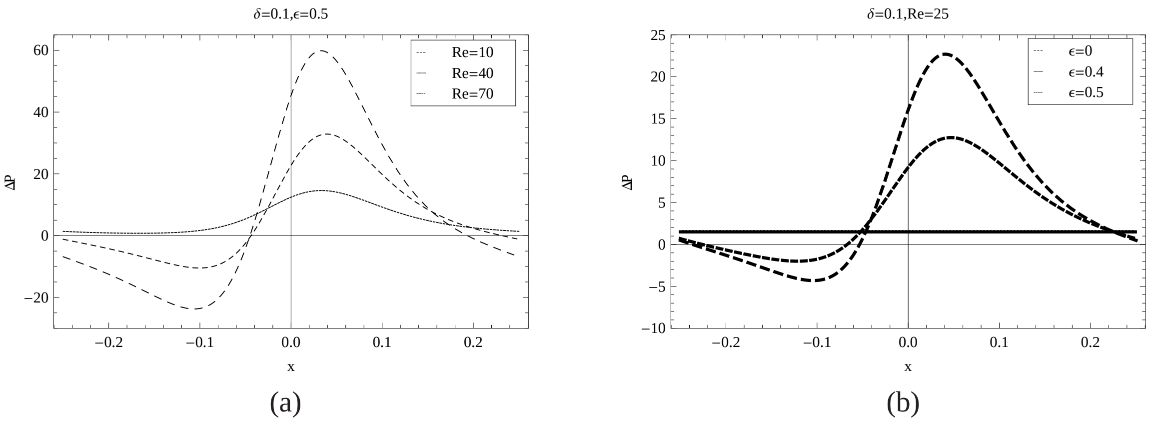
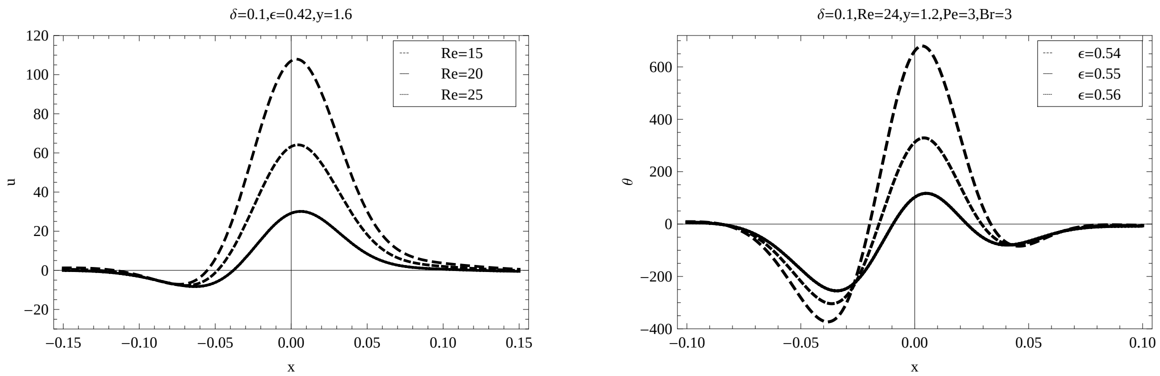


Figure 6. The pressure distribution for *Re* is shown in (a) and the pressure distribution for ϵ is shown in (b).

part. The straight line preserve Poiseuille flow as there is no stenosis.

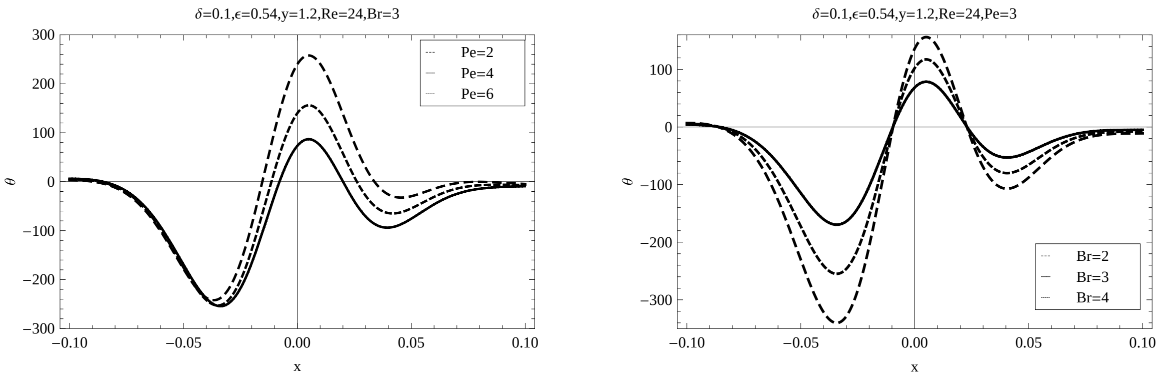
Figure 7(a) depict for various values of *Re* on axial component of velocity. It is observed that with the increase in *Re* the axial velocity is maximum over the obstructive region and becomes negative causing back flow in the converging and diverging sections of the channel. Figure 7(b) shows the effect of ϵ on velocity distribution. It is observed that as the ϵ increases the velocity increases over the stenosed region and decreases sharply in the converging section and then recover it in the diverging section of channel. Negative velocity indicates the back flow, due to separation and reattachment points in the channel.

Figure 8(a) shows the effect of *Pe* on temperature distribution. It is observed that with the increase in *Pe*, temperature increases over the stenosed region and becomes negative in the converging and diverging regions. The adverse temperature in the upstream and downstream sections



(a) (b)

Figure 7. The axial velocity distribution for *Re* is shown in (a) and the axial velocity distribution for ϵ is shown in (b).



(a) (b)

Figure 8. The effect of *Pe* on temperature distribution is shown in (a) and the effect of *Br* on temperature distribution is shown in (b).

describing flow separation and reattachment from the wall also confirm the results for velocity and wall shear stress. Temperature distribution across the stenosis has been described in figure 8(b) for different values of *Br*. Temperature increases steeply from its axial axis in the converging section of the stenosis to the peak value at the throat, then drop to a minimum value downstream behind the stenosis and again approaches to the axial axis in the region away from stenosis. The magnitude of adverse temperature in the diverging region of stenosis is smaller as compared to that in the converging section of the stenosis. The adverse temperature in these regions cause back flow as observed earlier in velocity and pressure fields.

7. Summary

In the present study, steady two-dimensional flow of incompressible Newtonian fluid with heat transfer between two parallel plates in the presence of a cosine shaped stenosis is presented. The underlying problem is solved with the help of the regular perturbation method. The results thus obtained are discussed graphically in terms of stream lines, pressure gradient, wall shear stress, separation and reattachment points and temperature distribution. It is observed that the general pattern of streamlines is similar as discussed in (Layek & Midya, 2007) - (Chow *et al.*, 1971), wall shear stress is same as given by (Morgan & Young, 1974) - (Haldar, 1991) and separation and reattachment points are in agreement with (Haldar, 1991). It is observed that:

- Stream lines for zeroth order and up to second order are similar due to small δ and first and second order shows rotational motion.
- Increase in Reynolds number increases the wall shear stress, velocity and pressure gradient.
- Increase in thickness of stenosis increases pressure gradient, temperature and wall shear stress causing separation and reattachment in the channel.
- Increase in the thickness of stenosis decreases the critical Reynolds number for separation and reattachment points, means even at low velocity separation occurs.
- By the increase in Peclet and Brinkman number increases the temperature between the channel.
- For $\epsilon = 0$ Poiseuille flow is recovered.

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Appendix 1

List of Mathematical Symbols

$\mathbf{V}$	Velocity vector (m/s)
∇	del operator
p	scalar pressure(Pa)
d/dt	material time derivative
c_p	specific heat(J/kgK)
T	temperature(C^o)
u, v	velocity components(m/s)
x, y	coordinate axis(m)
$\mathbf{A}_1$	first Rivlin-Ericksen tensor
$l_o/2$	length of stenosis(m)
$h(x)$	variable width between the stenosis(m)
h_o	radius of unobstructed channel(m)
u_o	average velocity(m/s)
Q	volume flow rate(m^3/s)
T_1, T_o	temperatures on boundary of stenosis and fluid(C^o)
Re	Reynolds number
Br	Brinkman number
Pe	Peclet number
$f(x)$	boundary profile
τ	extra stress tensor
ρ	density
κ	thermal conductivity
ϕ	viscous dissipation function
μ	dynamic viscosity(Pa/s)
$\top$	transpose
λ, ϵ	maximum height of stenosis
θ	dimensionless temperature
ν	kinematic viscosity(m^2/s)
ψ	stream function
δ	constant
η	ratio of y and f
τ_ω	wall shear stress

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Starlikeness and Convexity of Certain Classes of Meromorphically Multivalent Functions

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Abstract

The purpose of this paper is to investigate the problems of finding the order of starlikeness and the order of convexity of the products of certain meromorphically p-valent functions belonging to some interesting classes of β-uniformly p-valent starlike functions and β-uniformly p-valent convex functions in the open unit disk $\mathbb{U}$. The main results presented in the paper are capable of being specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

Keywords: Analytic functions, Meromorphically p-valent starlike functions, Meromorphically p-valent convex functions, Products of meromorphic functions, Uniformly starlike functions, Uniformly convex functions.
2010 MSC: Primary 30C45.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of all functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1.$$

A function $f(z) \in \mathcal{A}$ is said to be *uniformly convex* (or *uniformly starlike*) in $\mathbb{U}$ if, for every circular arc Γ contained in $\mathbb{U}$, with center at ω_0 also in $\mathbb{U}$, the arc $f(\Gamma)$ is convex (or starlike) with respect to the point $f(\omega_0)$. The classes of all uniformly convex function in $\mathbb{U}$ and all uniformly starlike

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functions in $\mathbb{U}$ are denoted by UCV and UST, respectively. These analytic function classes UCV and UST were introduced and studied by Goodman (Goodman, 1991*a*,*b*) who showed, among other things, that

$$f \in UCV \Longleftrightarrow \Re\left(1 + (z-\zeta)\,\frac{f''(z)}{f'(z)}\right) \geqq 0 \qquad (z,\zeta \in \mathbb{U})$$

and

$$f \in UST \Longleftrightarrow \Re\left(\frac{(z-\zeta)\,f'(z)}{f(z)-f(\zeta)}\right) \geqq 0 \qquad (z,\zeta \in \mathbb{U}).$$

Rønning (Rønning, 1993, 1994) and Ma and Minda (Ma & Minda, 1992) gave the following one-variable characterization of the class UCV of uniformly convex functions in $\mathbb{U}$.

Theorem A. *A function $f(z) \in \mathcal{A}$ is said to be in the class UCV of uniformly convex functions in $\mathbb{U}$ if it satisfies the following condition*:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqq \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}).$$

Since the Alexander type result that

$$f \in UCV \Longleftrightarrow zf'(z) \in UST$$

does not hold true (Rønning, 1994), the class $\mathcal{S}_{\mathrm{p}}$ defined by

$$\mathcal{S}_{\mathrm{p}} := \{f : zf'(z) \in UCV\}$$

was introduced by Rønning (Rønning, 1993). On the other hand, Shams *et al.* (Shams *et al.*, 2004) initiated a study of the class $SD(\alpha,\beta)$ of β-uniformly starlike functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ consisting of functions $f(z) \in \mathcal{A}$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (\beta \geqq 0;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

The class $KD(\alpha,\beta)$ of β-uniformly convex of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ is defined as follows:

$$f \in KD(\alpha,\beta) \Longleftrightarrow zf'(z) \in SD(\alpha,\beta).$$

Motivated by the above-defined function classes $SD(\alpha,\beta)$ and $KD(\alpha,\beta)$, Nishiwaki and Owa (Nishiwaki & Owa, 2007) introduced the class $MD(\alpha,\beta)$ consisting of all functions $f(z) \in \mathcal{A}$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) < \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (\beta \leqq 0;\ \alpha > 1;\ z \in \mathbb{U}).$$

The function class $ND(\alpha,\beta)$ may also be considered as a subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in MD(\alpha,\beta)$.

The class of uniformly convex functions and various other related function classes have been studied by several authors (see, for example, (Ali & Ravichandran, 2010; Frasin, 2011; Kanas & Srivastava, 2000; Kanas & Wisniowska, 1999, 2000; Murugusundaramoorthy & Magesh, 2004; Rønning, 1991); see also (Srivastava & Owa (Editors), 1992)).

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphically p-valent starlike functions of order α in $\mathbb{U}$ if and only if

$$\Re\left[\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < -\alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.2}$$

Also a function $f \in \Sigma_p$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex functions of order α in $\mathbb{U}$ if and only if

$$\Re\left[\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < -\alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$f(z) \in \Sigma C_p(\alpha) \Longleftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}$$

We note that the meromorphically p-valent function classes $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ were introduced by Kumar and Shukla (Kumar & Shukla, 1982).

We next denote by $\Sigma M_p(\alpha)$ and $\Sigma N_p(\alpha)$ the subclasses of the meromorphically p-valent function class Σ_p which satisfy the following inequalities:

$$\Sigma M_p(\alpha) := \left\{ f : f \in \Sigma_p \quad \text{and} \quad \Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < \alpha \quad (z \in \mathbb{U};\ \alpha > 1) \right\}$$

and

$$\Sigma N_p(\alpha) := \left\{ f : f \in \Sigma_p \quad \text{and} \quad \Re\left[-\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < \alpha \quad (z \in \mathbb{U};\ \alpha > 1) \right\},$$

respectively. The meromorphically p-valent function classes $\Sigma M_p(\alpha)$ and $\Sigma N_p(\alpha)$ are analogous, respectively, to the subclasses $M(\alpha)$ and $N(\alpha)$ of the analytic function class $\mathcal{A}$ which were introduced by Owa and Nishiwaki (Owa & Nishiwaki, 2002).

Recently, Kumar *et al.* (Kumar *et al.*, 2005) introduced the following subclass $\Sigma\mathcal{S}_p^*(\alpha,\beta)$ of meromorphically p-valent starlike functions $f \in \Sigma_p$ in $\mathbb{U}$, which is similar to the class $SD(\alpha,\beta)$, by means of the following inequality:

$$\Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] > \alpha\left|\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)+1\right|+\beta \tag{1.5}$$

$$(z \in \mathbb{U};\ \alpha \geqq 0;\ 0 \leqq \beta < 1).$$

Analogously, we define here the subclass $\Sigma\mathcal{C}_p(\alpha,\beta)$ of meromorphically p-valent convex functions in $\mathbb{U}$, which is similar to the class $KD(\alpha,\beta)$, consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)+1\right|+\beta \tag{1.6}$$

$$(z \in \mathbb{U};\ \alpha \geqq 0;\ 0 \leqq \beta < 1).$$

Similarly, for $-1 < \alpha \leqq 0$ and $\beta > 1$, we let $\Sigma\mathcal{M}_p(\alpha,\beta)$ be the subclass consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < \alpha\left|\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)+1\right|+\beta \tag{1.7}$$

$$(z \in \mathbb{U};\ -1 < \alpha \leqq 0;\ \beta > 1).$$

We also let $\Sigma\mathcal{N}_p(\alpha,\beta)$ be the subclass consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] < \alpha\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)+1\right|+\beta \tag{1.8}$$

$$(z \in \mathbb{U};\ -1 < \alpha \leqq 0;\ \beta > 1).$$

The main purpose of this paper is to investigate the problems of finding the order of starlikeness and the order of convexity of certain products of meromorphically p-valent functions belonging to some of the above-defined classes of β-uniformly p-valent starlike functions in $\mathbb{U}$ and β-uniformaly p-valent convex functions in $\mathbb{U}$. Our main results in Section 2 (stated as Theorems 1 to 4 *and* Corollaries 1 to 5) can indeed be specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

2. The main results and their consequences

Our first main result is asserted by Theorem 1 below.

Theorem 1. *Let* $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \textit{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the product $F_p(z)$ defined by

$$F_p(z) := z^{-p} \prod_{j=1}^{n} \left\{ z^p f_j(z) \right\} \tag{2.1}$$

is in the class $\Sigma S_p^*(\kappa)$ of meromorphically p-valent starlike functions of order κ in $\mathbb{U}$.

Proof. Clearly, $F_p(z) \in \Sigma_p$. By differentiating (2.1) logarithmically with respect to z, we obtain

$$\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right) = -1 + \sum_{j=1}^{n}\left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + 1\right], \tag{2.2}$$

which readily yields

$$\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right) = -1 + \left(1 - \gamma_j\right) + \sum_{j=1}^{n}\left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + \gamma_j\right]. \tag{2.3}$$

We thus find that

$$\Re\left[\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right)\right] = -1 + \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{n}\Re\left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + \gamma_j\right]. \tag{2.4}$$

Since, by hypothesis, $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, we have

$$\Re\left[\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right)\right] < -\left(1 - \sum_{j=1}^{n}\alpha_j\right) =: \kappa, \tag{2.5}$$

which evidently completes the proof of Theorem 1. □

Upon setting

$$f_j(z) = f(z), \quad \gamma_j = \gamma \quad \text{and} \quad \alpha_j = \alpha \qquad (j = 1, \cdots, n)$$

in Theorem 1, we have the following corollary.

Corollary 1. *Let $f \in \Sigma S_p^*(\gamma) \quad (\gamma := 1 - \alpha \geqq 0)$, where $\alpha \geqq 0$. Also let $1 - n\alpha \geqq 0$. Then the product $\Theta_p(z)$ defined by*

$$\Theta_p(z) := z^{-p}\left[z^p f(z)\right]^n$$

is in the class $\Sigma S_p^*(1 - n\alpha)$ of meromorphically p-valent starlike functions of order $1 - n\alpha$ in $\mathbb{U}$.

Corollary 2. *Let* $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the function $\Phi_p(z)$ *defined by*

$$\Phi_p(z) := -p \int_0^z t^{-p-1} \prod_{j=1}^{n} \left\{ t^p f_j(t) \right\} \mathrm{d}t \tag{2.6}$$

is in the class $\Sigma C_p(\kappa)$ *of meromorphically p-valent convex functions of order* κ *in* $\mathbb{U}$.

Proof. The result asserted by Corollary 2 follows immediately from Theorem 1, since

$$\Phi_p(z) \in \Sigma C_p(\kappa) \Longleftrightarrow -\frac{z\Phi'(z)}{p} =: F_p(z) \in \Sigma S_p^*(\kappa).$$

□

Corollary 3. *Let* $f_j \in \Sigma C_p(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the product $G_p(z)$ *defined by*

$$G_p(z) = z^{-p} \prod_{j=1}^{n} \left\{ -\left(\frac{z^{p+1} f_j'(z)}{p} \right) \right\} \tag{2.7}$$

is in the class $\Sigma S_p^*(\kappa)$ *of meromorphically p-valent starlike functions of order* κ *in* $\mathbb{U}$.

Proof. From the fact that

$$f_j(z) \in \Sigma C_p(\gamma_j) \Longleftrightarrow -\frac{z f_j'(z)}{p} \in \Sigma S_p^*(\gamma_j) \qquad (j = 1, \cdots, n),$$

by replacing $f_j(z)$ by $-\frac{z f_j'(z)}{p}$ in Theorem 1, we are led easily to Corollary 3. □

Corollary 4. *Let* $f_j \in \Sigma C_p(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the function $\Psi_p(z)$ *defined by*

$$\Psi_p(z) = -p \int_0^z t^{-p-1} \prod_{j=1}^{n} \left\{ -\left(\frac{t^{p+1} f_j'(t)}{p} \right) \right\} \mathrm{d}t \tag{2.8}$$

is in the class $\Sigma C_p(\kappa)$ *of meromorphically p-valent convex functions of order* κ.

Proof. The result asserted by Corollary 4 follows immediately from Corollary 3, since

$$\Psi_p(z) \in \Sigma C_p(\kappa) \Longleftrightarrow -\frac{z\Psi_p'(z)}{p} =: G_p(z) \in \Sigma S_p^*(\kappa).$$

□

By applying the same method and technique as in our proofs of Theorem 1 as well as of Corollaries 2, 3 and 4, we can establish Theorem 2 below.

Theorem 2. *Let* $f_j \in \Sigma_p \quad (j = 1, \cdots, n)$. *Suppose that*

$$\gamma_j := 1 + \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\sigma := 1 + \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then each of the following assertions holds true:

(i) *If* $f_j \in \Sigma M_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the product* $F_p(z)$ *defined by* (2.1) *is in the class* $\Sigma M_p(\sigma)$.

(ii) *If* $f_j \in \Sigma M_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the integral operator* Φ_p *defined by* (2.6) *is in the class* $\Sigma N_p(\sigma)$.

(iii) *If* $f_j \in \Sigma N_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the product* $G_p(z)$ *defined by* (2.7) *is in the class* $\Sigma M_p(\sigma)$.

(iv) *If* $f_j \in \Sigma N_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the integral operator* Ψ_p *defined by* (2.8) *is in the class* $\Sigma N_p(\sigma)$.

Theorem 3. *Let*

$$\alpha_j \geqq 0 \qquad \text{and} \qquad 0 \leqq \beta_j < 1 \qquad (j = 1, \cdots, n)$$

and suppose that

$$\delta := 1 - \sum_{j=1}^{n} \left(\frac{1 - \beta_j}{1 + \alpha_j} \right).$$

Also let the products $F_p(z)$ *and* $G_p(z)$ *be defined by* (2.1) *and* (2.7), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j = 1,\cdots,n)$, *then* $F_p(z) \in \Sigma S_p^*(\delta)$.

(ii) *If* $f_i \in \Sigma\mathcal{C}_p(\alpha_j,\beta_j) \quad (j = 1,\cdots,n)$, *then* $G_p(z) \in \Sigma S_p^*(\delta)$.

Proof. By following the lines as in (Kumar *et al.*, 2005), we first prove that

$$\Sigma\mathcal{S}_p^*(\lambda,\mu) \subset \Sigma S_p^*\left(\frac{\lambda+\mu}{1+\lambda}\right).$$

Indeed, if we let $f \in \Sigma\mathcal{S}_p^*(\lambda,\mu)$, then the quantity w defined by

$$w := \frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)$$

satisfies the following inequality:

$$-\Re(w) - \mu \geqq \lambda\,|w+1| \geqq \lambda\Re\,(w+1)\,,$$

which immediately yields

$$-\Re(w) \geqq \frac{\lambda+\mu}{1+\lambda}.$$

We thus have

$$f \in \Sigma\mathcal{S}_p^*(\lambda,\mu) \Longrightarrow f \in \Sigma S_p^*\left(\frac{\lambda+\mu}{1+\lambda}\right).$$

Next, since

$$f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j = 1,\cdots,n),$$

we have

$$f_j \in \Sigma S_p^*\left(\frac{\alpha_j+\beta_j}{1+\alpha_j}\right) \quad (j = 1,\cdots,n),$$

The assertion (i) of Theorem 3 now follows readily from an application of Theorem 1.

The proof of the assertion (ii) of Theorem 3 follows similarly by using Corollary 3. □

Corollary 5. *Let*

$$\alpha_j \geqq 0 \qquad \text{and} \qquad 0 \leqq \beta_j < 1 \qquad (j = 1,\cdots,n)$$

and suppose that

$$\delta := 1 - \sum_{j=1}^{n}\left(\frac{1-\beta_j}{1+\alpha_j}\right).$$

Also let the functions $\Phi_p(z)$ *and* $\Psi_p(z)$ *be defined by* (2.6) *and* (2.8), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j = 1,\cdots,n)$, *then* $\Phi_p(z) \in \Sigma\mathcal{C}_p(\delta)$.

(ii) *If* $f_j \in \Sigma\mathcal{C}_p(\alpha_j,\beta_j) \quad (j = 1,\cdots,n)$, *then* $\Psi_p(z) \in \Sigma\mathcal{C}_p(\delta)$.

Proof. The results asserted by Corollary 5 would follow immediately from Theorem 3, since

$$\Phi_p(z) \in \Sigma C_p(\delta) \Longleftrightarrow -\frac{z\Phi_p'(z)}{p} =: F_p(z) \in \Sigma S_p^*(\delta)$$

and

$$\Psi_p(z) \in \Sigma C_p(\delta) \Longleftrightarrow -\frac{z\Psi'(z)}{p} =: G_p(z) \in \Sigma S_p^*(\delta).$$

□

Finally, if we make use of the same method and technique as in our proofs of Theorem 3 and Corollary 5, we are led easily to Theorem 4 below.

Theorem 4. *Let*

$$-1 < \alpha_j \leqq 0 \qquad \textit{and} \qquad \beta_j > 1 \qquad (j = 1, \cdots, n)$$

and suppose that

$$\nu := 1 + \sum_{j=1}^{n} \left(\frac{\beta_j - 1}{1 + \alpha_j} \right).$$

Also let the products $F_p(z)$ *and* $G_p(z)$ *be defined by* (2.1) *and* (2.7), *respectively, and the functions* $\Phi_p(z)$ *and* $\Psi_p(z)$ *be defined by* (2.6) *and* (2.8), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{M}_p(\alpha_j, \beta_j) \quad (j = 1, \cdots, n)$, *then* $F_p(z) \in \Sigma M_p(\nu)$.

(ii) *If* $f_j \in \Sigma\mathcal{N}_p(\alpha_j, \beta_j) \quad (j = 1, \cdots, n)$, *then* $G_p(z) \in \Sigma M_p(\nu)$.

(iii) *If* $f_j \in \Sigma\mathcal{M}_p(\alpha_j, \beta_j) \quad (j = 1, \cdots, n)$, *then* $\Phi_p(z) \in \Sigma N_p(\nu)$.

(iv) *If* $f_j \in \Sigma\mathcal{N}_p(\alpha_j, \beta_j) \quad (j = 1, \cdots, n)$, *then* $\Psi_p(z) \in \Sigma N_p(\nu)$.

3. Concluding remarks and observations

In our present investigation, we have considered several interesting subclasses of the familiar class of meromorphically p-valent functions in the open unit disk $\mathbb{U}$. Our main purpose has been to successfully address the problems of finding the order of starlikeness and the order of convexity of the products of functions belonging to each of the various classes of β-uniformly p-valent starlike functions and β-uniformly p-valent convex functions in $\mathbb{U}$, which we have introduced here. The main results (stated as Theorems 1 to 4 *and* Corollaries 1 to 5) can indeed be specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

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Satellite Formation Control Using the Approximating Sequence Riccati Equations

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Abstract

In this study we develop a reliable algorithm to control the satellite formation using the Approximating Sequence of Riccati Equations(ASRE) minimizing the fuel consumption and the deviation of the orbit from the nominal orbit. The nonlinear Clohessy -Wiltshire(CW) equations of motions are used to describe the motion of the satellite formation about a virtual reference position maintained at the formation center. The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equations(ASRE) method. The technique is based on Linear Quadratic Regulator(LQR) with fixed terminal state, which guarantees closed loop solution.

Keywords: Nonlinear Feedback, Linear Quadratic Regulator, Approximation Sequence Riccati Equation, Satellite Formation.

1. Introduction

Satellite formation flying is one of the space dynamics branches which gained much consideration in recent years. Despite the topic evolved two decades ago, the implementation of formation flying is not yet mature.

A satellite formation consists of two or more satellite flying together in close proximity, cooperating together to achieve some space mission such as terrestrial or deep space one. This system of distributed satellites has several advantages over the single satellite system such as, larger capability, reliability, flexibility, and more importantly less cost. *Satellite formation in contrast to satellite constellation in which the satellites are moving independently, the satellites affecting each other in co orbital motion about a virtual reference position maintained at the formation center.* The

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nonlinear Clohessy -Wiltshire equations of motions are used to describe the motion of the satellite formation. CW equations are developed for rendezvous(Clohessy & Wiltshire, 1960). Later on the linear inhomogeneous CW are studied(Meirovitch, 1970). CW equations have been solved by simplifying the nonlinear equations of motion via coordinate transformation of the central gravity field dynamics in presence of quadratic drag force(Thomas Carter, 2002).

The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equations method. The most common way of solving the orbit rendezvous of a satellite is the low thrust orbit rendezvous approach, which is a nonlinear optimal control problem. In the open loop context the problem can be solved via indirect and then direct method. The indirect method was developed through Pontryagin Maximum Principle(PMP) (A. J. Bryson, 1975),(L. Pontryagin & Mishchenko, 1952) . The direct method was developed using the Karush-Kuhn-Tucker(KKT) algebric equation(Enright & Conway, 1992).

one of the most common methods for solving the nonlinear feedback optimal control problem in the is the State Dependent Riccati Equations(SDRE) (Cimen, 2006). The Approximating Sequence of Riccati Equations (Cimen, 2004) technique is an iterative approach to solve the nonlinear optimal control problem. The ASRE is developed(Topputo & Bernelli-Zazzera, 2012) using the state transition matrix. By the virtue of the closed-loop nature of this control law, a trajectory designed in this way has the property to respond to perturbations acting during the transfer that continuously alter the state of the spacecraft. The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation: this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers (A. J. Bryson, 1975). Recently, the nonlinear feedback control of circular coplanar low-thrust orbital transfers has been faced using continuous orbital elements feedback and Lyapunov functions(Chang & Marsden, 2002) and proved optimal by(Alizadah & Villac, 2011). Later on the problem has been solved using the primer vector approximation method(Haung, 2012). The problem is tackled using the Approximating Sequence Riccati Equation(ASRE) method based on Linear Quadratic Regulator(LQR) with fixed terminal state and the method is applied to GNSS circular constellation (Owis, 2013). In this work the control of the satellite formation described in the Earth Centered Earth Fixed Frame Fig. 1 is developed.

Linear Quadratic Regulator(LQR) with Fixed Terminal State

Consider the following system with linear dynamics and quadratic performance index as follows:

$$\dot{X} = AX + BU, \qquad X(t_0) = X_0 \in \mathbb{R}^n, \tag{1.1}$$

the following performance index

$$J = X_f^T Q_f X_f + \frac{1}{2}\int_{t_0}^{t_f} [X^T QX + U^T RU]dt, \tag{1.2}$$

Where A , B , Q , and R are constant coefficients matrices of the suitable dimen-

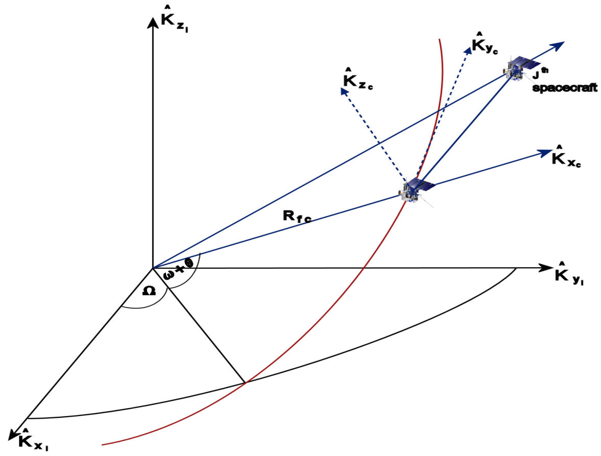

Figure 1. Satellite Formation Flying in the Earth Centered Earth Fixed Frame

sions. we have to find the m-dimensional control functions $U(t)$, $t \in [t_0\ t_f]$ which minimizes the J, which is an open loop (with t_0 fixed) optimal control. We optimize the performance index J, by adjoining the dynamics and the performance index (integrand) to form the Hamiltonian:

$$H(X, \lambda, U, t) = \frac{1}{2}(X^T QX + U^T RU) + \lambda^T (A(t)X + B(t)U),$$

where the Lagrange multiplier λ is called the adjoint variable or the costate. The necessary conditions for optimality are:

1. $\dot{X} = H_\lambda = A(t)X + B(t)U, \qquad X(t_0) = X_0,$
2. $\dot{\lambda} = -H_x = -QX - A^T\lambda, \qquad \lambda(t_f) = Q_f X_f,$
3. $H_u = 0 \Longrightarrow RU + B^T\lambda = 0 \Longrightarrow U^\star = -R^{-1}B^T\lambda.$

To find the minimum solution we have to check for $H_{uu} = \dfrac{\partial^2 H}{\partial \lambda^2} > 0$ or equivalently $R > 0$. Now we have that

$$\dot{X} = AX + BU^\star = AX - BR^{-1}B^T\lambda,$$

which can be combined to the the equation of the costate as follows

$$\begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ \lambda \end{bmatrix}, \tag{1.3}$$

which is called the Hamiltonian matrix, it represents a $2n$ boundary value problem with $X(t_0) = X_0$ and, $\lambda(t_f) = Q_f X_f$.

We can solve this $2n$ boundary value problem using the transition matrix method as follows. Let's define a transition matrix

$$\phi_(t_1, t_0) = \begin{bmatrix} \phi_{11}(t_1, t_0) & \phi_{12}(t_1, t_0) \\ \phi_{21}(t_1, t_0) & \phi_{22}(t_1, t_0) \end{bmatrix},$$

we use this matrix to relate the current values of X and λ to the final values X_f and λ_f as follows

$$\begin{bmatrix} X \\ \lambda \end{bmatrix} = \begin{bmatrix} \phi_{11}(t, t_f) & \phi_{12}(t, t_f) \\ \phi_{21}(t, t_f) & \phi_{22}(t, t_f) \end{bmatrix} \begin{bmatrix} X(t_f) \\ \lambda(t_f) \end{bmatrix},$$

so we have

$$\begin{aligned} X &= \phi_{11}(t, t_f)X(t_f) + \phi_{12}(t, t_f)\lambda(t_f) \\ &= [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]X(t_f), \end{aligned}$$

we can eliminate $X(t_f)$ to get

$$\begin{aligned} X &= [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f][\phi_{11}(t_0, t_f) + \phi_{12}(t_0, t_f)Q_f]^{-1}X(t_0) \\ &= X(t, X_0, t_0), \end{aligned}$$

now we can find $\lambda(t)$ in terms of $X(t_f)$ as

$$\lambda(t) = [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f]X(t_f),$$

then we can eliminate $X(t_f)$ to get

$$\begin{aligned} \lambda(t) &= [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f][\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]^{-1}X(t), \\ &= \phi_{\lambda x}X(t). \end{aligned}$$

Now we search a solution for $\phi_{\lambda x}$. By differentiating $\lambda(t)$ we get

$$\dot{\lambda}(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t).$$

Comparing the last equation with the Hamiltonian matrix we get

$$-QX(t) - A^T\lambda(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t),$$

then we have

$$
\begin{aligned}
-\dot{\phi}_{\lambda x}(t)X(t) &= QX(t) + A^T\lambda(t) + \phi_{\lambda x}\dot{X}(t) \\
&= QX(t) + A^T\lambda(t) + \phi_{\lambda x}(AX - BR^{-1}B^T\lambda(t)) \\
&= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\lambda(t) \\
&= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\phi_{\lambda x}X(t) \\
&= [Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}]X(t).
\end{aligned}
$$

Since this is true for arbitrary $X(t)$, $\phi_{\lambda x}$ must satisfy

$$-\dot{\phi}_{\lambda x}(t) = Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}, \tag{1.4}$$

which is the matrix differential Riccati Equation . We can solve for $\phi_{\lambda x}$ by solving Riccati Equation backwards in time from t_f with $\phi_{\lambda x}(t_f) = Q_f$. The optimal control is then given by

$$U^\star = -R^{-1}B^T\lambda(t) = -R^{-1}B^T\phi_{\lambda x}X = -K(t)X(t, X_0, t_0). \tag{1.5}$$

From 1.5 we notice that the optimal control is a linear full-state feedback control, therefore the linear quadratic terminal controller is feedback by default.

2. The Approximating Sequence of Riccati Equations(ASRE)

Assume that we have the following nonlinear system

$$\dot{X} = f(X, U, t) \tag{2.1}$$

$$X(t_0) = X_0, \quad X(t_f) = X_f \in R^n \tag{2.2}$$

with performance index

$$J = \phi(X_f, t_f) + \int_{t_0}^{t_f} L(X, U, t)dt \tag{2.3}$$

This system can be rewritten in the state dependent quasi-linear system as follows

$$\dot{X}^i = A(X^{i-1})X^i + B(X^{i-1})U^i \tag{2.4}$$

$$X(t_0) = X_0^0, \quad X(t_f) = X_f^n \in R^n \tag{2.5}$$

$$J = {X_f^i}^T Q(X_f^{i-1}) X_f^i + \frac{1}{2}\int_{t_0}^{t_f} [{X^i}^T Q(X^{i-1}) X^i \ + {U^i}^T R(X^{i-1}) U^i] dt, \tag{2.6}$$

where i represents the iteration step over the time interval $[t_i - 1, t_i]$ Fig. the technique is based of the previously introduced Linear Quadratic Regulator with fixed terminal state, which is a full state feedback and therefore the obtained solution will be a closed loop one, I.e. able to respond to the unexpected change in the inputs. The technique works as follows: the initial state is used to compute A_0, and B_0 and we solve for the first LQR iteration and compute X^1 and then used to compute new value of A_1, and B_1 for the second iteration until the final state error reaches a value below a set threshold.

3. Satellite formation control

Consider a satellite in the central gravity field. The eqaution of motion can be written in the cartesian frame as follows

$$\ddot{\mathbf{r}}) = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{f}}{m} \tag{3.1}$$

Where μ is the gravitational constant of the Earth$(3.986005 \times 10^{14} m^3/s^2)$. In the rotating coordinate frame along a circular orbit at a constant angular velocity, the position, velocity, and the acceleration become

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \delta\mathbf{r} = (R + x)i + yj + zk \\ \dot{\mathbf{r}} &= (\dot{x} - \omega y)i + [(\dot{y} + \omega(R + x)]j + \dot{z}k \\ \ddot{\mathbf{r}} &= [\ddot{x} - 2\omega\dot{y} - \omega^2(R + x)]i + [(\ddot{y} + 2\omega\dot{x}) - \omega^2 y]j + \ddot{z}k \end{aligned} \tag{3.2}$$

Plugging third equation of (3.2) into equ. (3.1) and substituting $r = \sqrt{[(R + x)^2 + y^2 + z^2]}$ we get

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - \omega^2(R + x) &= -\frac{\mu}{r^3}(R + x) + U_x \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y &= -\frac{\mu}{r^3}y + U_y \\ \ddot{z} &= -\frac{\mu}{r^3}z + U_z \end{aligned} \tag{3.3}$$

If we nondimensionalize the problem by setting the radius of the reference orbit $R = 1$ and reference time $\frac{1}{\omega}$ and in this system of units the gravitational constant μ is unity the nondimensionalized equation of motion can be written as

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + x)(\frac{1}{r^3} - 1) &= U_x \\ \ddot{y} + 2\dot{x} + y(\frac{1}{r^3} - 1) &= U_y \\ \ddot{z} + \frac{1}{r^3} z &= U_z \end{aligned} \tag{3.4}$$

where $r = \sqrt{[(1 + x)^2 + y^2 + z^2]}$, for simplicity we consider the in plan motion. We define the state vector of the system

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} \tag{3.5}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} \tag{3.6}$$

Then Equation (3.4) can be written in the form :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{3.7}$$

Choosing a suitable factorization equation (3.7) is rewritten in the factored state variable form :

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{3.8}$$

where :

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Gamma + \frac{\Gamma}{x_1} & 0 & 0 & 2 \\ 0 & \Gamma & 2 & 0 \end{bmatrix} \tag{3.9}$$

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.10}$$

where $\Gamma = \frac{1}{r^3} - 1$

4. Factored Controllability

For the factored system (3.8) the controllability is established by verifying that the controllability matrix

$$\mathbf{M}_{cl} = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \mathbf{A}^3\mathbf{B}]$$

has a rank equals to $n = 4\ \forall x$ in the domain.
Since **A** and **B** have nonvanishing rows the controllability matrix $\mathbf{M}_{cl}$ for the System (3.8) is of rank 4.

Nondimensionalization of the problem In order to simplify the calculation we dimensionalize the system by removing the units from the equations of motion via multiplying or dividing some constants. The two constant we divid by are the radial distance of the initial orbit and the gravitational constant μ in this case the radius of the initial orbit is unity and velocity is divided by the circular velocity of the initial orbit $\sqrt{\frac{\mu}{r_0^2}}$ and the time is multiplied by $\sqrt{\frac{\mu}{r_0^3}}$ In application we would like to make an optimal orbit transfer(i.e. from $(r = 1)$ to $(r = 1.2)$ in time $t_f = 4.469, 5.2231$ (time unit) Fig. 2 with optimal velocity Fig. 3 and optimal control function of both radial and tangential components Figs. 4, 5. The initial angle is $(\theta_0 = \frac{\pi}{2})$ and the final angle is $(\theta_f = \frac{3\pi}{2})$. $\dot{r}_0 = 0$ and $\dot{r}_f = 0$ for the initial and final orbits. $\dot{\theta}_0 = \sqrt{\frac{1}{r_0^3}} = 1$ and $\dot{\theta}_f = \sqrt{\frac{1}{r_f^3}} = 0.54433105395$. In the second $\theta_f = \frac{5\pi}{2}$ with $t_f = 6.866$.
in example the matrices **Q** and **R** are the identity matrices.

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

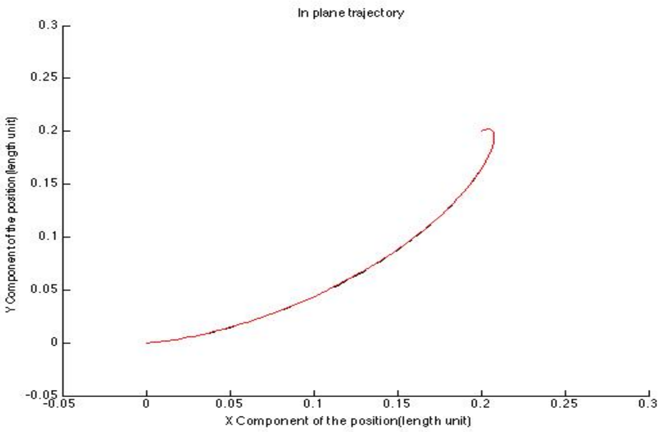

Figure 2. Trajectory of orbit rendezvous manoeuvre in the non dimensional coordinates

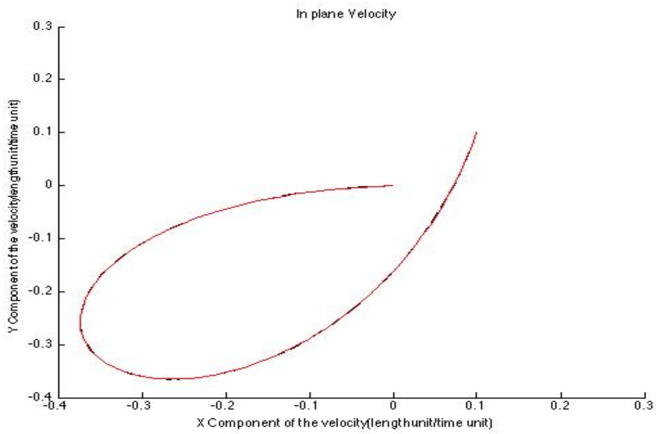

Figure 3. Velocity in the non dimensional coordinates

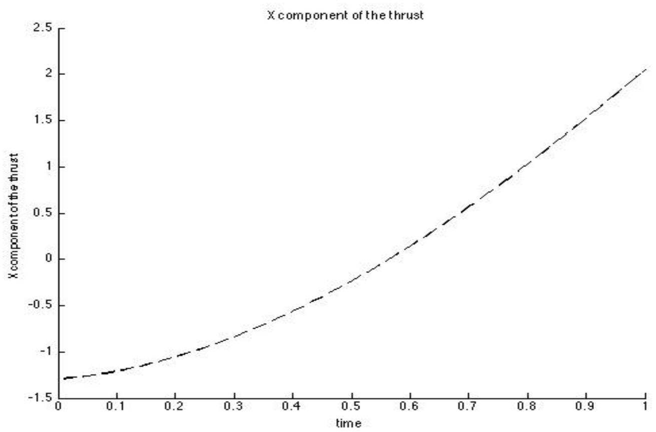

Figure 4. Control X component in the non dimensional coordinates

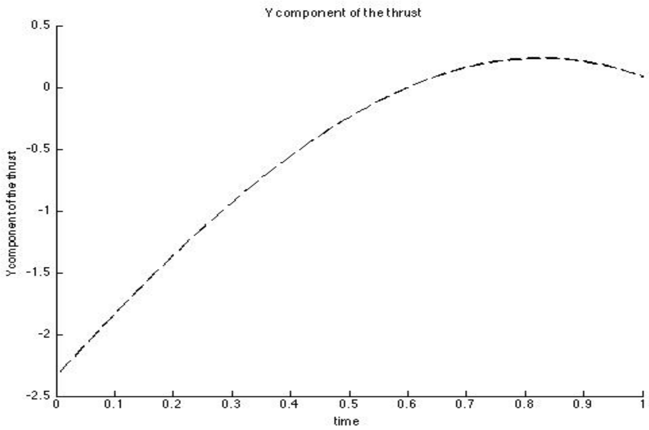

Figure 5. Control Y component in the non dimensional coordinates

5. Conclusion

The nonlinear orbital dynamics of the satellite formation with respect to the Earth Center Earth Fixed Coordinates are developed. The feedback optimal control of the satellite formation can be solved by factorizing the original nonlinear dynamics into accessible (weakly controllable) linear dynamics of state dependent factors. The factorized problem has been solved using the the Approximating Sequence Riccati Equations(ASRE) method. The technique is based on Linear Quadratic Regulator(LQR) with fixed terminal state, which guarantees closed loop solution. A computer simulation verified that the adopted technique is relaible.

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New Fractional Integral Results Using Euler Functions

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Abstract

In this paper, we use the the Riemann-Liouville fractional integral to develop some new results related to the Hermite-Hadamard inequality. Our results have some relationships with the paper of M.Z. Sarikaya et al. published in [Int. J. Open Problems Comput. Math., Vol. 5, No. 3, September, 2012]. Some interested inequalities of this paper can be deduced as some special cases.

Keywords: Integral inequalities, fractional integral, log convexity, Euler functions.
2010 MSC: 26D10, 26A33.

1. Introduction

Let us consider the famous Hermite-Hadamard inequality (Hadamard, 1893; Hermite, 1883) :

$$\frac{f(a+b)}{2} \leq \frac{2}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a convex function on $[a,b]$.
Many researchers have given considerable attention to (1.1) and a number of extensions and generalizations have appeared in the literature, see (Belaidi *et al.*, 2009; Dahmani, 2010; Dragomir & Pearse, 2000; Florea & Niculescu, 2007; Set *et al.*, 2010; Sarikaya *et al.*, 2012).
The aim of this paper is to present new extensions for a Hermite-Hadamard type inequality involving log-convex functions and using Euler Functions. Our results have some relationships with the work of M.Z. Sarikaya et al. (Sarikaya *et al.*, 2012). Some interested results of this reference can be deduced as particular cases.

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2. Preliminaries

We shall introduce the following definitions and properties which are used throughout this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function on $[a, b]$ is defined by:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, a \leq t \leq b, \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du.$

We give the semigroup property:

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \alpha > 0, \beta > 0. \tag{2.2}$$

For more details, one can consult (Gorenflo & Mainardi, 1997).

3. Main Results

Theorem 3.1. *Let f and g be two differentiable positive log-convex functions on I^0 (the interior of the interval I and $a, b \in I^0$.) Then, for $\alpha > 0$, the following inequalities hold.*

$$\begin{aligned} &2\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(b-a)J^{2\alpha-1} fg(b) \\ &\geq J^{\alpha}\Big[g(t) exp\Big(bA_b\Big)\Big] exp\Big[\tfrac{-J^{\alpha-1} bf(b) + J^{\alpha} bf'(b)}{J^{\alpha} f(b)}\Big] J^{\alpha} f(b) \\ &+ J^{\alpha}\Big[f(b) exp\Big(bD_b\Big)\Big] exp\Big[\tfrac{-J^{\alpha-1} g(b) + J^{\alpha} bg'(b)}{J^{\alpha} g(b)}\Big] J^{\alpha} g(b), \end{aligned} \tag{3.1}$$

where $A_b := \frac{-J^{\alpha-1} f(b) + J^{\alpha} f'(b)}{J^{\alpha} f(b)}, D_b := \frac{-J^{\alpha-1} g(b) + J^{\alpha} g'(b)}{J^{\alpha} g(b)}.$

Proof. Let us consider:
$K(x) := \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x), x \in [a, t], a < t \leq b, \alpha > 0.$
We remark immediately that, if $\alpha = 1$, then $K(x) = f(x)$ and hence, we can obtain the first main result of (Sarikaya *et al.*, 2012).
Now, let us take $\alpha \neq 1$. We can write

$$logK(x) - logK(y) \geq \frac{d}{dy}(logK(y))(x-y), x, y \in [a, t]. \tag{3.2}$$

Therefore,

$$log\frac{K(x)}{K(y)} \geq \frac{K'(y)}{K(y)}(x-y). \tag{3.3}$$

Hence,

$$\frac{K(x)}{K(y)} \geq exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2} f(y) + (t-y)^{\alpha-1} f'(y)}{(t-y)^{\alpha-1} f(y)}(x-y)\Big). \tag{3.4}$$

Consequently,

$$\frac{(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq \frac{(t-y)^{\alpha-1}f(y)g(x)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big). \quad (3.5)$$

Integrating the above inequality with respect to y over $[a,t], a < t \leq b$, yields

$$\frac{(t-a)(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq g(x)\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big[\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big]dy. \quad (3.6)$$

For the right hand side of (3.6) we use Jensen inequality. We obtain

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq \Big(\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}dy\Big)exp\Big[\frac{\int_a^t \frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{\Gamma(\alpha)}(x-y)dy}{\Big(\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}dy\Big)}\Big]. \quad (3.7)$$

Consequently,

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq exp\Big[\frac{-J^{\alpha-1}(x-t)f(t)+J^{\alpha}(x-t)f'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t). \quad (3.8)$$

That is

$$\int_a^t \frac{(t-y)^{\alpha-1}f(y)}{\Gamma(\alpha)}exp\Big(\frac{(1-\alpha)(t-y)^{\alpha-2}f(y)+(t-y)^{\alpha-1}f'(y)}{(t-y)^{\alpha-1}f(y)}(x-y)\Big)dy \geq exp\Big[\frac{J^{\alpha-1}tf(t)-J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]exp\Big[\frac{-J^{\alpha-1}f(t)+J^{\alpha}f'(t)}{J^{\alpha}f(t)}x\Big]J^{\alpha}f(t). \quad (3.9)$$

Thanks to (3.6) and (3.9), we obtain

$$\frac{(t-a)(t-x)^{\alpha-1}f(x)g(x)}{\Gamma(\alpha)} \geq g(x)exp\Big[\frac{J^{\alpha-1}tf(t)-J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]exp\Big[\frac{-J^{\alpha-1}f(t)+J^{\alpha}f'(t)}{J^{\alpha}f(t)}x\Big]J^{\alpha}f(t). \quad (3.10)$$

Then,

$$\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[g(t)exp\Big(tA_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tf(t)+J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t), \quad (3.11)$$

where

$$A_t := \frac{-J^{\alpha-1}f(t) + J^{\alpha}f'(t)}{J^{\alpha}f(t)}.$$

With the same arguments, we obtain:

$$\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[f(t)exp\Big(tB_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tg(t) + J^{\alpha}tg'(t)}{J^{\alpha}g(t)}\Big]J^{\alpha}g(t), \quad (3.12)$$

where

$$D_t := \frac{-J^{\alpha-1}g(t) + J^{\alpha}g'(t)}{J^{\alpha}g(t)}.$$

Adding (3.11) and (3.12), yields

$$\begin{aligned} 2\Gamma^{-2}(\alpha)\Gamma(2\alpha-1)(t-a)J^{2\alpha-1}fg(t) \geq J^{\alpha}\Big[g(t)exp\Big(tA_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tf(t) + J^{\alpha}tf'(t)}{J^{\alpha}f(t)}\Big]J^{\alpha}f(t) \\ +J^{\alpha}\Big[f(t)exp\Big(tD_t\Big)\Big]exp\Big[\frac{-J^{\alpha-1}tg(t)+J^{\alpha}tg'(t)}{J^{\alpha}g(t)}\Big]J^{\alpha}g(t). \end{aligned} \quad (3.13)$$

Taking $t = b$, we obtain the desired inequality (3.1). □

Theorem 3.2. *Let f and g be two differentiable positive log-convex functions on I^0 and $a, b \in I^0$. Then, for $\alpha > 0, \beta > 0, \alpha + \beta \neq 1$, we have:*

$$\begin{aligned} &2\Gamma(2\alpha+2\beta-3)(b-a)\frac{J^{2\alpha+2\beta-3}fg(b)}{\Gamma^2(\alpha)\Gamma^2(\beta)} \\ &\geq exp\Big[\frac{-J^{\alpha+\beta-2}bf(b) + J^{\alpha+\beta}bf'(b)}{J^{\alpha+\beta-1}f(b)}\Big]\frac{J^{\alpha+\beta-1}(g(b)exp[bE_b])}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(b)}{(\alpha+\beta-1)B(\alpha,\beta)} \\ &+exp\Big[\frac{-J^{\alpha+\beta-2}bg(b) + J^{\alpha+\beta}bg'(b)}{J^{\alpha+\beta-1}g(b)}\Big]\frac{J^{\alpha+\beta-1}(f(b)exp[bL_b])}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(b)}{(\alpha+\beta-1)B(\alpha,\beta)}, \end{aligned} \quad (3.14)$$

where

$$E_b := \frac{-J^{\alpha+\beta-2}f(b) + J^{\alpha+\beta-1}f'(b)}{J^{\alpha+\beta-1}f(b)}, L_b := \frac{-J^{\alpha+\beta-2}g(b) + J^{\alpha+\beta-1}g'(b)}{J^{\alpha+\beta-1}g(b)}.$$

Proof. We consider: $K(x) := \frac{(t-x)^{\alpha-1}(t-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}f(x), x \in [a,t], a < t \leq b, \alpha > 0, \beta > 0.$

We remark immediately that if $\alpha = 1, \beta = 1$, then we obtain the first main result in (Sarikaya *et al.*, 2012).

To prove Theorem 3.2, we need to take $\alpha + \beta \neq 1$. We have

$$\frac{K(x)}{K(y)} \geq exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y) + (t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big). \quad (3.15)$$

Then,

$$\begin{aligned} &\frac{(t-x)^{\alpha-1}(t-x)^{\beta-1}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)} \\ &\geq \frac{(t-y)^{\alpha+\beta-2}f(y)g(x)}{\Gamma(\alpha)\Gamma(\beta)}exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y) + (t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big). \end{aligned} \quad (3.16)$$

Integrating the above inequality with respect to y over $[a,t], a < t \leq b$, yields

$$\frac{(t-a)(t-x)^{\alpha+\beta-2}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\geq g(x)\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big[\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big]dy. \quad (3.17)$$

Thanks to Jensen inequality, we can write

$$\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big(\frac{(2-\alpha+\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big)dy$$

$$\geq \Big(\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)}dy\Big) exp\Big[\frac{\int_a^t \frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{\Gamma(\alpha)\Gamma(\beta)}(x-y)dy}{\Big(\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)}dy\Big)}\Big]. \quad (3.18)$$

By simple calculation, we can state that

$$\int_a^t \frac{(t-y)^{\alpha+\beta-2}f(y)}{\Gamma(\alpha)\Gamma(\beta)} exp\Big(\frac{(2-\alpha-\beta)(t-y)^{\alpha+\beta-3}f(y)+(t-y)^{\alpha+\beta-2}f'(y)}{(t-y)^{\alpha+\beta-2}f(y)}(x-y)\Big)dy$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta-1}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big] exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}x\Big]\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}, \quad (3.19)$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Thanks to (3.17) and (3.19), we obtain

$$\frac{(t-a)(t-x)^{\alpha+\beta-2}f(x)g(x)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta-1}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big] exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}x\Big]g(x)\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \quad (3.20)$$

Then,

$$\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$

$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(g(t)exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \quad (3.21)$$

With the same arguments, we obtain

$$\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$
$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tg(t)+J^{\alpha+\beta}tg'(t)}{J^{\alpha+\beta-1}g(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(f(t)exp\Big[\frac{-J^{\alpha+\beta-2}g(t)+J^{\alpha+\beta-1}g'(t)}{J^{\alpha+\beta-1}g(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \tag{3.22}$$

Adding (3.21) and (3.22), yields

$$2\Gamma(2\alpha+2\beta-3)(t-a)\frac{J^{2\alpha+2\beta-3}fg(t)}{\Gamma^2(\alpha)\Gamma^2(\beta)}$$
$$\geq exp\Big[\frac{-J^{\alpha+\beta-2}tf(t)+J^{\alpha+\beta}tf'(t)}{J^{\alpha+\beta-1}f(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(g(t)exp\Big[\frac{-J^{\alpha+\beta-2}f(t)+J^{\alpha+\beta-1}f'(t)}{J^{\alpha+\beta-1}f(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}f(t)}{(\alpha+\beta-1)B(\alpha,\beta)}$$
$$+exp\Big[\frac{-J^{\alpha+\beta-2}tg(t)+J^{\alpha+\beta}tg'(t)}{J^{\alpha+\beta-1}g(t)}\Big]\frac{J^{\alpha+\beta-1}\Big(f(t)exp\Big[\frac{-J^{\alpha+\beta-2}g(t)+J^{\alpha+\beta-1}g'(t)}{J^{\alpha+\beta-1}g(t)}t\Big]\Big)}{(\alpha+\beta-1)B(\alpha,\beta)}\frac{J^{\alpha+\beta-1}g(t)}{(\alpha+\beta-1)B(\alpha,\beta)}. \tag{3.23}$$

Taking $t = b$, we obtain (3.14). Theorem 3.2 is thus proved. □

Remark. Applying Theorem 3.2 for $\alpha = 1, \beta \neq 1$ or $\beta = 1, \alpha \neq 1$, we obtain Theorem 3.1.

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Possibility of Hypercomputation from the Standpoint of Superluminal Particles

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Abstract

In mathematics and computer science, an accelerated Turing machine is a hypothetical computational model related to Turing machines, which can perform the countable infinite number of computational steps within a finite time. But this machine cannot be physically realized from the standpoint of the Heisenberg uncertainty principle, because the energy required to perform the computation will be exponentially increased when the computational step is accelerated and it is considered that it is mere a mathematical concept and there is no possibility for its realization in a physical world. However, by using superluminal particles instead of subluminal particles including photons, it can be shown that the hypercomputation system which can perform infinite steps of computation within a finite time length and energy can be realized.

Keywords: Turing machine, Zeno machine, hypercomputation, superluminal particle, tachyon, halting problem.
2010 MSC: 68Q05, 81P68, 83A05.

1. Introduction

In mathematics and computer science, an accelerated Turing machine is a hypothetical computational model related to Turing machines which can perform the countable infinite number of computational steps within a finite time. It is also called a Zeno machine which concept was proposed by B. Russel, R. Blake and H. Weyl independently, which performs its first computational step in one unit of time and each subsequent step in half the time of the step before, that allows an infinite number of steps can be completed within a finite interval of time (Ord, 2006), (Hamkins & Lewis, 2000). However this machine cannot be physically realized from the standpoint of the Heisenberg uncertainty principle $\Delta E \cdot \Delta t \approx \hbar$, because the energy to perform the computation will be exponentially increased when the computational step is accelerated. Thus it is considered that the Zeno machine is mere a mathematical concept and there is no possibility to realize it in

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a physical world. Contrary to this conclusion, the author studied the possibility to realize it by utilizing superluminal particles instead of subliminal particles including photons.

2. Computational time required to perform infinite steps of computation by using ordinary particles

Feynman defined the reversible computer model as shown in Fig.1, which requires energy per step given by (Feynman, 2000):

$$\text{energy per step} = k_B T \frac{f-b}{(f+b)/2} \quad , \tag{2.1}$$

where k_B is Boltzmann's constant, T is a temperature, f is a forward rate of computation and b is backward rate.

Supposing that there in no energy supply and parameters f and b are fixed during the computation, we can consider the infinite computational steps given by:

$$E_1 = kE_0,\ E_2 = kE_1, \cdots, E_n = kE_{n-1}, \cdots, \tag{2.2}$$

where we let the initial energy of computation be $E_0 = k_B T$, $k = 2(f-b)/(f+b)$ and E_n is the energy for the n-th step computation.

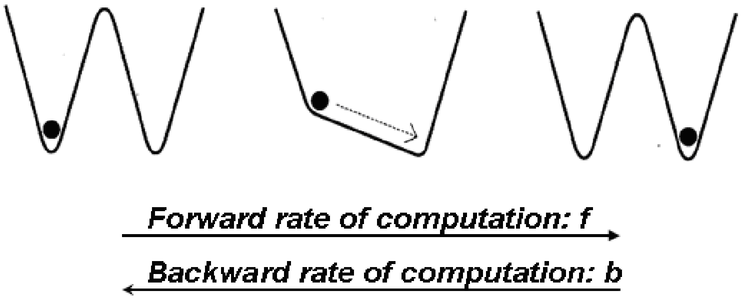

Figure 1. Computational steps for the reversible computation (Feynman, 2000).

From which, we have $E_n = k^n E_0$, then the energy loss for each computational step becomes:

$$\begin{aligned} \Delta E_1 = \ & E_0 - E_1 = (1-k)E_0 \\ \Delta E_2 = \ & E_1 - E_2 = (1-k)kE_0 \\ & \vdots \\ \Delta E_n = \ & E_{n-1} - E_n = (1-k)k^{n-1}E_0. \end{aligned} \tag{2.3}$$

According to the paper by (Lloyd, 2000), it is required for the quantum system with average energy ΔE to take time at least Δt to evolve to an orthogonal state given by:

$$\Delta t = \frac{\pi\hbar}{2\Delta E}, \tag{2.4}$$

From which, the total energy for the infinite steps yields E_0 if setting $E = \Delta E_i$ in equation (2.4), then the total time for the computation with infinite steps becomes:

$$T_n = \sum_{j=1}^{n} \Delta t_n = \frac{\pi\hbar}{2E_0} \sum_{j=1}^{n} \frac{1}{(1-k)k^{j-1}}. \tag{2.5}$$

As the infinite sum of equation (2.5) diverges to infinity as shown in Fig. 2, the Feynman model of computation requires infinite time to complete the calculation when satisfying $0 < k < 1$.

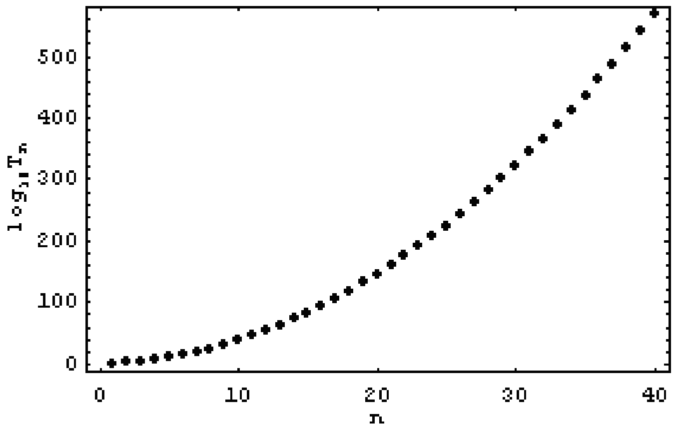

Figure 2. Computational time to complete the n-th step of computation by using subluminal particles (for the case, $k = 1/2\,, \gamma = 1.0$).

Hence it can be seen that a computer system utilizing subluminal particles including photons requires infinite time to complete infinite steps of computation.

3. Computational time by using superluminal elementary particles

3.1. Uncertanity Principle for superluminal particles

E. Recami claimed in his paper (Recami, 2001) that tunneling photons which travel in evanescent mode can move with superluminal group speed inside the barrier. Chu and S. Wong at AT&T Bell Labs measured superluminal velocities for light traveling through the absorbing material (Brown, 1995). Furthermore Steinberg, Kwait and Chiao measured the tunneling time for visible light through the optical filter consisting of the multilayer coating about 10^{-6} m thick. Measurement results by Steinberg and co-workers have shown that the photons seemed to have traveled at 1.7 times the speed of light (Steinberg *et al.*, 1993). Recent optical experiments at Princeton NEC have verified that superluminal pulse propagation can occur in transparent media (Wang *et al.*, 2000). These results indicate that the process of tunneling in quantum physics is superluminal as claimed by E. Recami. From relativistic equations of energy and momentum of the moving particle, shown as:

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, \tag{3.1}$$

and

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}, \tag{3.2}$$

the relation between energy and momentum can be shown as $p/v = E/c^2$.

From which, we have (Musha, 2012):

$$\frac{v\Delta p - p\Delta v}{v^2} = \frac{\Delta E}{c^2}, \tag{3.3}$$

Supposing that $\Delta v/v^2 \approx 0$, equation (3.3) can be simplified as:

$$\Delta p \approx \frac{v}{c^2}\Delta E \,. \tag{3.4}$$

This relation is also valid for the superluminal particle called a tachyon which has an imaginary mass im_* (Musha, 2012), the energy and the momentum of which are given by following equations, respectively.

$$E = \frac{m_* c^2}{\sqrt{v^2/c^2 - 1}}, \tag{3.5}$$

$$p = \frac{m_* v}{\sqrt{v^2/c^2 - 1}}. \tag{3.6}$$

According to the paper by M. Park and Y. Park (Park & Park, 1996), the uncertainty relation for the superluminal particle can be given by:

$$\Delta p \cdot \Delta t \approx \frac{\hbar}{v - v'}, \tag{3.7}$$

where v and v' are the velocities of a superluminal particle after and before the measurement. By substituting equation (3.4) into (3.7), we obtain the uncertainty relation for superluminal particles given by:

$$\Delta E \cdot \Delta t \approx \frac{\hbar}{\beta(\beta - 1)}, \tag{3.8}$$

when we let $v' = c$ and $\beta = v/c$.

3.2. *Computational time required for the superluminal particle*

Instead of subluminal particles including photons, the time required for the quantum system utilizing superluminal particles becomes

$$T_n = \sum_{j=1}^{n} \Delta t_j = \frac{\pi\hbar}{2E_0} \sum_{j=1}^{n} \frac{1}{\beta_j(\beta_j - 1)(1 - k)k^{j-1}}, \tag{3.9}$$

from the uncertainty principle for superluminal particles given by equation (3.8), where β_j can be given by:

$$\beta_j = \sqrt{1 + \frac{{m_*}^2 c^4}{{E_j}^2}} = \sqrt{1 + \frac{\gamma^2}{k^{2j}}}, \tag{3.10}$$

which is derived from equation (3.6), where $\gamma = m_* c^2 / E_0$.

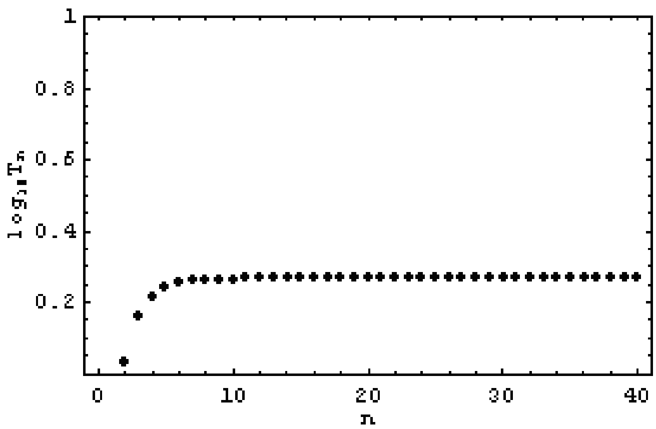

Figure 3. Computational time to complete the *n*-th step of computation by using superluminal particles (for the case, $k = 1/2$, $\gamma = 1.0$).

Hence it is seen that the computation time can be accelerated according to equation (3.10).

By the numerical calculation, it can be shown that the infinite sum of equation (3.9) converges to a certain value satisfying $0 < k < 1$ as shown in Fig.4.

In this figure, the horizontal line is for the parameter $\gamma = m_* c^2 / E_0$ and the vertical line is for the time to complete infinite step calculations. From these calculation results, an accelerated Turing machine can be realized by utilizing superluminal particles instead of subliminal particles for the Feynman's model of computation.

Thus, contrary to the conclusion for the Feynman's model of computation by using ordinary particles, it can be seen that superluminal particles permits the realization of an accelerated Turing machine.

It is known that an accelerate Turing machines allow us to be computed some functions which are not Turing-computable such as the halting problem (Kieu, 2004), described as "given a description of an arbitrary computer program, decide whether the program finishes running or continues to run forever".

This is equivalent to the problem of deciding, given a program and an input, whether the program will eventually halt when run with that input, or will run forever.

Halting problem for Turing machines can easily solved by an accelerated Turing machine using the following pseudocode algorithm (as shown in Fig.5). As an accelerated Turing machines are more powerful than ordinary Turing machines, they can perform computation beyond the Turing limit which is called hypercomputation, such as to decide any arithmetic statement that is infinite

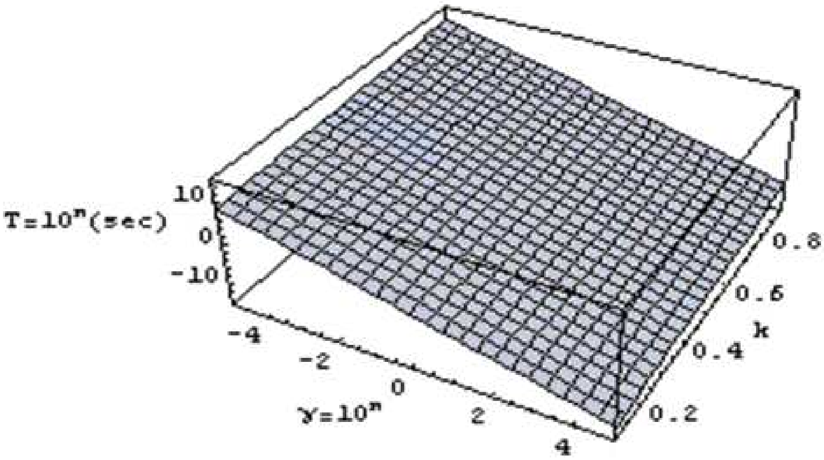

Figure 4. Computational time by using superluminal particles.

```
begin program
  write 0 on the first position of the output tape;
  begin loop
    simulate 1 successive step of the given Turing
    machine on the given input;
    if the Turing machine has halted, then write 1 on
    the first position of the output tape and break out
    of loop;
  end loop
end program
```

Figure 5. Psedocode algorithm to solve the halting problem (Wikipedia, 2009).

time decidable. From this result, we can construct an oracle machine (van Melkebeek, 2000) by using a superluminal particle, which is an abstract machine used to study decision problems. It can be conceived as a Turing machine with a black box, called an oracle, which is able to decide certain decision problems in a single operation.

4. Human mind from the standpoint of superluminal hyper computation

There are some papers on the hypothesis that the human mind is consisted of evanescent tunneling photons which has a property of superluminal particles called tachyons (Georgiev, 2003), (Musha, 2005, 2009).

Professor Dutheil proposed his hypothesis in his book titled, "L'homme superlumineux" (Dutheil

& Dutheil, 2006), that consciousness is a field of superluminal matter belonging to the true fundamental universe shown in Fig.6, and our world is merely a subluminal holographic projection of it.

He proposed the hypothesis based on superluminal consciousness shown as follows;

- The brain is nothing more than a simple computer that transmit information.
- Consciousness, or the mind is composed of a field of tachyons or superluminal matter, located on the other side of the light barrier in superluminal space-time.

If the human consciousness is consisted of superluminal particles as claimed by Prof. Dutheil, the superiority of the human brain to conventional silicon processors may be explained because it can perform infinite steps of computation within a finite time.

To further interpret this result, we consider S.Berkovich suggestion of a ”cloud computing paradigm”, in which is given an elegant constructive solution to the problem of the organization of mind. Within his article, he defines a situation where individual brains are not stand-alone computers but collective users whom have shared access to portions of a holographic memory of the Universe (Berkovich, 2010). He proposed that the cosmic background radiation (CMB) has nothing at all to do with the residual radiation leftover from the Big Bang.

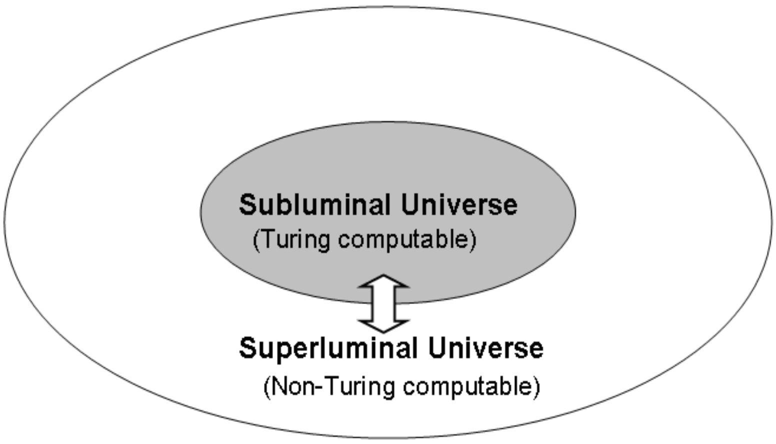

Figure 6. Superluminal Universe model proposed by Prof. Dutheil.

Instead, he claimed that CMB is nothing but noise from writing operations in the holographic memory of the Universe. Such holographic write operations would require some type of universal clocking rate for these operations. Since the virtual superluminal particle pairs are created and annihilated in the vacuum within a short, finite period of time according to the uncertainty principle, we could logically consider this duration as the clock rate for these operations (Fig.7).

From this standpoint, the extraordinary capability of a human brain such as the enigma of Srinivasa Ramanujan (Kanigel, 1991), who invented numerous remarkable and mysterious mathematical formulas from his inspiration without proofs, can be explained from the capability of superluminal consciousness which is superior to that of conventional Turing type computer systems.

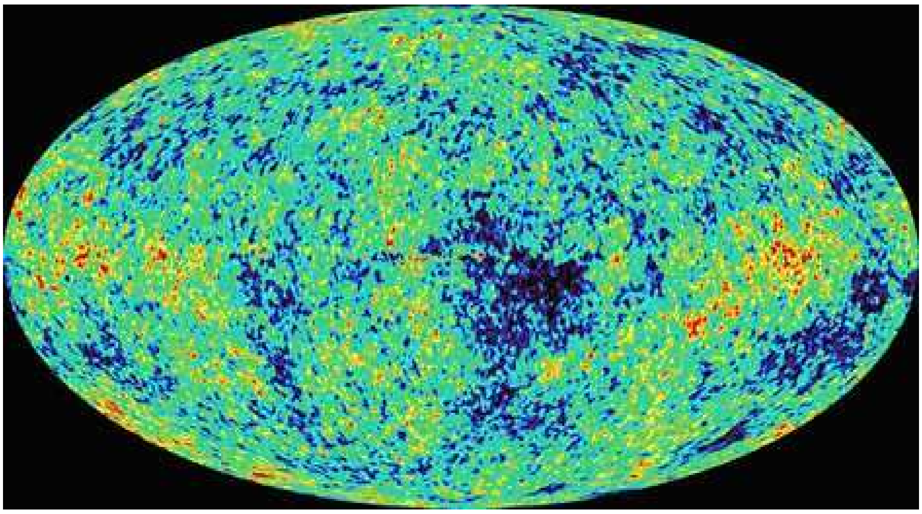

Figure 7. Is CBR an activity of zero-point energy fluctuations of vacuum which relates to the writing operations in the holographic memory of the Universe? (www.computus.org).

5. Conclusion

From the theoretical analysis, it is seen that a hypercomputational system which can complete infinite steps of computation within a finite time and energy can be realized by using superluminal particles from the standpoint of quantum mechanics. Thus an extraordinary capability of human consciousness such as intuition compared with the ordinary silicon processors might be explained if they are composed of superluminal particles, because they have a capability to function beyond the ordinary Turing machines.

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