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Fence-like Quasi-periodic Texture Detection in Images

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Abstract

The focus of this article is on automatic detection of fence or wire mesh (a form of quasi-periodic texture) in images through frequency domain analysis. Textures can be broadly classified in to two general classes: quasi-periodic and random. For example, a fence has a repetitive geometric pattern, which can be classified as a quasi-periodic texture. Quasi-periodic textures can be easily detected in the frequency spectrum of an image as they result in peaks in the frequency spectrum. This article explores a novel way of de-fencing viewed as a quasi-periodic texture segmentation by filtering in frequency domain to segregate the fence from the background. A resulting de-fenced image is followed by support vector machine classification. An interesting application of the proposed approach is the removal of occluding structures such as fence or wire mesh in animal enclosure photography.

Keywords: Frequency spectrum, quasi-periodic texture, texture segmentation
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1. Introduction

This article introduces an algorithm to detect automatically fence or wire mesh structures, which typically present in the foreground of the image. A region in an image has a constant texture, provided a set of local statistics or other local properties of the picture function are constant, slowly varying, or approximately periodic (Tuceryan & Jain, 1993). A fence can be classified as a texture in an image. Textures can be broadly classified in to two general classes: *periodic* or more generally *quasi-periodic textures* and *random textures*.

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According to (Rangayyan, 2004), if there is a repetition of a texture element at almost regular or quasi-periodic intervals, such textures can be classified as quasi-periodic or ordered and the smallest repetitive element is called a texton or a texel. In contrast if no such repetitive element can be identified, those textures can be classified as random.

(Ohm, 2004) classifies textures as *regular* and *irregular* textures. Regular textures refer to textures, which exhibits strong periodic or quasi-periodic behavior. According to (Ohm, 2004), exact periodicity is a very rare case mostly found in synthetic images. The regular structures in natural images are often quasi-periodic, which means that periodic pattern can clearly be recognized, but have slight variations of periods. As it will be shown in section 2, quasi-periodic textures are a generalization of periodic textures.

Based on the above classifications, a fence structure, which has a texture element repeating at quasi-periodic intervals can be categorized as a quasi-periodic texture. Hence, a fence-like texture can be modeled as a quasi-periodic signal, which shows peaks in its power spectrum. It is mentioned in (Chang & Kuo, 1993) that these kinds of quasi-periodic signals possess dominant frequencies located in the middle frequency channels.

The perception of texture has numerous dimensions. Thus, a number of different texture representations were introduced from time to time in order to accommodate a variety of textures. These representations are categorized in (Tuceryan & Jain, 1993) as statistical methods, which involves co-occurrence matrices and autocorrelation features, geometric methods, model based methods and signal processing methods. Signal processing methods are subdivided into spatial domain filtering (Malik & Perona, 1990) and frequency filtering.

Frequency analysis of the textured image is close to human perception of texture as human visual system analyzes the textured image by decomposing the image into its frequency and orientation components (Campbell & Robson, 1968). (Turner, 1986) and (Clark *et al.*, 1987) proposed to use the Gabor filters in texture analysis. The Gabor filter is a frequency and orientation selective filter. Another model, which is widely used for texture analysis is wavelet transform (Chang & Kuo, 1992, 1993; Wilscy & Sasi, 2010).

The focus of this article is on images, which are occluded with fence textures as shown in figure 1. In such cases, it is challenging to segment the fence from the rest of the image, especially when the image background is regular. Simple colour segmentations and edge detection does not work in this case.

The traditional frequency filters used for texture analysis, Gabor and Wavelet cannot be directly applied to extract fence texture in our scenario as the frequencies correspond to both fence and the background are present in the spectrum. Thus, we first perform frequency domain processing to isolate fence texture from the background and subsequently apply Wavelet transform.

An interesting application of the proposed algorithm can be detection and removal of fence-like textures obstructing the images in zoo photography. According to many web articles on photography (Stalking, 2010; Masoner, 2013), wire mesh and fences are a major challenge in zoo photography. The algorithm proposed in this article was tested for fences with different shapes, sizes, colours and orientations.

The rest of the article is organized as follows. Section 2 introduces quasi-periodic signals and provides the mathematical background to analyze quasi-periodic signals in images. Section 3 discusses the implementation of the quasi-periodic texture detection algorithm in three steps: (1)

frequency domain filtering for quasi-periodic texture detection, (2) multiresolution processing for fence mask formation and (3) fence segmentation through SVM classification. The experimental results of the proposed algorithm are given in Section 4 for some zoo images as well as for some challenging images from PSU NRT Database (Liu, 2007). A comparison of the proposed method with existing fence detection techniques is given in section 5 followed by future work and conclusion in sections 6 and 7 respectively.

Figure 1. Images Occluded with Fence Textures.

2. Quasi-periodic Signals

Before going into details of quasi-periodic texture detection in images, understanding the mathematical background of quasi-periodic signals is important.

Definition 2.1. Continuous-time Periodic Signal ((Proakis & Manolakis, 2006, §1, p. 13))
By definition, A continuous signal f(t) locally defined on the set $L^2(\mathfrak{R})$ of finite energy signals is fully periodic with period T, when the signal exactly satisfies

$$f(t) = f(t + T).$$

Definition 2.2. Continuous-time Quasi-periodic Signal ((Martin *et al.*, 2010))
A signal $f_{qp}(t)$ is quasi-periodic with k periods $T_1, ..., T_k$ when

$$f_{qp}(t) = g\{f_1(t), f_2(t), ..., f_k(t)\},$$

where the k signals $f_i(t)$ are continuous periodic signals with respect to each period T_i.

In the case of continuous functions locally defined on the set $L^2(\mathfrak{R})$ of finite energy signals, quasi-periodic signals are a generalization of periodic signals. All the periods are required to be strictly positive and to be rationally linearly independent (Martin *et al.*, 2010).

Definition 2.3. Discrete-time Periodic Signal((Proakis & Manolakis, 2006, §1, p. 15))
A discrete-time signal f(n) is periodic with period N, if and only if,

$$f(n) = f(n + N) \text{ for all n.}$$

Based on the definition of continuous-time quasi-periodic signals, the definition for discrete-time quasi-periodic signals can be derived.

Definition 2.4. Discrete-time Quasi-periodic Signal
A discrete-time signal $f_{qp}(n)$ is quasi-periodic with k periods $N_1, ...N_k$ when

$$f_{qp}(n) = g\{f_1(n), f_2(n), ..., f_k(n)\},$$

where $g : \mathbb{Z}^k \to \mathbb{Z}$ and the k signals $f_i(n)$ are discrete-time periodic signals with respect to each period N_i.

In the context of this paper, an image is considered as a 2D discrete-time signal. If we extend the definition of 1D quasi-periodic signal to 2D quasi-periodic signal;

Definition 2.5. 2D Discrete-time Periodic Signal((Woods, 2006, §1, p. 7))
A 2D discrete-time signal f(x,y) is periodic with period (M,N), if and only if,

$$f(x, y) = f(x + M, y) = f(x, y + N), \forall n, m \in \mathbb{Z}.$$

Definition 2.6. 2D Discrete-time Quasi-periodic Signal
A 2D discrete-time signal $f_{qp}(x, y)$ is quasi-periodic with k periods $(M_1, ...M_k, N_1, ...N_k)$ when

$$f_{qp}(x, y) = g\{f_1(x, y), f_2(x, y), ..., f_k(x, y)\},$$

where the k signals $f_i(x, y)$ are discrete-time periodic signals with respect to periods (M_i, N_i). Hence, a quasi-periodic signal can be defined as a combination of periodic signals with incommensurate (not rationally related) frequencies (Battersby & Porta, 1996). If the frequencies are commensurate, then f_{qp} becomes a periodic signal (Regev, 2006).

A discrete-time quasi-periodic signal can be expressed with a Fourier series as given in definition 2.8 as a generalization of definition 2.7. 1D case will be considered for simplicity and it can be extended to 2D.

Definition 2.7. Fourier Series of a Discrete-time Periodic Signal ((Proakis & Manolakis, 2006, §4, p. 242))

$$f(n) = \sum_{k=0}^{N-1} c_k exp\left(\frac{j2\pi kn}{N}\right).$$

Definition 2.8. Fourier Series of a Discrete-time Quasi-periodic Signal ((Regev, 2006, p. 156))
The Fourier series of a r-quasi-periodic signal is given by (Regev, 2006):

$$f_{qp}(n) = \sum_{k_1}\sum_{k_2}...\sum_{k_r} c_{k_1 k_2 ... k_r} exp\left[j\left(\frac{2\pi k_1 n}{N_1} + \frac{2\pi k_2 n}{N_2} + ... + \frac{2\pi k_r n}{N_r}\right)\right],$$

where k=1,2,...,r and the frequencies $\omega_k = 2\pi/N_k$ are incommensurate.

Theorem 2.1. *Let $f_{qp}(n)$ be a discrete-time quasi-periodic signal. Then the frequency spectrum of $f_{qp}(n)$ consists of a set of peaks determined by the fundamental frequencies of each discrete periodic signal component in the signal.*

Proof. With $\omega_i = 2\pi/N_i$, $f_{qp}(n)$ in definition 2.8 can be re-written as

$$f_{qp}(n) = \sum_K c_K exp\left[jK\Omega n\right],$$

where $K = (k_1, k_2, ..., k_r)$ and $\Omega = (\omega_i, \omega_2, ..., \omega_r)$. Thus, the frequency spectrum contains numerous peaks at all frequencies ν, satisfying

$$2\pi\nu = |K \cdot \Omega| = |k_1\omega_1 + k_2\omega_2 + ... + k_r\omega_r|,$$

for any combination of integers $k_1, k_2, ..., k_r$. □

3. Quasi-periodic Texture Detection in Frequency Domain

3.1. Frequency Domain Filtering for Quasi-periodic Texture Detection

As proven by theorem 2.1, the Fourier spectrum of a quasi-periodic signal consists of a discrete set of spikes or peaks at a number of frequencies depending on the number of periodic signals it is comprised of. Hence, based on theorem 2.1, the fence-like quasi-periodic structure should result in peaks in the frequency spectrum of the image. The objective of this section is to filter those spikes in the frequency spectra relevant to the quasi-periodic signal in order to extract the fence texture corresponding to the quasi-periodic signal from the rest of the image.

To achieve this, first start with the frequency domain representation of the 2D image. We will be considering the DFT of an image.

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)exp\left[-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)\right] \text{u=0,1,...M-1, v=0,1,...N-1.} \tag{3.1}$$

To filter the frequencies showing spikes in the frequency spectra, it is necessary to perform thresholding based on the magnitude of each frequency component. A filter function $H_1(u, v)$ in frequency domain can be defined for this purpose as given below.

$$H_1(u, v) = \begin{cases} 1 & \text{if } |F(u, v)| > T, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where T is a threshold to filter spikes in frequency.

Once the thresholding is applied to the frequency components:

$$F'(u, v) = H_1(u, v)F(u, v)$$

Although, we filtered the frequency components corresponding to peaks in the frequency spectra, it is necessary to filter peaks in frequencies resulted by other details in the image. For an example, the DC component F(0,0), which can be derived by substituting u=0 and v=0 in equation 3.1. $|F(0, 0)|$ typically is the largest component of the spectrum.

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN\overline{f}(x,y).$$

The quasi-periodic signal in our case is the fence. Fence-like textures typically result in qausi-periodic signals whose dominant frequencies are located in the middle frequency channels (Chang & Kuo, 1993). Therefore, by using a bandpass filter in frequency domain, the frequencies corresponding to the fence can be filtered.

$$H_2(u,v) = \begin{cases} 1 & \text{if } D_1 \leq D(u,v) \leq D_2, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

where D_1 and D_2 are constants and D(u,v) is the distance between a point (u,v) in the frequency domain and the center of the frequency spectrum.

Thus, the final result in frequency domain after applying the second filter would be:

$$\begin{aligned} F''(u,v) &= H_2(u,v)F'(u,v), \\ &= H_2(u,v)H_1(u,v)F(u,v), \\ &= H(u,v)F(u,v), \end{aligned}$$

where $H = H_2H_1$, since the application of H_1 and H_2 can be considered as a cascade system. When $F''(u,v)$ is transferred back into spatial domain, the resulting image is given by:

$$g(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F''(u,v)exp\left[j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)\right] \text{x=0,1,...M-1, y=0,1,...N-1.}$$

It is important to note that H_1 and H_2 are zero phase shift filters, which affect the magnitude of the frequency spectra, but do not alter the phase angle. These filters affect the real (Re(u,v)) and imaginary (Im(u,v)) parts equally, thus cancels out when calculating phase angle $\phi(u,v) = arctan[Im(u,v)/Re(u,v)]$.

Figure 2(d) illustrates the final result of frequency domain filtering explained above. It can be clearly seen that the fence texture is emphasized and other image details have been suppressed.

3.2. Multiresolution Processing for Fence Mask Formation

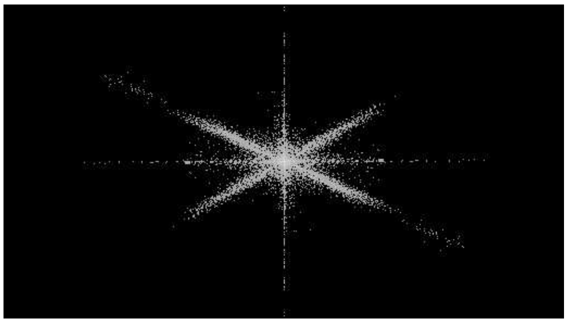
The human visual system analyzes the textured images by decomposing the image into its frequency and orientation components (Campbell & Robson, 1968). Wavelet transformation provides the ability to analyze images through multiresolution processing.

Wavelet transform in two dimension provides the two dimensional scaling function $\phi(x,y)$ and three two dimensional directionally sensitive wavelets $\psi^H(x,y)$, $\psi^V(x,y)$,$\psi^D(x,y)$ as given in (Gonzalez & Richard, 2002).

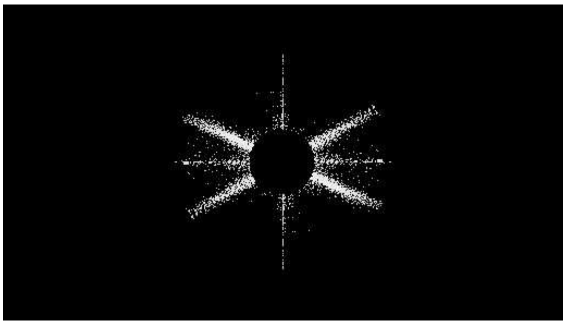
$$\phi_{j,m,n}(x,y) = 2^{\frac{j}{2}}\phi(2^j x - m, 2^j y - n).$$

$$\psi^i_{j,m,n}(x,y) = 2^{\frac{j}{2}}\psi^i(2^j x - m, 2^j y - n), i = \{H,V,D\}.$$

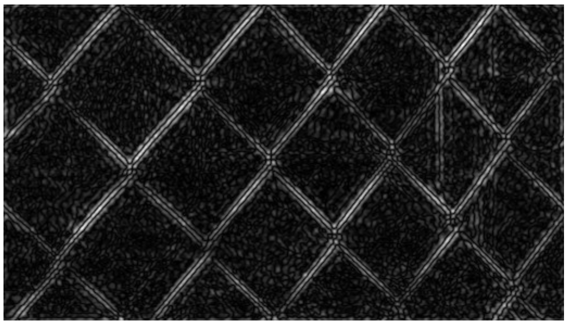
(a) Original Image (f(x,y)).



(b) Filtered Peak Frequencies (F'(u,v)).



(c) Bandpassed Spectrum (F"(u,v)).



(d) Spatial Domain (g(x,y)).

Figure 2. Frequency Domain Filtering for Fence Texture Segregation from Image Background.

These wavelets measure intensity variations for images along different directions: ψ^H measures variations along horizontal direction (along columns), ψ^V measures variations along vertical direction (along rows) and ψ^D corresponds to variations along diagonals.

The discrete transform of image f(x,y) is:

$$W_\phi(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\phi_{j_0,m,n}(x,y).$$

$$W_\psi^i(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\psi^i_{j,m,n}(x,y), i = \{H, V, D\},$$

where j_0 is an arbitrary starting scale and the $W_\phi(J_0, m, n)$ coefficients define an approximation of f(x,y) at scale j_0. The $W_\psi^i(j, m, n)$ coefficients add horizontal, vertical and diagonal details for scales $j \geq j_0$. $W_\psi^i(j_0, m, n)$ coefficients are called detail coefficients. Usually j_0 is set to zero.

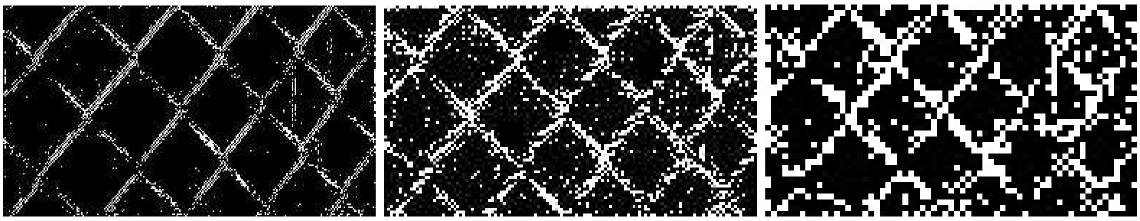
For each level j, thresholding is performed on the details coefficients $W_\psi^i(j, m, n)$ to extract the fence masks $M^i(j, m, n)$ at each level j.

$$M^i(j, m, n) = \begin{cases} 1 & \text{if } W_\psi^i(j, m, n) > T_j, \text{ where } T_j \text{ is the threshold for level } j, \\ 0 & \text{otherwise.} \end{cases}$$

The final fence mask at level j is obtained by performing *OR* operation of the vertical, horizontal and diagonal fence masks at level j.

$$M(j,m,n) = M^V(j,m,n) \oplus M^H(j,m,n) \oplus M^D(j,m,n).$$

The detected fence masks at 3 consecutive levels are shown in figure 3.



(a) M(1,m,n). (b) M(2,m,n). (c) M(3,m,n).

Figure 3. Detected Fence Masks at Three Different Levels.

Next, the fence masks at different levels of wavelet pyramid were combined by using a coarser to finer strategy. The objective is to reduce noise and extract pixels, which fall exactly on the fence. In order to make the resultant mask in the same size as the original image, a mask was created at the zero level by just thresholding the spatial domain result of frequency filtering (g(x,y)). Hence, altogether we have fence masks at 4 different levels in the pyramid.

First, the highest level fence mask (level 3) was considered and if a pixel belongs to the mask then we move to the next lower level (level 2) and check for the neighbouring children of the original pixel. If any of the neighbouring children are mask pixels, then recursively go and check for their neighbouring children in the subsequent lower levels. Finally, when the algorithms reaches the bottom most level (zero level), it marks the mask pixels as 1, given that the neighbouring children in the lowest level are mask pixels as well. The resultant fence mask is shown in Figure 4.

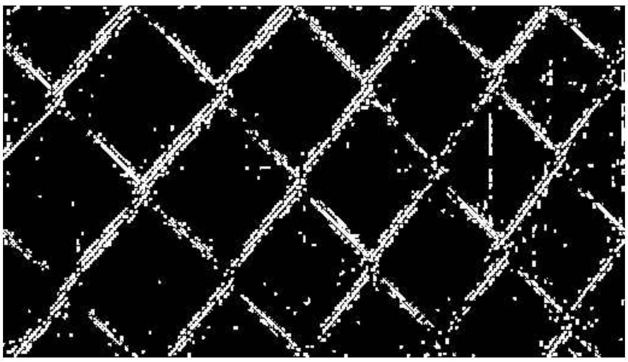


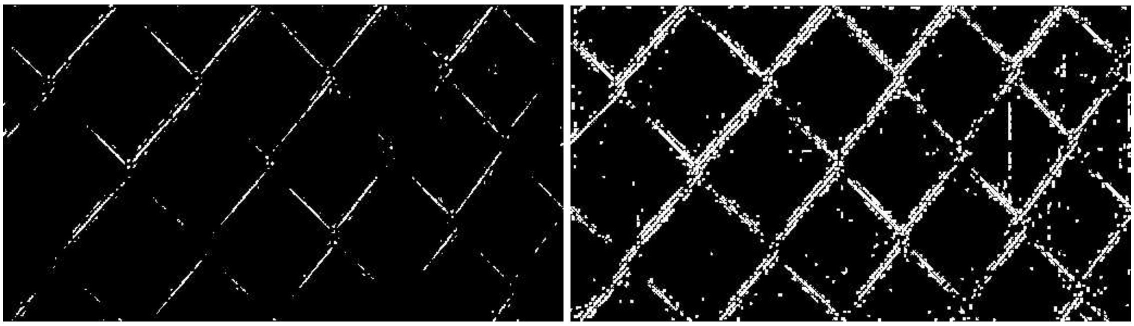
Figure 4. Fence Mask Formed by Combining Wavelet Decomposition Levels.

3.3. Fence Segmentation through SVM Classification

Although the noise is minimized and the fence is emphasized in the detected fence mask, it is not perfectly detected yet. However, the detected fence mask classifies a good number of pixels, which exactly falls on the fence in the image. This knowledge on fence pixels can be used to segment the fence. Hence, it was decided to pick some samples from the fence mask and use the features of those sample pixels to train a *Support Vector Machine (SVM) classifier* in order to segment the fence texture. A SVM classifier with a linear kernel is used in this case.

In addition to the samples from fence, it is necessary to pick samples from background to train the SVM classifier. For this purpose two root level fence masks were generated. One root level mask was generated by selecting a very high threshold and the other one is generated by using a very low threshold. These masks were used as the root level mask in the process of combining wavelet decomposition levels as explained in section 3.2 separately in order to generate two different final fence masks as shown in Figure 5.

As it can be clearly seen, the root level mask with high threshold generates a very thin final mask, resulting points, which exactly lie on the fence. On the other hand the root level mask with low threshold generates a thick fence mask, which has some points fall on the background as well.



(a) Thin Mask with High Threshold. (b) Thick Mask with Low Threshold.

Figure 5. Two Fence Masks used for SVM Classification.

The thin mask was used to pick random samples, which represent fence class and the negation of the thick mask (1-thick mask) is used to pick random samples, which represent the background class. The use of negation of thick mask for background sample selection reduces the chance of picking fence pixels as background pixels and hence improves the accuracy of classification.

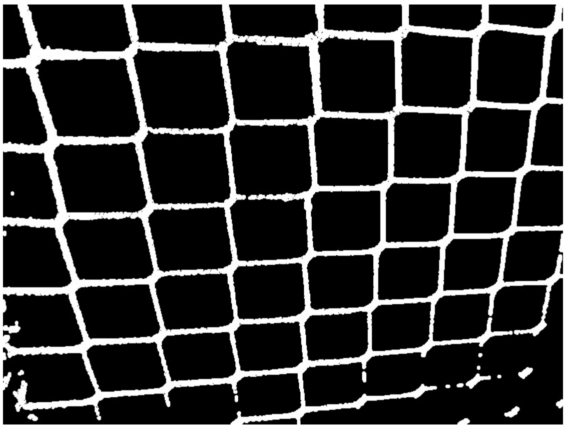
The feature vector selected for classification plays a very important role in this case as it affects the overall performance of the classification. The RGB colour channels and the gradient direction of the samples were used as the feature set for classification. The resultant fence mask can be further improved with the help of morphological operations.

The algorithm to achieve fence-like quasi-periodic texture detection in digital images is given in Algorithm 1.

Algorithm 1 Algorithm for fence-like quasi-periodic texture detection in images

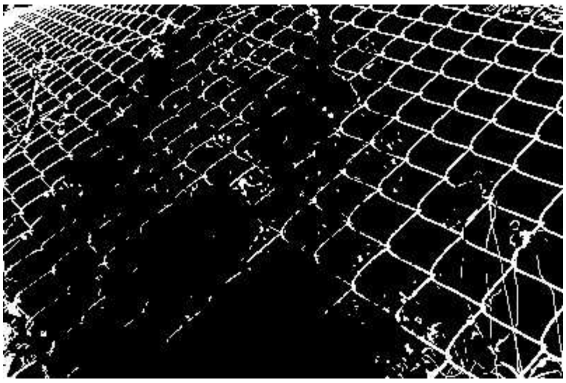
1: Read the fenced image I
2: Convert I into frequency domain using Discrete Fourier Transform (let the output be F)
3: Filter F using the peak frequency filter H_1 defined in equation 3.2 (let the output be $F1$)
4: Filter $F1$ using the band pass filter H_2 defined in equation 3.3 (let the output be $F2$)
5: Convert $F2$ back into spatial domain (let the output be *filtI*)
6: Perform Wavelet decomposition on *filtI* with three decomposition levels
7: **for** each Wavelet decomposition level **do**
8:     Find vertical (V), horizontal (H) and Diagonal (D) components
9:     Threshold V, H and D with the same threshold
10:     Combine thresholded V, H and D components using logical OR operation
11: **end for**
▹ %comment: Obtain fence mask by combining all three levels of the wavelet pyramid (let the output be fenceMask)%
12: Start from the highest Wavelet decomposition level (level 3)
13: **for** each pixel in level 3 **do**
14:     **if** a pixel belongs to the mask **then**
15:         Move to next lower level
16:         **if** current level == lowest level **then**
17:             Mark the pixel as mask pixels
18:             Mark the neighbouring children as mask pixels
19:         **else**
20:             Check neighbouring children
21:             **if** neighbouring children are mask pixels **then**
22:                 Go back to step 14
23:             **end if**
24:         **end if**
25:     **end if**
26: **end for**
27: Prepare the training data matrix using feature vectors of sample pixels fall on fence (fenceMask==1) and background (fenceMask==0).
28: Train the SVM classifier by using training data matrix of step 25.
29: Perform SVM classification by using the trained classifier in step 26 by giving original image as the input to obtain final fence mask.

(a) Child Image.



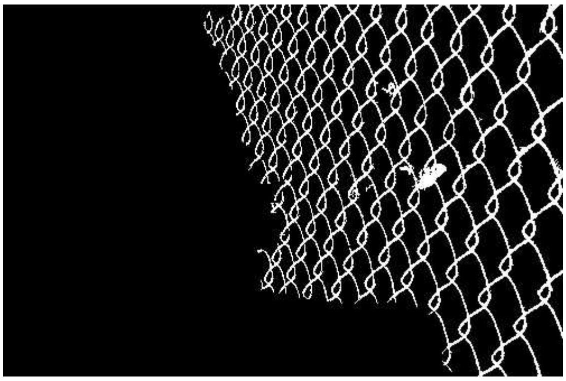
(b) Fence Mask for Child Image.

(c) Building Image.



(d) Fence Mask for Building Image.

(e) Flower Image.



(f) Fence Mask for Flower Image.

Figure 6. Results of Fence-like Texture Detection in Images from PSU NRT Database (Liu, 2007).

4. Experimental Results

The frequency domain-based fence-like quasi-periodic texture detection algorithm proposed in this article was implemented in Matlab R2013a and it was tested with a number of images with fence-like texture. Some test images were obtained from PSU Near-regular Texture database (Liu, 2007). Images with fences of different shapes (square and diagonal), sizes, colours and orientations were used for this experiment. Figure 6 illustrates results of some of the challenging cases encountered during experiments.

For the completion of the sample application chosen in this paper, once the fence texture was successfully detected and removed, the region, which belonged to the fence, should be filled with relevant information in order to obtain the final image. One of the techniques, which can serve this purpose is *inpainting*. According to (Bertalmio *et al.*, 2000), inpainting is the *modification of images in a way that is non-detectable for an observer who does not know the original image*. There are numerous inpainting techniques introduced in past literature.

For examples region filling and object removal by exemplar-based image inpainting by Criminisi et al. (Criminisi *et al.*, 2004), Fields of experts by Roth et al. (Roth & Black, 2009) and Image completion with structure propagation by Sun et al. (Sun *et al.*, 2005). Among these techniques, the exemplar based image inpainting technique (Criminisi *et al.*, 2004) was used to fill the fence region in this approach. The results are given in figure 7.

Interestingly, some image distortions can be observed after performing inpainting for some images. The region belonged to the fence texture is much more difficult to texture fill than large, circular regions of similar area. The fence texture in this case is usually wide spread in the whole image. Thus, it requires the inpainting algorithm to correctly propagate and join different types of structures in order to fill this wide spread fence region. Hence, mistakes in structure propagation can be quiet frequent in this case. The high ratio of foreground area to background area and the fragmented background source textures may become challenging for the inpainting technique.

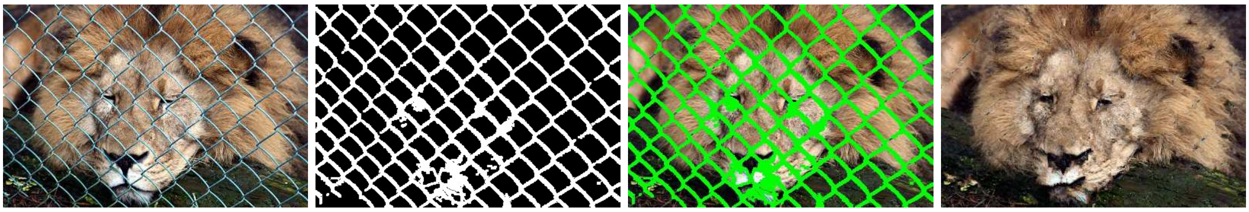
5. Comparison with Existing Fence Detection Techniques

Most of the articles, which investigated the image de-fencing problem, have used a texture based approach to detect the fence, based on the assumption that a fence is a near regular structure. (Liuy *et al.*, 2008) introduced an image de-fencing technique based on lattice structure of regular textures in their article. The de-fencing algorithm proposed in (Liuy *et al.*, 2008) consists of three steps. *(1) automatically finding the skeleton structure of a potential frontal layer in the form of a deformed lattice; (2) classifying pixels as foreground or background using appearance regularity as the dominant cue, and (3) inpainting the foreground regions using the background texture which is typically composed of fragmented source regions to reveal a complete, non-occluded image* (Liuy *et al.*, 2008).

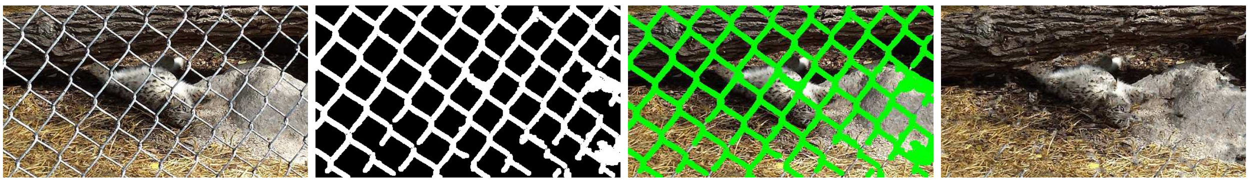
In the first step, to automatically detect the lattice of the fence, (Liuy *et al.*, 2008) uses the iterative algorithm explained in (Hays *et al.*, 2006), which tries to find the most regular lattice for a given image by assigning the neighbour relationships such that neighbors have maximum visual similarity. Step one results in a mesh of quadratiles, which contains repeated elements or texels. In the second step standard deviation of each colour channel and the color features are used for k-means clustering for background foreground separation. In order to obtain the standard deviation,

(a) Lynx. (b) Fence Mask. (c) Fence Mask Applied Image. (d) Inpainted Image.

(e) Puma. (f) Fence Mask. (g) Fence Mask Applied Image. (h) Inpainted Image.



(i) Lion (Schneider, 2010). (j) Fence Mask. (k) Fence Mask Applied Image. (l) Inpainted Image.



(m) Leopard. (n) Fence Mask. (o) Fence Mask Applied Image. (p) Inpainted Image.

(q) Puppies (Liu, 2007). (r) Fence Mask. (s) Fence Mask Applied Image. (t) Inpainted Image.

Figure 7. Results of Fence Removal from Zoo Images.

the texels were aligned and arranged in a stack and standard deviation is calculated along each vertical column of pixels. Finally, texture based inpainting technique introduced by Criminisi et al. (Criminisi *et al.*, 2004, 2003) is used to obtain the final de-fenced image.

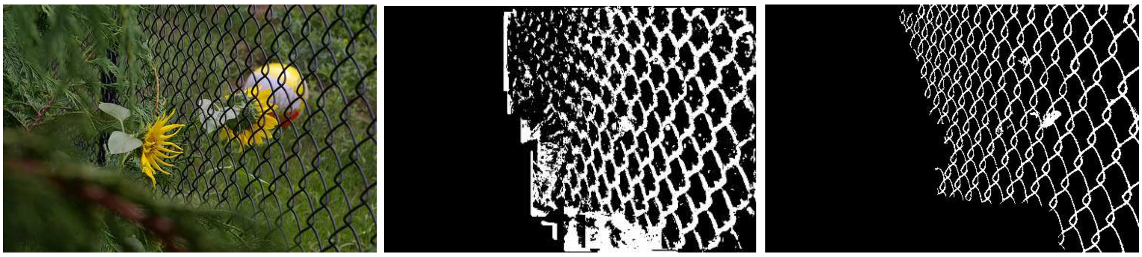
Park et al. revisits the image de-fencing problem in their paper (Park *et al.*, 2011). They no longer uses the lattice detection algorithm introduced in (Hays *et al.*, 2006), as they states its performance is far from practical due to inaccuracy and slowness. Rather the implementation of lattice detection algorithm in (Park *et al.*, 2011) is similar to (Park *et al.*, 2009). In their method, once the type of the repeating pattern is learnt, the irregularities are removed and the learned regularity is used in evaluating the foreground appearance likelihood during the lattice growth. They have improved the lattice detection algorithm by introducing an online learning and classification.

In essence, the de-fencing algorithms introduced in both of these articles uses a lattice detection algorithm in order to find the fence mask. Thus, the success of both algorithms depends on finding the repeated element or texel in the fence structure. The lattice detection algorithm used by (Liuy *et al.*, 2008) has no measures against irregularities in the lattice while the lattice detection algorithm used by (Park *et al.*, 2011) takes some measures to remove irregularities during lattice growth. However, both these approaches depend on the regularity of the fence as well as the irregularity of the background of the image. Although (Park *et al.*, 2011) takes measures against irregularities in the fence, it does not take in to account the possibility of regularities in the background. Furthermore, the lattice detection process itself is very complex and time consuming.

In contrast to the two methods discussed above, the method explained in this article uses a frequency domain approach to address the fence detection problem. Due to the uncertainty principle, the global wide spread fence texture in spatial domain becomes local to a set of frequencies in the frequency domain. So the processing required to extract the fence texture in frequency domain is simpler and faster compared to spatial domain processing. This becomes advantageous in the proposed method compared to the existing techniques. Moreover, the band pass filtering in frequency domain used in the proposed method helps to avoid other periodic structures (regularities) in the background, which is not possible in existing techniques. The proposed method is robust against deformations and irregularities in the fence texture due to SVM classification used in fence segmentation phase.

The existing near regular lattice detection approaches work well for some images and on the other hand fail for some cases. They have observed that the failure cases are often accompanied by sudden changes of colors in the background and obscuring objects in front of the fence. For examples in (Liuy *et al.*, 2008) method, the lattice detection fails for images (a) and (c) in Figure 6 and for image (q) in Figure 7. The proposed method is successful in detecting fence texture in all those images. A comparison of fence mask detected in Flower image by (Liuy *et al.*, 2008) method and proposed method is given in Figure 8.

However, the proposed method fails to provide satisfactory results for blurred images, especially when the fence is very much blurred. In such cases preprocessing to sharpen the fence may give better results. Furthermore, fence segmentation becomes challenging when the visual similarity between fence pixels and background pixels becomes high. Feature set used for segmentation has to be tuned to overcome such problems. Determining the correct feature set is challenging in such scenarios.



(a) Flower Image. (b) (Liuy *et al.*, 2008). (c) Proposed Method.

Figure 8. Comparison of Fence Mask Detected for Flower Image.

6. Future Work

Fence texture segmentation becomes challenging, when there are pixels with features similar to fence pixels in the background. SVM classification used for final segmentation of the fence texture in this article can be replaced with descriptive motif pattern generation described in (Peters & Hettiarachichi, 2013). The accuracy of this phase can be further improved with help of near set theory (Peters, 2013; Peters & Naimpally, 2012; Peters, 2014; Peters *et al.*, 2014).

7. Conclusion

Fence-like texture present in the foreground of the image occludes the points of interest in an image and is difficult to segment by directly applying conventional frequency filters used for texture analysis. The proposed approach in this article segregates each fence texture by frequency domain processing prior to wavelet transformation and the segmentation is achieved through support vector machine classification.

The proposed method works well for fence texture with different shapes, sizes, colours and orientations. Fence texture detection was successful not only for images having fence in the foreground but also for images having fence in the background.

As a sample application of the proposed approach, removal of fences from zoo animal enclosure images is presented. In addition to this, the proposed approach to de-fencing can be used for any application, where the images are occluded with fence-like texture.

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Counting Sets of Lattice Points in the Plane with a Given Diameter under the Manhattan and Chebyshev Distances

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Abstract

In this paper we present new algorithms for counting the sets of lattice points in the plane whose diameter is a given value D, under the Manhattan (L_1) and Chebyshev (L_∞) distances. We consider two versions of the problem: counting all sets within a given lattice $U \times V$, and counting all sets that are not equivalent under translations.

Keywords: lattice points, Chebyshev distance, Manhattan distance, diameter.
2010 MSC: 52Cxx, 11P21.

1. Introduction

In this paper we present new algorithms for counting the sets of lattice points in the plane with a given diameter, under the Manhattan (L_1) and Chebyshev (L_∞) distances. We consider two versions of the problem. In the first version we assume that a fixed size 2D grid is given and the sets must be placed inside this grid. Two sets are different if they have a different number of points or the positions of their points inside the grid are not all identical. In the second version we assume that two sets are considered identical (and, thus, need to be counted only once) if one can be obtained from another by translation operations.

The rest of this paper is structured as follows. In Section 2 we present the problems in more details, together with some preliminaries required by the algorithms presented in the other sections. In Section 3 we present an algorithm with $O(D \cdot log(D))$ arithmetic operations for the Chebyshev (L_∞) distance which can solve both versions of the problem. In Section 4 we present a more efficient algorithm, with only $O(log(D))$ arithmetic operations, for the Chebyshev distance, but only for the second version of the problem. In Sections 5 and 6 we present algorithms with a similar number of arithmetic operations for the Manhattan distance and for the same versions of

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the problem. In Section 7 we present experimental results regarding the two algorithms for the Manhattan distance. In Section 8 we discuss related work. In Section 9 we conclude and discuss future work.

2. Problem Statement and Preliminaries

In this paper we consider sets of lattice points in the plane. A lattice point is a point with integer coordinates. The diameter of a set of points is the maximum distance between any two points in the set. In this paper we will consider two distances. The Manhattan distance (also called the L_1 distance) between two points (x_1, y_1) and (x_2, y_2) is defined as $|x_1 - x_2| + |y_1 - y_2|$. The L_∞ distance (also called the Chebyshev distance) between two points (x_1, y_1) and (x_2, y_2) is defined as $max\{|x_1 - x_2|, |y_1 - y_2|\}$. When the coordinates of the points are integer (i.e. when we consider only lattice points) both the L_1 and the L_∞ distances are integers.

We consider two versions of the problem for counting sets of lattice points having exactly a given diameter D (under the L_1 or L_∞ distances). The first version assumes that a 2D grid of fixed size $U \times V$ is given (U is the number of points along the OX axis and V is the number of points along the OY axis). We may assume that the points of the grid have coordinates (x, y) with $0 \leq x \leq U - 1$ and $0 \leq y \leq V - 1$. In this case two sets of points are considered different if they consist of a different number of points or if the positions of their points are not all identical. The dimensions of the grid (U and V) are part of the input of the algorithms presented for this version.

For the second version we assume that two sets A and B are identical if one can be obtained from another by translation operations. To be more precise, set A is considered identical to B if there exist the integer numbers TX and TY such that by adding TX to the x-coordinate of each point of A and TY to the y-coordinate of each point of A we obtain exactly the set B (note that this automatically implies that A and B have the same number of points). In this case the sets are not constrained to be located within a fixed size grid, so the parameters U and V from the first version of the problem do not exist here.

We are interested in computing the number of sets of points with a given value of the diameter D ($D \geq 1$) under both versions of the problem and considering either the L_1 or the L_∞ distance. Let's consider, for instance, the second version of the problem. For $D = 1$ there are two sets of lattice points for the Manhattan distance, each consisting of two adjacent lattice points. In the first set the two points are horizontally adjacent and in the second set the two points are vertically adjacent. On the other hand, there are 9 sets of lattice points for $D = 1$ and the Chebyshev distance.

In order for a set of points in the plane to have diameter D under the L_∞ distance all the points must be located inside a square of side length D and for at least one pair of opposite sides there must be at least one point from the set located on each of the two sides.

The diameter of a set of points A under the Manhattan distance is equivalent to the diameter under the L_∞ distance of a modified set of points B (Indyk, 2001). B is obtained by transforming each point (x, y) of A into the point $(x - y, x + y)$ in B. Thus, the two problems considered in this paper are strongly connected to each other. The transformed coordinates correspond to *diagonal coordinates*.

In a 2D plane we have two types of diagonals: *main diagonals* (running from north-east to south-west) and *secondary diagonals* (running from north-west to south-east). All the points (x, y)

on the same *main diagonal* have the same value of $x - y$ and all the points (x, y) on the same *secondary diagonal* have the same value of $x + y$. The *index of a main diagonal* is the difference $x - y$ of all the points (x, y) on it. Similarly, the *index of a secondary diagonal* is the sum $x + y$ of all the points (x, y) on it. The *parity* of a diagonal (main or secondary) is defined as the parity of its index. The distance between two diagonals of the same type (main or secondary) is defined as the absolute difference of their indices.

After the transformation to *diagonal coordinates* we can easily see that in order for a set of points to have exactly diameter D under the Manhattan distance one of the following conditions must hold:

1. It should have at least two points located on main diagonals at distance D apart, while the other pairs of main diagonals of all the points are located at distance at most D apart and the pairs of secondary diagonals of all the points are at distance strictly less than D or
2. It should have at least two points located on secondary diagonals at distance D apart while the other pairs of secondary diagonals of all the points are located at distance at most D apart and the pairs of main diagonals of all the points are at distance strictly less than D or
3. It should have at least two points located on main diagonals at distance D apart and at least two points located on secondary diagonals at distance D apart and all the other pairs of main and secondary diagonals of the points are at distance at most D apart.

The three cases correspond to different types of sets of points (i.e. each set of points having diameter D under the Manhattan distance belongs to exactly one of the three cases). Note that there is a bijection between the sets of points corresponding to cases 1 and 2. Each set of points corresponding to case 1 can be transformed into a set of points corresponding to case 2 (by switching the order of the diagonals). Similarly, each set of points corresponding to case 2 can be transformed into a set of points corresponding to case 1.

Thus, for the second version of the problem, it will be enough to count the number of sets of points corresponding to case 1 (C_1) and the number of sets of points corresponding to case 3 (C_3) in order to obtain the total number of sets of points having diameter D under the Manhattan distance. Then, the total number of sets of lattice points having diameter D (under the Manhattan distance) is equal to $2 \cdot C_1 + C_3$.

3. Algorithm 1 for Counting Sets of Lattice Points of Diameter D under the Chebyshev (L_∞) Distance

In this section we will present an algorithm which computes the number of sets of lattice points having diameter D under the Chebyshev distance for both versions of the problem. The algorithm will make use of a function denoted by $CNTSETS(LX, LY)$ which will compute the number of sets of lattice points contained in a rectangle having side length LX along the OX axis and side length LY along the OY axis and such that each counted set has at least one point on each of the 4 sides of the rectangle. Moreover, the corners of the rectangle are lattice points.

We will start with some simple cases. We have $CNTSETS(0, 0) = 1$ and $CNTSETS(P, 0) = CNTSETS(0, P) = 2^{P-1}$. When both LX and LY are greater than or equal to 1 we will use the following approach. We will first identify the 4 corners of the rectangle. We will consider each of the 2^4 binary configurations of 4 bits. Let BC denote the current binary configuration and $BC(i)$ will denote bit i in the configuration ($0 \leq i \leq 3$). Each bit will correspond to one of the 4 corners. If $BC(i)$ is 1 we will assume that the corresponding corner is selected to be part of the set; otherwise we will assume that it is not selected. Lets consider now each of the horizontal sides of the rectangle. If at least one corner located on the considered side was selected, then there are 2^{LX-1} possibilities left for selecting the remaining points (non-corners) of the horizontal side (note that the side contains $LX + 1$ points overall, out of which 2 are corners). If none of the corners of the side are selected, then there are only $2^{LX-1} - 1$ possibilites left for selecting the remaining points of the horizontal side. The situation is similar for the vertical sides: if at least one corner is selected from a vertical side, there are 2^{LY-1} possibilities of selecting the remaining points of the vertical side; otherwise the number of possibilities is only $2^{LY-1} - 1$. After considering all the 4 sides of the rectangle we need to consider the points located strictly inside the rectangle. There are $NIN = (LX - 1) \cdot (LY - 1)$ points located strictly inside the rectangle. Each of these inner points may be selected or not, meaning that there are 2^{NIN} possibilities of selecting these points. For a given binary configuration BC the number of possibilities of selecting points according to it is equal to the product of five terms: four of which are the number of possibilities corresponding to each of the 4 sides of the rectangle and the 5^{th} term corresponds to the number of possibilities of selecting the inner points of the rectangle. The value returned by $CNTSETS(LX, LY)$ is equal to the sum of the numbers of possibilities of selecting points corresponding to each of the 2^4 binary configurations.

We will use a variable C ranging from 0 to D. For each value of C we will first compute $CNTSETS(D, C)$. Note that this value corresponds to the number of sets of lattice points having diameter D and which are contained in a minimum bounding rectangle of side lengths D (along the OX axis) and C (along the OY axis). For the first version of the problem, each set counted by $CNTSETS(C, D)$ may appear multiple times inside the grid - in fact, it may appear $(U-D)\cdot(V-C)$ times (as that's the number of possibilities of placing a $D \cdot C$ rectangle inside the grid). Thus, we will add the term $CNTSETS(D, C) \cdot (U - D) \cdot (V - C)$ to the final answer for the first version of the problem (or 0, if $D > U$ or $C > V$). In the second version of the problem we simply need to add $CNTSETS(D, C)$ to the final answer for the second version of the problem. This is because all the sets counted by $CNTSETS(D, C)$ are different under translation operations.

If $C < D$ we will also compute $CNTSETS(C, D)$ (which is identical in value to $CNTSETS(D, C)$). For the first version of the problem we will add to the final answer the value $CNTSETS(C, D) \cdot (U - C) \cdot (V - D)$ (or 0, if $C > U$ or $D > V$). For the second version of the problem we will add to the final answer the value $CNTSETS(C, D)$.

The algorithm presented in this section uses $O(D \cdot log(D))$ arithmetic operations, because it considers $O(D)$ cases and for each case it needs to perform a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$. All the exponentiations raise 2 to a given exponent. If D is not very large we may consider precomputing all the powers of 2 from 0 to D (we may achieve this with only $O(D)$ multiplications because we can write $2^P = 2^{P-1} \cdot 2$ for $P \geq 1$ and we can consider the values of P in ascending order). However, NIN is of the order $O(D^2)$. If D is sufficiently small then we may precompute powers of 2 up to D^2 (using $O(D^2)$ multiplications). If, however, D^2 is too large, then we need to notice that, as C increases from 0 to D, NIN also increases. We will assume that our algorithm considers the values of C in ascending order (note that NIN is the same for both $CNTSETS(D, C)$ and $CNTSETS(C, D)$). Let's assume that $PREVNIN$ is equal to the value of NIN for the case $C-1$ and $RESPREVNIN = 2^{PREVNIN}$. We will initially (for $C = 0$) have $PREVNIN = 0$ and $RESPREVNIN = 1$. When we need to compute 2^{NIN} for a case we will first compute the difference $DIFNIN = NIN - PREVNIN$. We will always have $DIFNIN = D - 1$. Thus, we can compute 2^{NIN} with only one multiplication, as $RESPREVNIN \cdot 2^{DIFNIN}$ (note that 2^{DIFNIN} is taken from the table of precomputed powers of two). After handling the current value of C we will update $PREVNIN = NIN$ and $RESPREVNIN = 2^{NIN}$ (where 2^{NIN} was just computed by the method we presented). Using this approach we only need $O(D)$ arithmetic operations instead of $O(D \cdot log(D))$.

So far we assumed that we want to compute the number of sets of lattice points exactly. In this case we will need to work with numbers which have $O(4 \cdot D)$ bits. However, there are many situations when the exact numbers are not required. For instance, if we are only interested in computing the number of sets modulo a given number M, then we only need numbers having $O(2 \cdot log(M))$ bits for storing intermediate and final results. If M is sufficiently small (e.g. a 32-bit number) then we can practically assume that on the current machine architectures the numbers we use have a constant number of bits. However, the exponents to which 2 is raised can still be pretty large numbers (having $O(log(D))$ bits). This may not necessarily be a problem, but we may inadvertently face some challenging algorithmic problems. For instance, when multiplying $(LX - 1)$ by $(LY - 1)$ in the $CNTSETS$ function we need to multiply together two numbers having $O(log(D))$ bits. The naive algorithm would use $O(log^2(D))$ time for computing the result. In order to speed up the multiplication we may need to use more complicated algorithms (Schonhage & Strassen, 1971), (Furer, 2009) which reduce the time complexity to $O(log(D) \cdot log(log(D)) \cdot log(log(log(D))))$ or slightly better. Nevertheless, there is a simple situation when all these complications are not needed: when M is an odd prime. In this case we know that $A^{M-1} = 1$ (modulo M) for any natural number $1 \leq A \leq M - 1$. Since we only need to raise 2 at some powers (modulo M), we notice that we only need the remainder of the exponent when divided by $M - 1$ in order to compute the required result. Thus, instead of using exact exponents we will only use the exponents modulo $M - 1$. This way we can avoid the complicated multiplication of $(LX - 1)$ by $(LY - 1)$ and replace it with the multiplication of *((LX-1) mod (M-1))* by *((LY-1) mod (M-1))*. This way we will need to spend $O(log(D))$ time in order to compute the remainders of numbers having $O(log(D))$ bits when divided by $M - 1$, but we do not need to multiply together two large numbers.

4. Algorithm 2 for Counting Sets of Lattice Points of Diameter D under the Chebyshev (L_∞) Distance

The algorithm presented in this section can only solve the second version of the problem (i.e. when two sets are identical if one can be obtained from another by using translation operations). We will first define the following function: $NSETS(LX, LY)$=the number of sets of lattice points contained in a rectangle of horizontal side length LX and vertical side length LY such that at least one point is located on each of the opposite vertical sides (for this function we will ignore the fact the two sets are identical if one can be obtained from another by translation operations). We assume $LX \geq 1$ and $LY \geq 0$, both numbers are integers and the corners of the rectangle are lattice points. Such a rectangle contains $(LX + 1) \cdot (LY + 1)$ lattice points inside of it or on its borders. It is easy to see that $NSETS(LX, LY) = (2^{LY+1} - 1)^2 \cdot 2^{(LX+1)\cdot(LY+1)-2\cdot(LY+1)}$. This formula corresponds to the following cases. On each of the two opposite vertical sides we must have one selected point. Thus, there are $2^{LY+1} - 1$ possibilities of choosing lattice points on each of these two sides. Each of the remaining $(LX + 1) \cdot (LY + 1) - 2 \cdot (LY + 1)$ lattice points can be selected or not to be part of the set. Thus, we have $2^{(LX+1)\cdot(LY+1)-2\cdot(LY+1)}$ possibilities for selecting these points. If $LY < 0$, by definition, we will have $NSETS(LX, LY) = 0$.

In order for a set of points in the plane to have diameter D under the Chebyshev distance all the points must be located inside a square of side length D, such that at least one pair of opposite sides has at least one point from the set on each side from the pair. We will consider three cases:

1. both of the vertical opposite sides of the square contain points from the set on them, but not both horizontal sides of the square contain points from the set: the number of sets corresponding to this case is $NSETS(D, D-1) - NSETS(D, D-2)$ (this forces every set to have a point selected on the bottom side of the square of side length D)
2. both of the horizontal opposite sides of the square contain points from the set on them, but not both vertical sides of the square contain points from the set: the number of sets corresponding to this case is also $NSETS(D, D - 1) - NSETS(D, D - 2)$.
3. both of the horizontal opposite sides and both of the vertical opposite sides of the square contain points from the set on them: the number of sets corresponding to this case is $NSETS(D, D) - 2 \cdot NSETS(D, D - 1) + NSETS(D, D - 2)$. We actually made use of the inclusion-exclusion principle here. From all the sets of lattice points with points on both opposite vertical sides ($NSETS(D, D)$) we subtracted the sets of lattice points which do not have points on the top or bottom horizontal side ($2 \cdot NSETS(D, D - 1)$). In doing this we over-subtracted the sets of lattice points which do not have points on any of the horizontal sides ($NSETS(D, D - 2)$) thus, we need to add this number back.

By adding together the numbers corresponding to the cases 1, 2 and 3, we obtain the total number of sets of lattice points having diameter D under the Chebyshev distance: $2 \cdot (NSETS(D, D - 1) - NSETS(D, D - 2)) + NSETS(D, D) - 2 \cdot NSETS(D, D - 1) + NSETS(D, D - 2)$, which simplifies to $NSETS(D, D) - NSETS(D, D - 2)$.

This method requires $O(log(D^2)) = O(log(D))$ arithmetic operations in order to compute the answer (this number corresponds to raising 2 to a power whose value is of the order $O(D^2)$). In case exact results are not needed, the same discussion from the previous section applies to this case, too, because in the $NSETS$ function we need to multiply two numbers of $O(log(D))$ bits each: $(LX + 1)$ and $(LY + 1)$.

5. Algorithm 1 for Counting Sets of Lattice Points of Diameter D under the Manhattan (L_1) Distance

In this section we present an algorithm similar in essence to the one from section 3. The algorithm can compute the number of sets of lattice points having diameter D under the Manhattan distance for both versions of the problem. The algorithm will make use of a function $CNTEQ(C, X)$ which computes the number of sets of lattice points such that:

- the main diagonals of at least two points are at distance exactly D apart

- all the other pairs of main diagonals of the points are at distance at most D apart
- the secondary diagonals of at least two points are at distance exactly C apart
- all the other pairs of secondary diagonals of the points are at distance at most C apart
- $X = 0$ means that the parity of the first secondary diagonal is equal to the parity of the first main diagonal, while $X = 1$ means that these parities differ (the first diagonal of each type is the one with the smallest index)

The algorithm will simply iterate through all the values of C (from 0 to D), and for each value of C, through all the values of X (from 0 to 1).

For the first version of the problem we will need to compute the minimum bounding rectangle for the sets counted by $CNTEQ(C, X)$ ($X = 0, 1$). Let's assume that the minimum bounding rectangle has side length $MBRX$ along the OX axis and $MBRY$ along the OY axis. We will add to the final answer the value $CNTEQ(C, X) \cdot (U - MBRX) \cdot (V - MBRY)$ ($X = 0, 1$), or 0 if $MBRX > U$ or $MBRY > V$. If $C < D$ then we have a set of symmetric sets of lattice points by switching the role of main and secondary diagonals. These sets have a minimum bounding rectangle with side length $MBRY$ along the OX axis and $MBRX$ along the OY axis. Thus, we will also add to the final answer the value $CNTEQ(C, X) \cdot (U - MBRY) \cdot (V - MBRX)$ ($X = 0, 1$), or 0 if $MBRX > V$ or $MBRY > U$.

For the second version of the problem C_1 will be equal to the sum of the values $CNTEQ(C, X)$ ($0 \leq C \leq D - 1$, $0 \leq X \leq 1$) and C_3 will be equal to $CNTEQ(D, 0) + CNTEQ(D, 1)$.

When computing $CNTEQ(C, *)$, we need to consider a figure containing lattice points enclosed by a pair of main diagonals at distance D and a pair of secondary diagonals at distance C. We will denote the first main diagonal as the *left* diagonal, the second main diagonal as the *right* diagonal, the first secondary diagonal as the *bottom* diagonal and the second secondary diagonal as the *top* diagonal. We will need to compute the following numbers:

- NLEFT=the number of lattice points on the left diagonal of the figure
- NRIGHT=the number of lattice points on the right diagonal of the figure
- NUP=the number of lattice points on the top diagonal of the figure
- NDOWN=the number of lattice points on the bottom diagonal of the figure
- NTOTAL=the total number of lattice points inside the figure and on its borders

Then, we will need to identify the corners of the figure. A corner is a lattice point which belongs to two adjacent diagonals (a main diagonal and a secondary diagonal). Note that we may have 0, 2 or 4 corners. Let's assume that we have NC corners. We will make sure to decrease the corresponding numbers ($NLEFT$, $NRIGHT$, NUP, $NDOWN$) by the number of corners among the set of lattice points which were counted (e.g. if the left diagonal has Q corners on it, we will decrease $NLEFT$ by Q).

In Fig. 1, 2, 3, 4 we present all the cases which may occur during the computation of the $CNTEQ(C, X)$ function (the remaining cases are reducible to these 4 cases by symmetry). Lattice points on the main diagonals are drawn in green, lattice points on the secondary diagonals are drawn in red, corners are drawn in cyan and inner lattice points are drawn in yellow. In Fig. 1 we have $D = 10$, $C = 8$ and $X = 0$. Notice that we obtain $NC = 4$ corners. Note that two adjacent diagonals form a corner if they have the same parity. In Fig. 2 we have $D = 10$, $C = 7$ and $X = 0$. In this case we obtain only $NC = 2$ corners. This is because the top diagonal has a different parity from both the left and the right diagonals, thus forming no corners with them. In Fig. 3 we have $D = 9$, $C = 7$ and $X = 0$; we obtain $NC = 2$ corners. In Fig. 4 we have $D = 12$, $C = 8$ and $X = 1$; no corner is formed in this case.

The main algorithm for computing $CNTEQ(C, X)$ is as follows. We will consider each possible binary configuration of NC bits. If bit i ($0 \leq i \leq NC - 1$) is set to 1 we will assume that the corresponding corner (i) belongs to the set of lattice points; otherwise, it doesn't belong to the set. After deciding the states of the corners we will check which of the first and second main and secondary diagonals have no selected corners on them. For each such diagonal we will have $2^{NP} - 1$ possibilities of choosing lattice points on it (where NP is the number of lattice points on it, excluding the corners). This equation makes sure that at least one lattice point is selected on each such diagonal. For each of the other diagonals we will have 2^{NP} possibilities of choosing lattice points on them (for these diagonals it is possible to not select any of the lattice points on them, because they already have a selected corner). Then each of the interior lattice points of the figure can be selected as part of the set or not (there are $NIN = NTOTAL - (NUP + NDOWN + NLEFT + NRIGHT + NC)$ lattice points inside the figure and, thus, there are 2^{NIN} possibilities of choosing the inner points). The answer for each binary configuration of the corners is the product between the number of possibilities for each of the 4 diagonals and for the inner points of the figure. $CNTEQ(C, X)$ is the sum of all the answers for each binary configuration of corners. Note that this algorithm works even when $NC = 0$ (there is one binary configuration of 0 bits).

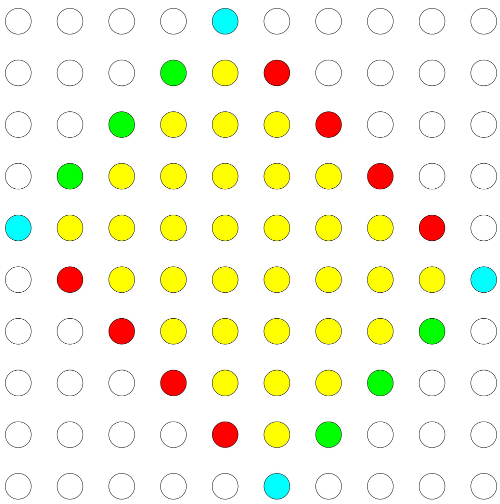

Figure 1. D=10, C=8, X=0, NC=4.

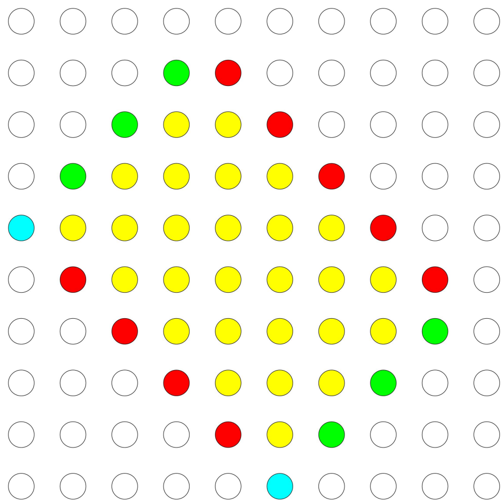

Figure 2. D=10, C=7, X=0, NC=2.

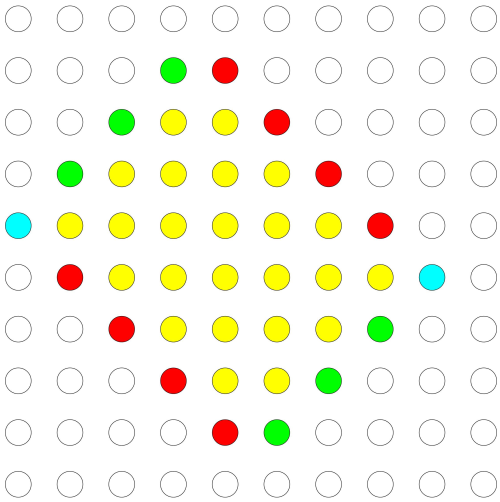

Figure 3. D=9, C=7, X=0, NC=2.

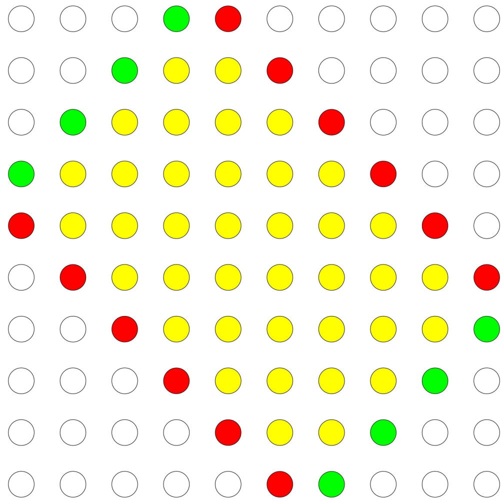

Figure 4. D=12, C=8, X=1, NC=0.

What is left is to identify the values NUP, $NDOWN$, $NLEFT$, $NRIGHT$, $NTOTAL$ and the corners depending on the values of C, X and the parity of D. We will also define the parameter Y, which is defined similarly as X, but for the second secondary diagonal (i.e. $Y = 0$ if the second secondary diagonal has the same parity as the first main diagonal, and $Y = 1$ otherwise). Note that $Y = X$ if C is even, and $Y = 1 - X$ if C is odd. From now on we will assume that the value of Y is computed when evaluating the function $CNTEQ(C, X)$.

We will first consider the case when D is even. If $C = 0$ and $X = 0$ then $CNTEQ(0, 0) = 2^{D/2-1}$. If $C = 0$ and $X = 1$ then $CNTEQ(0, 1) = 0$. Let's consider now that $C \geq 1$. If $X = 0$ then $NDOWN = (D/2) + 1$ and if $X = 1$ then $NDOWN = D/2$. Note that whenever we use the division operator "/" in this paper we refer to integer division. Similarly, if $Y = 0$ then $NUP = (D/2) + 1$, and if $Y = 1$ then $NUP = D/2$. If $X = 0$ then we have $NLEFT = NRIGHT = (C/2) + 1$; otherwise, if $X = 1$ then we have $NLEFT = NRIGHT = (C + 1)/2$. $NTOTAL$ is equal to $NLEFT \cdot ((D/2) + 1) + (C + 1 - NLEFT) \cdot (D/2)$.

If D is odd then we have the following values. $NUP = NDOWN = (D + 1)/2$. If $X = 0$ then $NLEFT = (C/2) + 1$ and $NRIGHT = (C + 1)/2$; otherwise, if $X = 1$ then $NLEFT = (C + 1)/2$ and $NRIGHT = (C/2) + 1$. $NTOTAL$ is equal to $(C + 1) \cdot ((D + 1)/2)$.

The exact formulas we presented for NUP, $NDOWN$, $NLEFT$, $NRIGHT$ and $NTOTAL$ can be easily derived by a careful analysis of all the relevant cases. Let's consider now the cases from Fig. 1, 2, 3, 4 and verify the formulas for those cases. In Fig. 1 we have $NLEFT = (4/2) + 1 = 5$, $NRIGHT = (4/2) + 1 = 5$, $NDOWN = (10/2) + 1 = 6$, $NUP = (10/2) + 1 = 6$ and $NTOTAL = 5 \cdot ((10/2) + 1) + (8 + 1 - 5) \cdot (10/2) = 50$. In Fig. 2 we have $NLEFT = (7/2) + 1 = 4$, $NRIGHT = (7/2) + 1 = 4$, $NDOWN = (10/2) + 1 = 6$, $NUP = 10/2 = 5$ and $NTOTAL = 4 \cdot ((10/2) + 1) + (7 + 1 - 4) \cdot (10/2) = 44$. In Fig. 3 we have $NLEFT = (7/2) + 1 = 4$, $NRIGHT = (7 + 1)/2 = 4$, $NDOWN = (9 + 1)/2 = 5$, $NUP = (9 + 1)/2 = 5$ and $NTOTAL = (7 + 1) \cdot ((9 + 1)/2) = 40$. In Fig. 4 we have $NLEFT = (8 + 1)/2 = 4$, $NRIGHT = (8 + 1)/2 = 4$, $NDOWN = 12/2 = 6$, $NUP = 12/2 = 6$ and $NTOTAL = 4 \cdot ((12/2) + 1) + (8 + 1 - 4) \cdot (12/2) = 58$.

We will show now how to compute the sizes $MBRX$ and $MBRY$ of the minimum bounding rectangle corresponding to the sets counted by $CNTEQ(C, X)$ ($X = 0, 1$). $MBRX = NLEFT - 1 + NUP - 1 + Y$ and $MBRY = NLEFT - 1 + NDOWN - 1 + X$. Let's verify now these formulas for the cases presented in Fig. 1, 2, 3, 4. In Fig. 1 we have $MBRX = 5 - 1 + 6 - 1 + 0 = 9$ and $MBRY = 5 - 1 + 6 - 1 + 0 = 9$. In Fig. 2 we have $MBRX = 4 - 1 + 5 - 1 + 1 = 8$ and $MBRY = 4 - 1 + 6 - 1 + 0 = 8$. In Fig. 3 we have $MBRX = 4 - 1 + 5 - 1 + 1 = 8$ and $MBRY = 4 - 1 + 5 - 1 + 0 = 7$. In Fig. 4 we have $MBRX = 4 - 1 + 6 - 1 + 1 = 9$ and $MBRY = 4 - 1 + 6 - 1 + 1 = 9$.

After initializing the NUP, $NDOWN$, $NLEFT$, $NRIGHT$ and $NTOTAL$ values, we need to identify the corners. We will consider each pair of (main diagonal, secondary diagonal) and check if they have the same parity (note that the parity of each main and secondary diagonal can be uniquely determined relative to the parity of the first main diagonal from the values X, Y and D; for instance, if $X = 0$ the first main diagonal and the first secondary diagonal have the same parity, if $Y = 0$ the first main diagonal and the second secondary diagonal have the same parity, if ($X = 0$) and (D is even) the second main diagonal and the first secondary diagonal have the same parity, if ($Y = 0$) and (D is even) the second main diagonal and the second secondary diagonal have the same parity). Whenever a main diagonal and a secondary diagonal have the same parity, they form a corner. Whenever a corner is identified, the number of lattice points corresponding to the two diagonals is decremented by 1. For instance, if the first main diagonal and the first secondary diagonal form a corner then $NLEFT$ and $NDOWN$ are both decremented by 1. If the first main diagonal and the second secondary diagonal form a corner then both $NLEFT$ and NUP are decremented by 1. If the second main diagonal and the first secondary diagonal form a corner then $NRIGHT$ and $NDOWN$ are both decremented by 1. If the second main diagonal and the second secondary diagonal form a corner then both $NRIGHT$ and NUP are decremented by 1. NC is set to the number of identified corners and the corners are placed in an array on positions 0 to $NC - 1$, so that we know exactly to which diagonals each corner i ($0 \leq i \leq NC - 1$) belongs to.

The algorithm presented in this section uses $O(D \cdot log(D))$ arithmetic operations, because it considers $O(D)$ cases and for each case it needs to perform a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$. In order to reduce the number of arithmetic operations to $O(D)$ we can use the same approach as in section 3. We will assume that our algorithm considers the values of C in ascending order and for each value of C it first computes $CNTEQ(C, 0)$ and then $CNTEQ(C, 1)$. Let's assume that $PREVNIN$ is equal to the value of NIN for the case $C - 1$ and $X = 1$ and $RESPREVNIN = 2^{PREVNIN}$. We will initially have $PREVNIN = 0$ and $RESPREVNIN = 1$. When we need to compute 2^{NIN} for a case we will first compute the difference $DIFNIN = NIN - PREVNIN$. We will always have $0 \leq DIFNIN \leq D$. Thus, we can compute 2^{NIN} with only one multiplication, as $RESPREVNIN \cdot 2^{DIFNIN}$ (note that 2^{DIFNIN} is taken from the table of precomputed powers of two). After handling the case ($C, X = 1$) we will update $PREVNIN = NIN$ and $RESPREVNIN = 2^{NIN}$ (where 2^{NIN} was just computed by the method we presented). Using this approach we only need $O(D)$ arithmetic operations instead of $O(D \cdot log(D))$. The same discussion as in Section 3, regarding the computation of exact numbers or of numbers modulo a given number M, applies here, too. In this case $NTOTAL$ is the value of order $O(D^2)$ which is obtained by multiplying together two numbers which are of the order $O(D)$.

6. Algorithm 2 for Counting Sets of Lattice Points of Diameter D under the Manhattan (L_1) Distance

Our second algorithm for the Manhattan distance (and only for the second version of the problem) will use a function $CNTLEQ(C, X)$, where $C \leq D$ and $X = 0$ or 1. $CNTLEQ(C, X)$ computes the number of sets of lattice points such that:

- the main diagonals of at least two points are at distance exactly D apart
- all the other pairs of main diagonals of the points are at distance at most D apart
- all the pairs of secondary diagonals of the points are at distance at most C apart
- $X = 0$ means that the parity of the first secondary diagonal is equal to the parity of the first main diagonal, while $X = 1$ means that these parities differ (the first diagonal of each type is the one with the smallest index)

For $C < 0$ we have $CNTLEQ(C, 0) = CNTLEQ(C, 1) = 0$, by definition. We will present solutions for $C \geq 0$ depending on the parity of D.

We will first consider the case when D is even. In this case we have $CNTLEQ(0, 0) = 2^{D/2-1}$ and $CNTLEQ(0, 1) = 0$ (note that every time we use division we consider integer division). Let's consider now the case $C \geq 1$. All the points must be contained between two main diagonals located at distance D apart and between two secondary diagonals located at distance C apart.

For $X = 0$ this figure has $P = (C/2) + 1$ lattice points on each of the main diagonals and has $R = ((C/2) + 1) \cdot ((D/2) + 1) + (C - (C/2)) \cdot (D/2)$ lattice points in total inside of it and on its borders. $CNTLEQ(C, 0)$ is equal to $(2^P - 1)^2 \cdot 2^{R-2 \cdot P}$.

For $X = 1$ the figure has $P = (C + 1)/2$ lattice points on each of the main diagonals and has $R = ((C + 1)/2) \cdot ((D/2) + 1) + (C + 1 - (C + 1)/2) \cdot (D/2)$ lattice points in total inside of it and on its borders. $CNTLEQ(C, 1)$ is defined identically as $CNTLEQ(C, 0)$, except that we use these new values for P and R.

Let's consider now the case when D is odd. We have $CNTLEQ(C, 0) = CNTLEQ(C, 1)$. The figure defined by the main and secondary diagonals has $P = (C/2) + 1$ lattice points on the first main diagonal and $Q = (C + 1)/2$ lattice points on the second main diagonal. In total, the figure contains $R = (C + 1) \cdot ((D + 1)/2)$ lattice points inside of it and on its borders. We have $CNTLEQ(C, 0) = CNTLEQ(C, 1) = (2^P - 1) \cdot (2^Q - 1) \cdot 2^{R-P-Q}$.

Note that the $CNTLEQ$ function ignores the fact that two sets are identical if one can be obtained from another by translation operations. Instead, it considers two sets to be different if they correspond to different subsets of points belonging to the figure. However, this aspect will be considered when deriving the final formula for the number of sets of lattice points with a given diameter, by using the inclusion-exclusion principle.

The total number of sets of lattice points corresponding to case 1 is equal to $C_1 = CNTLEQ(D-1, 0) + CNTLEQ(D-1, 1) - CNTLEQ(D-2, 0) - CNTLEQ(D-2, 1)$. The total number of sets of lattice points corresponding to case 3 is equal to $C_3 = (CNTLEQ(D, 0) - CNTLEQ(D-1, 0) - CNTLEQ(D-1, 1) + CNTLEQ(D-2, 1)) + (CNTLEQ(D, 1) - CNTLEQ(D-1, 0) - CNTLEQ(D-$

$1, 1) + CNTLEQ(D-2, 0))$. Again we made use of the inclusion-exclusion principle when computing C_1 and C_3.

It is easy to see that this algorithm uses $O(log(D))$ arithmetic operations (from a constant number of exponentiations where the base 2 logarithm of the exponent is of the order $O(log(D))$). The same discussion as in Section 3, regarding the computation of exact numbers or of numbers modulo a given number M, applies to this case, too. In this case R is the value which is obtained by multiplying together two numbers having $O(log(D))$ bits each.

7. Experimental Results

We implemented the two algorithms for the Manhattan distance and the second version of the studied problem, presented in Sections 5 and 6. For the algorithm from Section 5 we used its $O(D)$ optimized version. We computed the values modulo a prime number $M = 10^9 + 7$, in order to make use of all the computation optimizations possible. We also implemented a backtracking algorithm which generates every set independently (i.e. it enumerates all the valid sets of lattice points having diameter D). We used several values of D in order to compare the running times of the three algorithms. Note that for some values of D some of the algorithms were too slow and we stopped them after a running time of 5 minutes. The running times are presented in Table 1 (a ”-” is shown where the running time exceeded the 5 minutes threshold). All the three algorithms were implemented in C/C++ and the code was compiled using the G++ compiler version 3.3.1. The tests were run on a machine running Windows 7 with an Intel Atom N450 1.66 GHz CPU and 1 GB RAM.

As expected, the $O(log(D))$ algorithm is much faster than the other two algorithms. The $O(D)$ algorithm is faster for odd values of D than for even values. This is because, when D is odd, we can never obtain a figure with 4 corners (in order to have 4 corners both the main and secondary diagonals would need to have the same parity, but when D is odd the main diagonals have different parities).

8. Related Work

There is a large body of work in the scientific literature concerned with counting lattice points in various multidimensional structures. In (Loera, 2005) the general problem of counting lattice points in polytopes was considered. The general problem of counting lattice points in a bounded subset of the Euclidean space was considered in (Widmer, 2012). Harmonic analysis is applied in (Chamizo, 2008) for counting lattice points in large parts of space.

A problem concerned with counting configurations of lattice points obtained when translating a convex set in the plane was considered in (Huxley & Zunic, 2009), (Huxley & Zunic, 2013). Two configurations were considered identical under similar conditions as the ones used in this paper. Counting arrangements of connected polyominoes (equivalent under translation) and other figures was considered in (Rechnitzer, 2000). The problem of counting directed lattice walkers in horizontal strips of finite width was considered in (Chan & Guttman, 2003). Counting lattice triangulations was studied in (Keibel & Ziegler, 2003).

As far as we are aware, the problems we considered in this paper have not been considered before in any other publication.

Table 1. Running time (in sec) of the three algorithms for several values of D.

D	Backtracking Algorithm	O(D) Algorithm	O(log(D)) Algorithm
1	0.002	0.002	0.002
2	0.002	0.002	0.002
3	0.004	0.002	0.002
4	0.065	0.002	0.002
5	3.91	0.002	0.002
6	-	0.002	0.002
10	-	0.002	0.002
11	-	0.002	0.002
10^4	-	0.09	0.003
$10^4 + 1$	-	0.08	0.003
10^5	-	0.81	0.003
$10^5 + 1$	-	0.54	0.003
10^6	-	7.84	0.003
$10^6 + 1$	-	5.2	0.003
10^7	-	78.3	0.003
$10^7 + 1$	-	51.9	0.003
10^8	-	-	0.003
$10^8 + 1$	-	-	0.003
10^9	-	-	0.004
$10^9 + 1$	-	-	0.004

9. Conclusions

In this paper we presented novel, efficient algorithms for computing the number of sets of lattice points in the plane whose diameter is exactly equal to D, when considering the Manhattan (L_1) or the Chebyshev (L_∞) distance. We considered two versions for defining the equivalence of two such sets of lattice points. The first version forces the sets of points to be fully included inside a given 2D grid. The second version defines two sets of lattice points to be equivalent if one can be obtained from another by using translation operations. Our algorithms require $O(D \cdot log(D))$ or $O(D)$ arithmetic operations (additions, multiplications) for the first version of the problem and only $O(log(D))$ arithmetic operations for the second version of the problem for both distances. We also discussed the possibility of computing the results modulo a given number M, as a way of simplifying some parts of the algorithms (in particular, in order to use numbers with a number of bits independent of D).

As future work we intend to approach the same problems described in this paper but for a number of dimensions greater than 2. Note that in the 1D case the two problems are identical and very simple to solve (for instance, the answer is always 2^{D-1} for the second version of the problem, because we must have two points in the set at distance D and all the other $D - 1$ points between them may be selected or not to be part of the set).

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Proximate Growth and Best Approximation in L^p-norm of Entire Functions

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Abstract

Let $0 < p \leq +\infty$ and $V_K = \sup\left\{\frac{1}{d}\ln|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}$ the Siciak extremal function of a L-regular compact K. The aim of this paper is the characterization of the proximate growth of entire functions of several complex variables by means of the best polynomial approximation in L_p-norm on a L-regular compact K.

Keywords: Extremal function, L-regular, proximate growth, best approximation of entire function, L^p-norm.
2010 MSC: Primary 30E10; Secondary 41A21, 32E30.

1. Introduction

The classical growth have been characterized in term of approximation errors for a function continuous on $[-1, 1]$ by A.R. Reddy(see (Reddy, 1972*a*), and a compact K of positive capacity by T. Winiarski (see (Winiarski, 1970) with respect to maximum norm. For a nonconstant entire function $f(z) = \sum_{k=0}^{+\infty} a_k.z^{\lambda_k}$ and $M(f, r) = \max_{|z|=r} \left|f(z)\right|$, it is well known that the function $r \to \log(M(f, r))$ is indefinitely increasing convex function of $\log(r)$. To estimate the growth of f precisely, R.P. Boas, (see (Boas, 1954)), has introduced the concept of order, defined by the number ρ $(0 \leq \rho \leq +\infty)$:

$$\rho = \limsup_{r\to+\infty} \frac{\log\log(M(f, r))}{\log(r)}.$$

The concept of type has been introduced to determine the relative growth of two functions of same nonzero finite order. An entire function, of order ρ $(0 < \rho < +\infty)$, is said to be of type σ $(0 \leq \sigma \leq +\infty)$ if

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$\sigma = \limsup_{r\to+\infty} \frac{\log(M(f,r))}{r^\rho}$.

If f is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function cannot be precisely measured by the above concept. However S.K. Bajpai, O.P. Juneja and G.P. Kapoor (see (Bajpai *et al.*, 1976)) have introduced the concept of index-pair of an entire function. Thus, for $p \geq q \geq 1$, they defined also the number

$$\rho(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p]}(M(f,r))}{\log^{[q]}(r)}$$

$b \leq \rho(p,q) \leq +\infty$ where $b = 0$ if $p > q$ and $b = 1$ if $p = q$.

The function f is said to be of index-pair (p,q) if $\rho(p-1,q-1)$ is nonzero finite number. The number $\rho(p,q)$ is called the (p,q)-order of f.

S.K. Bajpai, O.P. Juneja and G.P. Kapoor defined also the concept of the (p,q)-type $\sigma(p,q)$, for $b < \rho(p,q) < +\infty$, by

$$\sigma(p,q) = \limsup_{r\to+\infty} \frac{\log^{[p-1]}((M(f,r)))}{\left(\log^{[q-1]}(r)\right)^{\rho(p,q)}}$$

In their works, the authors established the relationship of (p,q)-growth of f in term of the coefficients a_k in the Maclaurin series of f.

We have also many results in terms of polynomial approximation in classical case. Let K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity and f be a complex function defined and bounded on K. For $k \in \mathbb{N}$ put

$$E_k(K,f) = \left\|f - T_k\right\|_K$$

where the norm $\left\|.\right\|_K$ is the maximum on K and T_k is the $k-th$ Chebytchev polynomial of the best approximation to f on K.

S.N. Bernstein showed (see (Bernstein, 1926), p. 14), for $K = [-1,1]$, that there exists a constant $\rho > 0$ such that

$$\lim_{k\to+\infty} k^{1/\rho}\sqrt[k]{K_k(K,f)}$$

is finite, if and only if, f is the restriction to K of an entire function of order ρ and some finite type.

This result has been generalized by A.R. Reddy (see (Reddy, 1972*a*) and (Reddy, 1972*b*)) as follows:

$$\lim_{k\to+\infty} \sqrt[k]{E_k(K,f)} = (\rho.e.\sigma)2^{-\rho}$$

if and only if f is the restriction to K of an entire function g of order ρ and type σ for $K = [-1,1]$.

In the same way T. Winiarski (see (Winiarski, 1970)) generalized this result for a compact K of the complex plane $\mathbb{C}$, of positive logarithmic capacity noted $c = cap(K)$ as follows:

If K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity then

$$\lim_{k\to+\infty} k^{\frac{1}{\rho}}\sqrt[k]{E_k(K,f)} = c(e\rho\sigma)^{\frac{1}{\rho}}$$

if and only if f is the restriction to K of an entire function of order ρ $(0 < \rho < +\infty)$ and type σ.

Recall that the capacity of $[-1, 1]$ is $cap([-1, 1]) = \frac{1}{2}$ and the capacity of a unit disc is $cap(D(O, 1)) = 1$.

The authors considered respectively the Taylor development of f with respect to the sequence $(z_n)_n$ and the development of f with respect to the sequence $(W_n)_n$ defined by

$$W_n(z) = \prod_{j=1}^{j=n}(z - \eta_{nj}), \quad n = 1, 2, ...$$

where $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, ..., \eta_{nn})$ is the $n - th$ extremal points system of K (see (Winiarski, 1970), p. 260). We remark that the above results suggest that rate at which the sequence $\left(\sqrt[k]{K_k(K, f)}\right)_k$ tends to zero depends on the growth of the entire function (order and type). For a compact K the Siciak's extremal function of K (see (Siciak, 1962) and (Siciak, 1981)) is defined by:

$$V_K = \sup\left\{\frac{1}{d} \log |P_d|, P_d \text{ polynomial of degree} \leq d, \|P_d\|_K \leq 1\right\}.$$

It is known that the regularity of a compact K (we say K is L-regular) is equivalent to the continuity of V_K in $\mathbb{C}^n$.

Let K be a compact L-regular of $\mathbb{C}^n$. For an entire function f in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see (Zeriahi, 1983)), M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions α and β (see (Harfaoui, 2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{z \in \mathbb{C}^n, \ exp(V_K)(z) < r\right\}.$$

M. Harfaoui (see (Harfaoui, 2010) and (Harfaoui, 2011)) obtained a result of generalized order and generalized type ((α,β)-order and (α,β)-type) in term of approximation in L^p-norm for a compact of $\mathbb{C}^n$. Later M. Harfaoui an M. El Kadiri (see (Kadiri & Harfaoui, 2013)) obtained the results in term of (p, q)-order and (p, q)-type for the entire functions .

These results will bee used to establish the generalized growth in terms of best approximation in L_p-norm for $p \geq 1$.

Let f be a function defined and bounded on K. For $k \in \mathbb{N}$ put

$$\pi_k^p(K, f) = \inf\left\{\left\|f - P\right\|_{L^p(K,\mu)}, \ P \in \mathcal{P}_k(\mathbb{C}^n)\right\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu = (dd^c V_K)^n$ associated to a L-regular compact K) (see (Zeriahi, 1987)) and $L^p(K, \mu)$, $p \geq 1$, is the class of all functions such that:

$$\left\|f\right\|_{L^p(K,\mu)} = \left(\int_K | f |^p \, d\mu\right)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$, M. Harfaoui and M. El Kadiri established a precise relationship between the (p, q)-growth and the general growth ((α, α)-growth) with respect to the set (see

((Harfaoui, 2010), (Harfaoui, 2011), (Kadiri & Harfaoui, 2013) and (Harfaoui & Kumar, 2014)) and the coefficients of the development of f with respect to the sequence $(A_k)_k$. He used these results to give the relationship between the generalized growth of f and the sequence $(\pi_k^p(K, f))_k$.

To our knowledge no work is discussed in term of best approximation in L_p-norm with respect to the proximate growth.

The aim of this paper is to give the proximate growth and the $(m, 1)$- proximate growth of entire functions in $\mathbb{C}^n$ ($m \in \mathbb{N}^*$) by means of the best polynomial approximation in term of L^p-norm, with respect to the set

$$\Omega_r = \left\{z \in \mathbf{C}^n; \exp V_K(z) \leq r\right\}.$$

In the paper of A. R. Reddy and T. Winiarski (see (Reddy, 1972*a*), (Reddy, 1972*b*) and (Winiarski, 1970)) the authors use the development of f in the basis $(z_n)_n$ and $(W_n)_n$ and used the Cauchy inequality.

In our work we use a new basis of extremal polynomial and we replace the the Cauchy inequality by an inequality given by A. Zeriahi (see (Zeriahi, 1983)).

So we establish relationship between the rate at which $\left(\pi_k^p(K, f)\right)^{1/k}$, for $k \in \mathbb{N}$, tends to zero in terms of best approximation in L^p-norm, and the proximate growth growth of entire functions of several complex variables for a L-regular compact K of $\mathbb{C}^n$.

2. Notations aned auxiliary results

Before we give some definitions and results which will be frequently used.

For $p \in \mathbb{Z}$ put

$$\log^{[p]}(x) = \log(\log^{[p-1]}(x)); \;\; \log^{[0]}(x) = x; \Lambda_{[p]} = \prod_{k=1}^{p} \log^{[k]}(x).$$

$$\exp^{[p]}(x) = \exp(\exp^{[p-1]}(x)); \;\; \exp^{[0]}(x) = x \text{ and } \; E_{[p]}(x) = \prod_{k=0}^{p} \exp^{k}(x).$$

Lemma 2.1. *(see (Bajpai et al., 1976))*

With the above notations we have the following results

$$(RR_1) \;\; E_{[-p]}(x) = \frac{x}{\bigwedge_{[p-1]}(x)} \text{ and } \bigwedge_{[-p]}(x) = \frac{x}{E_{[p-1]}(x)}$$

$$(RR_2) \;\; \frac{d}{dx} \exp^{[p]}(x) = \frac{E_{[p]}(x)}{x} = \frac{1}{\bigwedge_{[-p-1]}(x)}$$

$$(RR_3) \;\; \frac{d}{dx} \log^{[p]}(x) = \frac{E_{[-p]}(x)}{x} = \frac{1}{\bigwedge_{[p-1]}(x)}$$

$$(RR_4) \;\; E_{[p]}^{-1}(x) = \begin{cases} x, & \text{if } p = 0 \\ \log^{[p-1]}\{\log(x) - \log^{[2]}(x) + o(\log_{[3]}(x))\}, & \text{if } p = 1, 2, ... \end{cases}.$$

(RR$_5$) $\lim_{x\to+\infty} \exp\left(E_{[p-2]}(x)\right) = \begin{cases} e & \text{if } p = 2 \\ 1 & \text{if } p \geq 3 \end{cases}$

(RR$_6$) $\lim_{x\to+\infty} \left[\exp^{[p-1]}\left(E^{-1}_{[p-2]}(x)\right)\right]^{\frac{1}{x}} = \begin{cases} e & \text{if } p = 2 \\ 1 & \text{if } p \geq 3 \end{cases}$

It is known that if K is a compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see (Siciak, 1962) and (Siciak, 1981)), in particular, we have:

(P_1) Bernstein-Markov inequality:

$\forall \epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \left\|P_d\right\|_K = C(1+\varepsilon)^{s_k}\left\|P_d\right\|_{L^2(K,\mu)}, \tag{2.1}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) inequality:

For every set L-regular K and every real $r > 1$ we have:

$$\left\|f\right\|_K \leq M.r^{deg(f)}\left(\int_K \mid f \mid^p .d\mu\right)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

For $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ..., s_n(k))$ be a bijection such that

$$|s(k+1)| \geq |s(k)| \;\text{ where }\; |s(k)| = s_1(k) + ... + s_n(k),$$

A. Zeriahi (see (Zeriahi, 1983)) constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see (Siciak, 1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that

$$\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}.$$

We need the following notations which will be used in the sequel:

(N_1) $\nu_k = \nu_k(K) = \left\|A_k\right\|_{L^2(K,\mu)}$.

(N_2) $a_k = a_k(K) = \left\|A_k\right\|_K = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$,

where $s_k = deg(A_k)$.

With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$.

Lemma 2.2. *(see (Zeriahi, 1983))*
Let K be a compact L-regular subset of $\mathbb{C}^n$. Then

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_K(z)), \tag{2.5}$$

for every $z \in \mathbb{C}^n \setminus \widehat{K}$ the connected component of $\mathbb{C}^n \setminus K$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_K}{\nu_k}\right]^{1/s_k} = 1. \tag{2.6}$$

3. Growth with respect to the proximate order and coefficient with respect to extremal polynomial.

Before we give some definitions and results which will be frequently used in this paper.

Definition 3.1.
Let ρ be a positive real such that $0 < \rho < +\infty$. A proximate order for ρ is a function $\rho(r)$ defined in $\mathbb{R}^+$ and verified:

1. $\lim\limits_{r\to+\infty} \rho(r) = \rho$;
2. $\lim\limits_{r\to+\infty} r\rho' \log(r) = 0$.

Example 3.1. The function $\rho(r)$ defined by

$$r^{\rho(r)} = r^{\rho}(\ln(r))^{\beta_1}.(\ln^{[2]}(r))^{\beta_2})...(\ln^{[m](r)})^{\beta_m}$$

is a proximate order for ρ, where $\log^{[m](r)}$ is defined by:

$$\log^{[0]}(r) = r, \;\; \log^{[m](r)} = \ln^+(\log^{[m-1]}(r)) \;\text{ and }\; \ln^+(t) = 1_{[1;+\infty[}\ln(t)$$

Theorem 3.1. *If $h(r)$ is a positive function for $r > 0$ such that*

$$\lim_{r\to+\infty}\frac{\log(h(r))}{\log(r)} = \rho < +\infty,$$

then the proximate order $\rho(r)$ maybe chosen such that for every $r > 0$: $h(r) \leq r^{\rho(r)}$, and for some sequence $r_n^{\rho(r_n)}$, $h(r_n) \leq r_n^{\rho(r_n)}$, for n sufficiently large.

For an entire function in $\mathbb{C}^n$ we define the K-type for the proximate order as follows:

Definition 3.2.
Let K be a L-regular of $\mathbb{C}^n$. If for an entire function in $\mathbb{C}^n$

$$\limsup_{r\to+\infty} \frac{\log (M_K(f,r))}{r^{\rho(r)}} \tag{3.1}$$

is finite not zero then the function $\rho(r)$ is called proximate order

$$\sigma_K = \lim_{r\to+\infty} \frac{\ln (M_K(f,r))}{r^{\rho(r)}} \tag{3.2}$$

is called K-type of f with respect to the proximate order $\rho(r)$, where

$$M_K(f,r) = \sup_{z\in\Omega_r} \left|f(z)\right|.$$

Let K be a compact L-regular and f an entire function of several variables and $f(z) = \sum_{k=0}^{+\infty} f_k.A_k$ the development of f with respect to the sequence of extremal polynomials.

2.1. K-type of f with respect to the proximate order

Theorem 3.2.
If $\rho(r)$ is a proximate order for ρ then the K-type of f with respect to the proximate order is given by the formula:

$$\sigma_K = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}, \tag{3.3}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.
We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Lemma 3.1. *[7, p.42(1.58)]*
For every $k > 0$ we have

$$\limsup_{t\to+\infty} \frac{\varphi(k.t)}{\varphi(t)} = k^{1/\rho}.$$

Proof of theorem 3.2.
Put $\sigma = \frac{1}{e.\rho} \limsup_{k\to+\infty} \left(\varphi(s_k)\tau_k\right)^{\rho}.\left|f_k\right|^{\rho/s_k}$ and show that $\sigma = \sigma_K$.
Show that $\sigma \le \sigma_K$.

We have for every $\theta > 1$ $\sigma_K = \lim_{r\to+\infty} \frac{\ln (M_K(f,r\theta))}{r^{\rho(r\theta)}}$, then for every $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for every $r > r(\varepsilon)$

$$\log (\|f\|_{\overline{\Omega}_{r\theta}}) \le (r\theta)^{r\theta}(\sigma(K,f) + \varepsilon). \tag{3.4}$$

But $(r+1)^{N_\theta}\|f\|_{\overline{\Omega}_{r\theta}} \le \exp\left((\sigma_{K,f} + \varepsilon)(r\theta)^{r\theta}\right)$, where $N_\theta \in \mathbb{N}$ such that

$$\left|f_k\right|\nu_k \le C_\theta.r^{-s_k}.\frac{(r+1)^{N_\theta}}{(r-1)^{2n-1}}\|f\|_{\overline{\Omega}_{r\theta}} \tag{3.5}$$

then

$$\left|f_k\right|\nu_k \le C_\theta.r^{-s_k}.\exp\left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right),$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$ or

$$\log\left(\left|f_k\right|\nu_k\right) \le \log(C_\theta) - s_k\log(r) + \left((\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}\right), \tag{3.6}$$

for $r > r(\varepsilon)$ and $k > k(\varepsilon)$.

Chose r such that $s_k = [(\sigma(K,f)+\varepsilon)(r\theta)^{r\theta}.]$, where $[x]$ means the integer part of x. Then $s_k \le (\sigma(K,f)+\varepsilon)(r\theta)^{r\theta} < s_k+1$. Replacing in the relation (3.6) we get

$$\log\left(\left|f_k\right|\nu_k\right) \le \log(C_\theta) - s_k\log(r) + s_k\log(\theta) + \frac{s_k+1}{\rho}. \tag{3.7}$$

Since $\dfrac{s_k}{\rho(\sigma(K,f)+\varepsilon)} \le (r\theta)^{r\theta}$, then $\varphi(\dfrac{s_k}{\rho(\sigma(K,f)+\varepsilon)}) \le r\theta$, thus

$$\log\left[(\tau_k.\varphi(s_k))^\rho(\left|f_k\right|)^{\rho/s_k}\right] \le \frac{\rho}{s_k}\log(C_\theta) + \rho\log\left(\frac{\varphi(s_k)}{\varphi(\dfrac{s_k}{\rho.(\sigma(K,f)+\varepsilon)})}\right) + 1 + \frac{1}{s_k}.$$

After passing to the upper limit and applying the lemma 2.1, the relation (2.6) of the lemma 2.2 and the lemma 3.1 we get

$$\limsup_{k\to+\infty}\log\left[(\varphi(s_k))^\rho(\nu_k.\left|f_k\right|)^{\rho/s_k}\right] \le \log\left(\rho.\sigma(K,f)\right) + 1 = \log\left(e.\rho.(\sigma(K,f))\right). \tag{3.8}$$

which gives the result

$$\limsup_{r\to+\infty}(\tau_k.\varphi(s_k))^\rho(\left|f_k\right|)^{\rho/s_k} \le e.\rho.(\sigma(K,f)). \tag{3.9}$$

Show that $\sigma \ge \sigma_K$. If $\sigma < \sigma_K$ let σ_1 and σ_2 such that $\sigma < \sigma_1 < \sigma_2 < \sigma_K$. There exists $k_1$such that for every $k > k_1$:

$$(\tau_k)^{s_k}.|f_k| \le \frac{e.\rho.(\sigma_1)^{1/\rho}}{\varphi(s_k)} \tag{3.10}$$

as we have also for k sufficiently large ($k > q_2$), $(\sigma_1.\rho)^{1/\rho}.\dfrac{\varphi(\frac{s_k}{\sigma_1.\rho})}{\varphi(s_k)}$, then for $k_0 = \max(q_1,q_2)$ we have

$$M_K(f,r) \le \sum_{k=0}^{k_0}|f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty}|f_k|.\|A_k\|_{\overline{\Omega}_r}. \tag{3.11}$$

According to the Bernstein-Walsh inequality we have

$$\|A_k\|_{\overline{\Omega}_r} \le a_k(K).r^{s_k},$$

and according to the Bernstein-Markov inequality we have

$$a_k(K) \le A_\epsilon.(1+\epsilon)^{s_k}a_k(K).\tau_k^{s_k}.$$

Thus

$$M_K(f,r) \leq C_0.r^{s_k} + A_\epsilon.\sum_{k=k_0+1}^{+\infty}\Big(\frac{e^{1/\rho}}{\varphi(s_k/\sigma_1.\rho)}\Big)^{s_k}.((1+\epsilon))^{s_k}. \tag{3.12}$$

If we put $\delta = \dfrac{\sigma_1}{\sigma_2}$ $(\delta < 1)$ then

$$M_K(f,r) \leq C_0.r^{s_k} + A_\epsilon.\sum_{k=k_0+1}^{+\infty}\Big(\frac{1+\epsilon}{1-\epsilon}\Big)^{s_k}.\sup_{k>k_0} e^{\Psi(s_k)}.$$

where

$$\Psi(x) = x\log(r) - \frac{x}{\rho} - x\log\Big(\varphi(x/\sigma_2.\rho)\Big).$$

If we choose ϵ such that $0 < \epsilon < \dfrac{1-\delta}{1+\delta}$ then $\dfrac{\delta(1+\epsilon)}{1-\epsilon} < 1$ and thus

$$M_K(f,r) \leq C_0.r^{s_k} + C.\sup_{k>k_0} e^{\Psi(s_k)}.$$

We note that $\Psi(x) = 0$ is equivalent to

$$\log(r) + \frac{1}{\rho} - \frac{x}{\sigma_2.\rho}.\frac{\varphi'(x/\sigma_2.\rho)}{\varphi(x/\sigma_2.\rho)} - \log\Big(\varphi(x/\sigma_2.\rho)\Big) = 0, \tag{3.13}$$

then the solution x_r of the equation (3.13) verify

$$\log(r) - \frac{\rho}{\epsilon} < \varphi(\frac{x_r}{\sigma_2.\rho}) < \log(r) + \frac{\rho}{\epsilon} \text{ for } r > r_1$$

and thus

$$\begin{cases} \Psi(x) \leq \dfrac{x_r}{\rho} + x_r\Big(\log(r) - \log\big(\varphi(\dfrac{x_r}{\sigma_2.\rho})\big)\Big) \leq (1+\epsilon)\dfrac{x_r}{\rho} \\ \dfrac{x_r}{\sigma_2} \leq (\theta.r)^{\varphi(\theta)} \text{ where } \theta = e^{\epsilon/\rho} \end{cases}$$

Since for every $\theta > 1$ we have $(\theta.r)^{\varphi(\theta)} \leq (\theta.r)^{\rho+\epsilon}.r^{\rho(r)}$ then

$$e^{\Psi(x_r)} \leq e^{(1+\epsilon).\theta^{\rho+\epsilon}}.\sigma_2.r^{\rho(r)} \text{ for } r > r_1$$

and consequently, for $r > r_1$,

$$M_K(f,r) \leq C_0.r^{s_{k_0}} + A.\theta^{\rho+\epsilon}.\sigma_2.r^{\rho(r)}.$$

whence

$$\frac{\log\left(M_K(f,r)\right)}{r^{\rho(r)}} \leq \sigma_1 + o(1),$$

passing to the upper limit we get $\sigma(K,f) \leq \sigma_1$. Which leads a contradiction and this shows the result.

2.2.(K,m)-type of f with respect to the proximate order

For the entire functions infinite order we introduce the notion of m-order defined by:

$$\rho_m = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r))}{\log(r)}, \tag{3.14}$$

for $m \geq 2$. The function f is said to be of index-pair $(m,1)$ if $\rho_{m-1} = +\infty$ and $\rho_m < +\infty$. The number ρ_m is called the m-order of f.

Definition 3.3.
If $\rho(r)$ is a proximate order associated to the m-order ρ_m, the (K,m)-type with respect to the proximate order $\rho(r)$ is defined by:

$$\sigma_m(K,f) = \limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r)}{r^{\rho(r)}} \tag{3.15}$$

Let $f = \sum_{k=0}^{+\infty} f_k(f).A_k$ the development of f with respect to the sequence of extremal polynomials.

Theorem 3.3.
The (K,m)-type of f with respect to the proximate order is given by the formula:

$$\sigma_m(K,f) = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m-2]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}, \tag{3.16}$$

for $m > 2$.

Proof of theorem 3.3.
Put $\rho_m = \rho$ and $\sigma = \limsup_{k\to+\infty} \Big(\varphi(\log^{[m]}(s_k))\tau_k\Big)^{\rho}.\big|f_k\big|^{\rho/s_k}$.
Show that $\sigma_m(K,f) \leq \sigma$.
We have for every $\epsilon > 0$ there exists k_0 such that for every $k > k_0$

$$\varphi(\log^{[m-2]}(s_k))\tau_k.\big|f_k\big|^{1/s_k} \leq \sigma^{1/\rho} + \epsilon, \tag{3.17}$$

thus

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + \sum_{k=0}^{k_0} |f_k|.\|A_k\|_{\overline{\Omega}_r} + \sum_{k=k_0+1}^{+\infty} \Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k}. \tag{3.18}$$

For $\sigma_1 > \sigma$ we have

$$\Big(\frac{\sigma^{1/\rho}+\epsilon}{\varphi(\log^{[m-2]}(s_k))}\Big)^{s_k}.r^{s_k} \leq \Big(\frac{\sigma^{1/\rho}+\epsilon}{\sigma_1^{1/\rho}+\epsilon}\Big)^{s}.\sup_{k>k_0} e^{\Psi(s_k)},$$

where

$$\Psi(x) = x\log(r) + x\log(\sigma^{1/\rho}+\epsilon) + x\log(\varphi(\log^{[m-2]}(x)).$$

The solution x_r of the equation $\Psi'(x) = 0$ verify, for r sufficiently large ($r > r_1$)

$$\Psi(x_r) \leq \epsilon.exp^{[m-2]}((1+\epsilon).\theta^{\rho+\epsilon}.r^{\rho(r)}), \text{ where } \theta = (\sigma^{1/\rho}+\epsilon).e^{\epsilon}$$

therefore

$$M_K(f,r) \leq C_0 r^{s_{k(r)}} + A.e^{\Psi(x_r)}, \text{ where A is a constant.}$$

This gives $\limsup_{r\to+\infty} \dfrac{\log^{[m]}(M_K(f,r))}{r^{\rho(r)}} \leq \sigma_1$ and since this is true for every $\sigma_1 > \sigma$ then

$$\limsup_{r\to+\infty} \frac{\log^{[m]}(M_K(f,r))}{r^{\rho(r)}} \leq \sigma.$$

Show now that $\sigma_m(K,f) \geq \sigma$.

By definition of $\sigma_m(K,f)$ we have for every $\epsilon > 0$ there exists $r_0(\epsilon)$ such that for every $r > r_0(\epsilon)$,

$$M_K(f,r) \leq \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right], \text{ and } \theta > 1,$$

thus

$$\left|f_k\right|.\tau_k^{s_k} \leq C'_0.\sup_{k>k_0} \exp^{\Psi(s_k)}.$$

where

$$\Psi(x) = -s_k \log\left(\frac{r}{1+\epsilon}\right) + \exp^{[m-2]}\left[(\sigma_m(K,f)+\epsilon)(r\theta)^{r\theta}\right].$$

For r sufficiently large the solution of the equation $\Psi'(x) = 0$ verify

$$E^{-1}_{[m-2]}\Big(s_k(\frac{1}{\rho}-1)\Big) \leq (\sigma_m(K,f)+\epsilon)(r_k\theta)^{r_k\theta} \leq E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}+1)). \tag{3.19}$$

Using the relation (3.19) an elementary calculus gives

$$\left|f_k\right|.\tau_k^{s_k}.\varphi\Big(E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}-1))\Big) \leq (\sigma_m(K,f)+\epsilon)^{1/\rho}.\exp^{[m-1]}\left[E^{-1}_{[m-2]}(s_k(\frac{1}{\rho}+1))\right]. \tag{3.20}$$

Therefore passing to the upper limit and using the propriety of the function $x \to E_{[m-2]}(x)$ we obtain the result.

4. Best polynomial approximation in terms of L^p-norm.

The object of this section is to study the relationship of the rate of the best polynomial approximation of f in L^p-norm with the -growth with respect to the proximate order of an entire function g such that $g_{/K} = f$.

More precisely we show the following theorem:

Theorem 4.1.

If $\rho(r)$ is a proximate order for $\rho$and f and let $f \in L^p(K,\mu)$ for $p > 0$. Then f is μ-almost-surely the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero order ρ and K-type $\sigma(K, f_1) \in]0, +\infty5$ with respect to the proximate order $\rho(r)$ for ρ if and only if

$$\sigma(K, f_1) = \frac{1}{e.\rho} \limsup_{k\to+\infty} (\varphi(k))^{\rho}.(\mathcal{E}_k^p)^{\rho/k}, \tag{4.1}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.

We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Proof of theorem 4.1.

Suppose that f is μ-almost-surely the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero order ρ and K-type $\sigma(K, f_1) \in]0, +\infty5$ with respect to the proximate order $\rho(r)$ for ρ. We have $f_1 \in L^2(K,\mu)$ and

$$f_1 = \sum_{k=0}^{+\infty} f_k.A_k.$$

Put $\sigma = \dfrac{1}{e.\rho} \limsup\limits_{k\to+\infty} \Big(\varphi(s_k)\tau_k\Big)^{\rho}.\Big|f_k\Big|^{\rho/s_k}$

By the relation (92) for $p \geq 2$ and the relation (96) for $p \in [1, 2[$ of the paper of M. El Kadiri and M. Harfaoui (see (Kadiri & Harfaoui, 2013))

$$\Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.|f_k|)^{\rho/s_k} \leq (A_\epsilon)^{\rho/s_k}\Big(\varphi(s_k)\Big)^{\rho}(1+\epsilon)^{\rho}.(\mathcal{E}_k^p)^{\rho/s_k} \tag{4.2}$$

then

$$\Big(\varphi(s_k)\tau_k\Big)^{\rho}.(\Big|f_k\Big|)^{\rho/s_k} \leq \Big(\varphi(s_k)\Big)^{\rho}(\Big|f_k\Big|.\nu_k)^{\rho/s_k}.\Big(\frac{\tau_k^{s_k}}{\nu_k}\Big)^{\rho/s_k} \tag{4.3}$$

By the relation 3.6 we have

$$\Big(\mathcal{E}_k^p\Big)^{1/s_k} \leq (A_\epsilon)^{\rho/s_k}.\Big[\Big|f_k\Big|.\nu_k.(1+\epsilon)^{s_k+1} + ...\Big]. \tag{4.4}$$

But

$$\sigma' = \limsup_{k\to+\infty} \Big(\varphi(s_k)\Big)^{\rho}.(\nu_k.\Big|f_k\Big|^{\rho/s_k} = e.\rho.\sigma.$$

Thus, for k sufficiently large

$$\varphi(s_k).(\nu_k.\Big|f_k\Big|)^{1/s_k} \leq (\sigma')^{1/\rho} + \epsilon \Leftrightarrow \nu_k.\Big|f_k\Big| \leq \Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{s_k}.$$

Hence for every $j \in \mathbb{N}$;

$$\nu_{k+j}.\Big|f_{k+j}\Big| \leq \Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_{k+j})}\Big]^{s_{k+j}}.$$

Then, if we put $S = (1+\epsilon)^{s_k}.\nu_k.\big|f_k + (1+\epsilon)^{s_{k+1}}.\nu_{k+1}.\big|f_{k+1}....$, we have

$$S \leq \sum_{j=0}^{+\infty} \epsilon)^{s_{k+j}} [\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_{k+j})}]^{s_{k+j}}$$

which is equivalent to

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_{k+j})}\Big]^{s_k} \sum_{j=0}^{+\infty} (1+\epsilon)^{s_{k+j}} \frac{\big[(\sigma')^{1/\rho}+\epsilon\big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}} \Big[\frac{[\varphi(s_k)]^{s_k}}{[\varphi(s_{k+j})]^{s_{k+j}}}\Big]^{s_{k+j}}$$

or

$$S \leq \Big[\frac{(\sigma)^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^{s_k} \sum_{j=0}^{+\infty} (1+\epsilon)^{s_{k+j}} \frac{\big[(\sigma')^{1/\rho}+\epsilon\big]^{s_k+j}}{\big[(\sigma)^{1/\rho}+\epsilon\big]^{s_k}} \frac{\big[\varphi(s_k)\big]^{s_k}}{\big[\varphi(s_k+j)\big]^{s_k}+j}$$

Since $\dfrac{\varphi(s_k)}{\varphi(s_k+j)} \leq 1$ we get also

$$S \leq (1+\epsilon)^{s_k}.\frac{\big((\sigma')^{1/\rho}+\epsilon\big)^{s_k}}{\varphi(s_k)}.\sum_{j=0}^{+\infty}\Big[(1+\epsilon).\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(j)}\Big]^j.$$

As for k sufficiently large $\dfrac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k+j)} < 1$ the series is convergent to a finite sum L and we will get finally

$$\big(\mathcal{E}_k^p\big)^{1/s_k} \leq (1+\epsilon)^\rho (A_\epsilon)^{1/s_k}.L^{\rho/s_k}.\Big[\frac{(\sigma')^{1/\rho}+\epsilon}{\varphi(s_k)}\Big]^\rho$$

which equivalent to

$$(\varphi(s_k))^\rho.\big(\mathcal{E}_k^p\big)^{1/s_k} \leq (1+\epsilon)^\rho (A_\epsilon)^{\rho/s_k}.L^{\rho/s_k}.((\sigma')^{1/\rho}+\epsilon^\rho).$$

Passing to the upper limit get

$$\sigma(K,f_1) = \frac{1}{e.\rho}\limsup_{k\to+\infty}(\varphi(k))^\rho.(\mathcal{E}_k^p)^{\rho/k} \leq \sigma.$$

Conversely, suppose now that f satisfies the relation 4.6. We show the result by three steps.

If $f \in L^p(K,\mu)$ with $p \geq 2$ then $f \in L^2(K,\mu)$ and we have $\sum_{k=0}^{+\infty} f_k A_k$ with convergence in $L^2(K,\mu)$, where

$$f_k = \frac{1}{\nu_k^2}\int_K f.\overline{A}_k \quad (k \geq 0).$$

We verify easily by the relations 3.3, 3.6 and the inequality (B.M):

$$\limsup_{k\to+\infty}\big(\varphi(s_k)\tau_k\big)^\rho.\big|f_k\big|^{\rho/s_k} = \limsup_{k\to+\infty}(\varphi(k))^\rho.(\mathcal{E}_k^p)^{\rho/k}. \tag{4.5}$$

By this inequality the series $\sum f_k A_k$ considered in $\mathbb{C}^n$ converges normally on every compact of $\mathbb{C}^n$ to a function denoted f_1 by the inequality (B.M) and the inequality of the coefficient of $|f_k|$. We have obviously $f_1 = f$ μ-a.s on K and the proof is completed by the theorem3.1.

If $p \in [0, 2[$ we take p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $p' \geq 2$. Applying the previous arguments of the first step to p' and Hölder and Bernstein inequality we obtain the result.

If $0 < p < 1$, of course, for $0 < p < 1$ the L_p-norm does not satisfy the triangle inequality. But our relations (4.2) and (4.3) are also satisfied for $0 < p < 1$ (see (Harfaoui & Kumar, 2014)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed)

$$\| f \|_{L^p(K,\mu)} \leq M. \| f \|_{L^r(K,\mu)} .$$

Using the inequality

$$\int_K | f |^p \, d\mu \leq \| f \|_K^{p-r} . \int_K | f |^r \, d\mu$$

we get

$$\| f \|_{L^p(K,\mu)} \leq \| f \|_K^{1-(r/p)} . \| f \|_{L^r(K,\mu)}^{r/p} .$$

We deduce that (K, μ) satisfies the Bernstein-Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials P we have

$$\| P \|_K \leq C(1 + \epsilon)_{deg(P)}. \ \| P \|_{L^p(K,\mu)} .$$

Thus if (K, μ) satisfies the Bernstein-Markov inequality for one $p > 0$ then (4.2) and (4.3) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step.1 and step.2

Theorem 4.2.

If $\rho(r)$ is a proximate order for $\rho_m \in]0, +\infty[$ ($m > 2$), and f and let $f \in L^p(K, \mu)$ for $p > 0$. Then f is μ-almost-surely (μ-a.s) the restriction to K of an entire function in $\mathbb{C}^n$, f_1, of finite nonzero m-order ρ_m and (K, m)-type $\sigma_m(K, f_1) \in]0, +\infty 5$ with respect to the proximate order $\rho(r)$ for ρ if and only if

$$\sigma_m(K, f_1) = \limsup_{k \to +\infty} \Big(\varphi((\log^{[m-2]}(k)))\Big)^{\rho_m} .(\mathcal{E}_k^p)^{\rho_m/k}, \tag{4.6}$$

where φ is the inverse function of the function $r \to r^{\rho(r)} = \psi(r)$.
We have so $\psi(r) = y \Leftrightarrow \varphi(y) = r$.

Proof of theorem 4.2.

The theorem can be proved on similar lines as those of the proof of the theorem 4.1 because the relations (4.2) and (4.3) are still valid by iteration of logarithm . Hence we omit the proof.

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Some Results in Connection with the Bounds for the Zeros of Entire Functions in the Light of Slowly Changing Functions

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Abstract

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function. In this paper we would like to establish the bounds for the moduli of zeros of entire functions on the basis of slowly changing functions.

Keywords: Zeros of entire functions, proper ring shaped region, slowly changing functions.
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1. Introduction, Definitions and Notations.

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0$$

be a polynomial of degree n. Datt and Govil (Datt & Govil, 1978); Govil and Rahaman(Govil & Rahaman, 1968); Marden (Marden, 1966); Mohammad (Mohammad, 1967); Chattopadhyay, Das, Jain and Konwer (Chattopadhyay, 2005); Joyal, Labelle and Rahaman (Joyal, Labelle & Rahaman 1967) Jain (Jain, 1976), (Jain, 2006) Sun and Hsieh (Sun & Hsie, 1996); Zilovic, Roytman, Combettes and Swamy (Zilovic, Roytman); Das and Datta (Das & Datta, 2008) etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions on the basis of slowly changing functions.

The following definitions are well known :

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Definition 1.1. (Valiron, 1949)The order ρ and lower order λ of an entire function f are defined as

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker (Singh & Barker, 1977) defined it in the following way:

Definition 1.2. (Singh & Barker, 1977) A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\ (> 0)$,

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \text{ for } r > r(\varepsilon) \quad \text{and}$$

uniformly for $k (\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to $\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0$.

Somasundaram and Thamizharasi (Somasundaram & Thamizharasi, 1988) introduced the notions of L-order and L-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 1.3. (Somasundaram & Thamizharasi, 1988) The L-order ρ^L and the L-lower order λ^L of an entire function f are defined as

$$\rho^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[rL(r)]}.$$

The more generalised concept for L-order and L-lower order are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 1.4. The L^*-order ρ^{L^*} and the L^*-lower order λ^{L^*} of an entire function f are defined as

$$\rho^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log[re^{L(r)}]}.$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *If $f(z)$ is an entire function of L-order ρ^L, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [rL(r)]^{\rho^L + \varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [rL(r)]$.

Proof. Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros z_i's of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^L suppose to be finite.

Now we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(R\, e^{i\phi}\right)\right| d\phi. \tag{2.1}$$

Let us replace R by $2r$ and n by $N(2r)$ in (2.1).

$$\therefore \log|f(0)| = -\sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi.$$

Since $f(0) = 1, \therefore \log|f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi. \tag{2.2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \geq N(r)\log 2 \tag{2.3}$$

because for large values of r, $\log \frac{2r}{|z_i|} \geq \log 2$.

$$\text{R.H.S} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\, e^{i\phi}\right)\right| d\phi \leq \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \tag{2.4}$$

Again by definition of order ρ^L of $f(z)$ we have fore every $\varepsilon > 0$, and as $L(2r) \sim L(r)$,

$$\log M(2r) \leq [2rL(2r)]^{\rho^L+\varepsilon/2} \log M(2r) \leq [2rL(r)]^{\rho^L+\varepsilon/2}. \tag{2.5}$$

Hence from (2.2) by the help of (2.3), (2.4) and (2.5) we have

$$N(r)\log 2 \leq [2rL(r)]^{\rho^L+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^L+\varepsilon/2}}{\log 2} \cdot \frac{[rL(r)]^{\rho^L+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [rL(r)]^{\rho^L+\varepsilon}.$$

This proves the lemma. □

In the line of Lemma 2.1, we may state the following lemma:

Lemma 2.2. *If $f(z)$ is an entire function of L^*-order ρ^{L^*}, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [re^{L(r)}]$.

Proof. With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros z_i's of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^{L^*} supposed to be finite.

In view of (2.1), (2.2), (2.3) and (2.4), by definition of ρ^{L^*} and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \leq [2re^{L(2r)}]^{\rho^{L^*}+\varepsilon/2}, i.e., \log M(2r) \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}.$$

Hence by the help of (2.3), (2.4) and (2) we obtain from (2.2) that

$$N(r)\log 2 \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}, N(r) \leq \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2} \cdot \frac{[re^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}.$$

Thus the lemma is established. □

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L-order ρ^L. Also for all sufficiently large r in the disc $|z| \leq [rL(r)]$, $\left|a_{N(r)}\right| \neq 0$, $|a_0| \neq 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$$

and t_0' is the greatest positive root of

$$f(t) \equiv |a_0|\, t^{N(r)+1} - (|a_0| + M')\, t^{N(r)} + M' = 0$$

$$\text{where } M = max\left\{|a_0|, |a_1|, ... \left|a_{N(r)-1}\right|\right\}$$

$$\text{and } M' = max\left\{|a_1|, |a_2|, ... \left|a_{N(r)}\right|\right\}.$$

Proof. Now

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$

because $N(r)$ exists for $|z| \leq [rL(r)]$; r is sufficiently large and $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region given in Theorem 3.1 which we are to prove.

Now

$$\begin{aligned} |P(z)| &\approx \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right|. \end{aligned}$$

Also

$$\begin{aligned} &\left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \leq |a_0| + ... + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \leq M\left(1 + |z| + ... + |z|^{N(r)-1}\right) \\ &= M \frac{|z|^{N(r)} - 1}{|z| - 1} \text{ if } |z| \neq 1. \end{aligned} \tag{3.1}$$

Therefore using (3.1) we obtain that

$$|P(z)| \geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1}\right| \geq \left|a_{N(r)}\right| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1}.$$

Hence

$$|P(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} - M \frac{|z|^{N(r)} - 1}{|z| - 1} > 0$$

$$\begin{aligned} &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)} > M \frac{|z|^{N(r)} - 1}{|z| - 1} \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} > M \left(|z|^{N(r)} - 1\right) \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left|a_{N(r)}\right| |z|^{N(r)} - M\ |z|^{N(r)} + M > 0 \\ &\text{i.e., if } \left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0. \end{aligned}$$

Therefore on $|z| \neq 1$, $|P(z)| > 0$ if $\left|a_{N(r)}\right| |z|^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) |z|^{N(r)} + M > 0$. Now let us consider

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0. \tag{3.2}$$

Clearly the maximum number of changes in sign in (3.2) is two. So the maximum number of positive roots of $g(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of (3.2). So $g(t) = 0$ must have another positive root t_1(say).

Let us take $t_0 = max\{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not, for some $t = t_2 > t_0$, $g(t_2) < 0$.

Now $g(t_2) < 0$ and $g(\infty) > 0$ imply that $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction.

Therefore for $t > t_0$, $g(t) > 0$ and so $t_0 > 1$.

Hence $|P(z)| > 0$ for $|z| > t_0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in the disc } |z| \leq t_0. \tag{3.3}$$

Again let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right) \approx z^{N(r)} \left\{a_0 + \frac{a_1}{z} + ... + \frac{a_{N(r)}}{z^{N(r)}}\right\} = a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}$$

$$\text{i.e., } |Q(z)| \geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \text{ for } |z| \neq 1.$$

Now

$$\left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|\,|z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \leq M'\left(|z|^{N(r)-1} + ... + 1\right)$$

$$= M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1. \tag{3.4}$$

Using (3.4) we get that

$$|Q(z)| \geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \geq |a_0|\,|z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) \text{ for } |z| \neq 1.$$

Therefore for $|z| \neq 1$,

$$|Q(z)| > 0 \text{ if } |a_0|\,|z|^{N(r)} - M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right) > 0$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)} > M'\left(\frac{|z|^{N(r)} - 1}{|z| - 1}\right)$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)+1} - |a_0|\,|z|^{N(r)} - M'\,|z|^{N(r)} + M' > 0$$

$$\text{i.e., if } |a_0|\,|z|^{N(r)+1} - (|a_0| + M')\,|z|^{N(r)} + M' > 0.$$

So for $|z| \neq 1$, $|Q(z)| > 0$ if $|a_0|\,|z|^{N(r)+1} - (|a_0| + M')\,|z|^{N(r)} + M' > 0$. Let us consider

$$f(t) \equiv |a_0|\,t^{N(r)+1} - (|a_0| + M')\,t^{N(r)} + M' = 0.$$

Since the maximum number of changes of sign in $f(t)$ is two, the maximum number of positive roots of $f(t) = 0$ is two and by Descartes' rule of sign if it is less, less by two. Clearly $t = 1$ is one positive root of $f(t) = 0$. So $f(t) = 0$ must have another positive root t_2 (say).

Let us take $t'_0 = max\{1, t_2\}$. Clearly for $t > t'_0$, $f(t) > 0$. If not, for some $t_3 > t'_0$, $f(t_3) < 0$. Now $f(t_3) < 0$ and $f(\infty) > 0$ implies that $f(t) = 0$ have another positive root in the interval (t_3, ∞) which is a contradiction.

Therefore for $t > t'_0, f(t) > 0$.

Also $t'_0 \geq 1$. So $|Q(z)| > 0$ for $|z| > t'_0$.

Therefore $Q(z)$ does not vanish in $|z| > t'_0$.

Hence all the zeros of $Q(z)$ lie in $|z| \leq t'_0$.

Let $z = z_0$ be a zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that $Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)}.P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)}.0 = 0$. Therefore $Q\left(\frac{1}{z_0}\right) = 0$. So

$z = \frac{1}{z_0}$ is a root of $Q(z) = 0$. Hence $\left|\frac{1}{z_0}\right| \le t'_0$ implies that $|z_0| \ge \frac{1}{t'_0}$.
As z_0 is an arbitrary root of $P(z) = 0$.

$$\text{Therefore all the zeros of } P(z) \text{ lie in } |z| \ge \frac{1}{t'_0}. \tag{3.5}$$

From (3.3) and (3.5) we get that all the zeros of $P(z)$ lie in the proper ring shaped region $\frac{1}{t'_0} \le |z| \le t_0$ where t_0 and t'_0 are the greatest positive roots of the equations $g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$ and $f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$ where M and M' are given in the statement of Theorem 3.1. This proves the theorem. □

In the line of Theorem 3.1, we may state the following theorem in view of Lemma 2.2:

Theorem 3.2. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}. Also for all sufficiently large r in the disc $|z| \le [re^{L(r)}]$, $\left|a_{N(r)}\right| \ne 0$, $|a_0| \ne 0$. and also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t'_0} \le |z| \le t_0$$

where t_0 is the greatest positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)+1} - \left(\left|a_{N(r)}\right| + M\right) t^{N(r)} + M = 0$$

and t'_0 is the greatest positive root of

$$f(t) \equiv |a_0| t^{N(r)+1} - (|a_0| + M') t^{N(r)} + M' = 0$$
$$\text{where } M = max\left\{|a_0|, |a_1|, ... \left|a_{N(r)-1}\right|\right\}$$
$$\text{and } M' = max\left\{|a_1|, |a_2|, ... \left|a_{N(r)}\right|\right\}.$$

The proof is omitted.

Theorem 3.3. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L-order ρ^L, $a_{N(r)} \ne 0$, $a_0 \ne 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \le [rL(r)]$ when r is sufficiently large. Further, for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \ge |a_1| (\rho^L)^{N(r)-1} \ge ... \ge \left|a_{N(r)-1}\right| \rho^L \ge \left|a_{N(r)}\right|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right).$$

Proof. For the given entire function

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with $a_n \to 0$ as $n > N(r)$, where r is sufficiently large, $N(r)$ exists and $N(r) \leq [rL(r)]^{\rho^L + \epsilon}$.
Therefore

$$P(z) \approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}$$

as $a_0 \neq 0, a_{N(r)} \neq 0$ and $a_n \to 0$ for $n > N(r)$.
Let us consider

$$\begin{aligned} R(z) &= (\rho^L)^{N(r)} P\left(\frac{z}{\rho^L}\right) \approx (\rho^L)^{N(r)} \left(a_0 + a_1 \frac{z}{\rho^L} + a_2 \frac{z^2}{(\rho^L)^2} + ... + a_{N(r)} \frac{z^{N(r)}}{(\rho^L)^{N(r)}}\right) \\ &= \left(a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)} z^{N(r)}\right). \end{aligned}$$

Therefore

$$|R(z)| \geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right|. \quad (3.6)$$

Now by the given condition $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq ...$ provided $|z| \neq 0$,we obtain that

$$\begin{aligned} &\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \leq |a_0| (\rho^L)^{N(r)} + ... + \left|a_{N(r)-1}\right| \rho^L |z|^{N(r)-1} \\ &\leq |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right). \end{aligned}$$

Therefore on $|z| \neq 0$,

$$-\left|a_0 (\rho^L)^{N(r)} + a_1 (\rho^L)^{N(r)-1} z + ... + a_{N(r)-1} \rho^L z^{N(r)-1}\right| \geq -|a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right). \quad (3.7)$$

Therefore using (3.7) we get from (3.6) that

$$\begin{aligned} |R(z)| &\geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right) \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| (\rho^L)^{N(r)} |z|^{N(r)} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}} + ...\right) \\ &= |z|^{N(r)} \left[\left|a_{N(r)}\right| - |a_0| (\rho^L)^{N(r)} \left\{\sum_{k=1}^{\infty} \frac{1}{|z|^k}\right\}\right]. \end{aligned}$$

Clearly $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is a geometric series which is convergent for $\frac{1}{|z|} < 1$ i.e., for $|z| > 1$ and converges to$\frac{1}{|z|} \frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1}$ if $|z| > 1$. Hence we get from above that for $|z| > 1 |R(z)| >$

$|z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right)$. Now for $|z| > 1$,

$$\begin{aligned} |R(z)| > 0 \text{ if } & |z|^{N(r)}\left(\left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1}\right) \geq 0 \\ \text{i.e., if } & \left|a_{N(r)}\right| - (\rho^L)^{N(r)}|a_0|\frac{1}{|z|-1} \geq 0 \\ \text{i.e., if } & \left|a_{N(r)}\right| \geq (\rho^L)^{N(r)}\frac{|a_0|}{|z|-1} \\ \text{i.e., if } & |z| - 1 \geq (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|} \\ \text{i.e., if } & |z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|} > 1. \end{aligned}$$

Therefore $|R(z)| > 0$ if $|z| \geq 1 + (\rho^L)^{N(r)}\frac{|a_0|}{\left|a_{N(r)}\right|}$. So all the zeros of $R(z)$ lie in $|z| < 1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}$. Let z_0 be an arbitrary zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$. Putting $z = \rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0) = (\rho^L)^{N(r)}.P(z_0) = (\rho^L)^{N(r)}.0 = 0$.

Hence $z = \rho^L z_0$ is a zero of $R(z)$. Therefore$\left|\rho^L z_0\right| < 1+\frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}$i.e., $|z_0| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right)$. Since z_0 is any zero of $P(z)$ therefore all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{\rho^L}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^L)^{N(r)}\right). \tag{3.8}$$

Again let us consider $F(z) = (\rho^L)^{N(r)}z^{N(r)}P\left(\frac{1}{\rho^L z}\right)$. Now $F(z) = (\rho^L)^{N(r)}z^{N(r)}P\left(\frac{1}{\rho^L z}\right)$ $\approx (\rho^L)^{N(r)}z^{N(r)}\left\{a_0 + \frac{a_1}{\rho^L z} + ... + \frac{a_{N(r)}}{(\rho^L z)^{N(r)}}\right\} = a_0(\rho^L)^{N(r)}z^{N(r)} + a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}$. Therefore $|F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right|$. Again

$$\begin{aligned} & \left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \\ & \leq |a_1|(\rho^L)^{N(r)-1}\left(|z|^{N(r)-1} + ... + |z| + 1\right) \end{aligned}$$

provided $|z| \neq 0$. So $\left|a_1(\rho^L)^{N(r)-1}z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)$. So for $|z| \neq 0, |F(z)| \geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - |a_1|(\rho^L)^{N(r)-1}|z|^{N(r)}\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)$ $= (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\left(\frac{1}{|z|} + ... + \frac{1}{|z|^{N(r)}}\right)\right]$. Therefore for $|z| \neq 0$,

$$|F(z)| > (\rho^L)^{N(r)-1}|z|^{N(r)}\left[|a_0|\rho^L - |a_1|\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right]. \tag{3.9}$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$*i.e.*, for $|z| > 1$ and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ if } |z| > 1. \tag{3.10}$$

Using (3.9) and (3.10) we have for $|z| > 1$, $|F(z)| > (\rho^L)^{N(r)-1} |z|^{N(r)} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right]$. Hence for $|z| > 1$,

$$|F(z)| > 0 \text{ if } |z|^{N(r)} (\rho^L)^{N(r)-1} \left[|a_0| \rho^L - \frac{|a_1|}{|z|-1}\right] \geq 0$$

$$\begin{aligned} &\text{i.e., if } |a_0| \rho^L - \frac{|a_1|}{|z|-1} \geq 0 \\ &\text{i.e., if } |a_0| \rho^L \geq \frac{|a_1|}{|z|-1} \\ &\text{i.e., if } |z| \geq 1 + \frac{|a_1|}{|a_0| \rho^L} > 1. \end{aligned}$$

Therefore $|F(z)| > 0$ for $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. So $F(z)$ does not vanish in $|z| \geq 1 + \frac{|a_1|}{|a_0|\rho^L}$. Equivalently all the zeros of $F(z)$ lie in $|z| < 1 + \frac{|a_1|}{|a_0|\rho^L}$. Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $a_0 \neq 0$ and $z_0 \neq 0$.

Now let us put $z = \frac{1}{\rho^L z_0}$ in $F(z)$. So we have $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)} \left(\frac{1}{\rho^L z_0}\right)^{N(r)} .P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a root of $F(z)$.

Hence

$$\left|\frac{1}{\rho^L z_0}\right| < 1 + \frac{|a_1|}{|a_0| \rho^L}$$

$$\begin{aligned} &\text{i.e., } \frac{1}{|z_0|} < \rho^L \left(1 + \frac{|a_1|}{|a_0| \rho^L}\right) \\ &\text{i.e., } |z_0| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}. \end{aligned}$$

As z_0 is an arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie on

$$|z| > \frac{1}{\rho^L \left(1 + \frac{|a_1|}{|a_0|\rho^L}\right)}. \tag{3.11}$$

From (3.8) and (3.11) we get that all the zeros of $P(z)$ lie on the proper ring shaped region $\frac{1}{\rho^L\left(1+\frac{|a_1|}{|a_0|\rho^L}\right)} < |z| < \frac{1}{\rho^L} \left(1 + \frac{|a_0|}{|a_{N(r)}|} (\rho^L)^{N(r)}\right)$ where $|a_0| (\rho^L)^{N(r)} \geq |a_1| (\rho^L)^{N(r)-1} \geq ... \geq \left|a_{N(r)}\right|$ for $\rho^L > 0$.

This proves the theorem. □

In the line of Theorem 3.3, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.4. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}, $a_{N(r)} \neq 0$, $a_0 \neq 0$ and also $a_n \to 0$ for $n > N(r)$ for the disc $|z| \leq [re^{L(r)}]$ when r is sufficiently large. Further, for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \geq |a_1| (\rho^{L^*})^{N(r)-1} \geq ..... \geq \left|a_{N(r)-1}\right| \rho^{L^*} \geq \left|a_{N(r)}\right|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^{L^*}\left(1 + \frac{|a_1|}{|a_0|\rho^{L^*}}\right)} < |z| < \frac{1}{\rho^{L^*}}\left(1 + \frac{|a_0|}{\left|a_{N(r)}\right|}(\rho^{L^*})^{N(r)}\right).$$

The proof is omitted.

Corollary 3.1. *From Theorem 3.3 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

of degree n, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

just on putting $\rho^L = 1$.

Corollary 3.2. *From Theorem 3.4 we can easily conclude that all the zeros of*

$$P(z) = a_0 + a_1 z + ... + a_n z^n$$

of degree n, $|a_n| \neq 0$ with the property $|a_0| \geq |a_1| \geq ... \geq |a_n|$ lie in the proper ring shaped region

$$\frac{1}{\left(1 + \frac{|a_1|}{|a_0|}\right)} < |z| < \left(1 + \frac{|a_0|}{|a_n|}\right)$$

just on putting $\rho^{L^*} = 1$.

Theorem 3.5. *Let $P(z)$ be an entire function with L-order ρ^L. For sufficiently large values of r in the disk $|z| \leq [rL(r)]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1} z^{p_1} + a_{p_2} z^{p_2} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, a_0 \neq 0$$

be such that $1 \leq p_1 < p_2 ... < p_m \leq N(r) - 1$, p_i's are integers and for $\rho^L > 0$,

$$|a_0| (\rho^L)^{N(r)} \geq \left|a_{p_1}\right| (\rho^L)^{N(r)-p_1} \geq ... \geq \left|a_{p_m}\right| (\rho^L)^{N(r)-p_m}.$$

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^L t_0'} < |z| < \frac{1}{\rho^L} t_0$$

where t_0 and t_0' are the unique positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| \left(\rho^L\right)^{N(r)} = 0 \text{ and}$$
$$f(t) \equiv |a_0| \left(\rho^L\right)^{p_1} t^{p_1} - |a_0| \left(\rho^L\right)^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively.

Proof. Let

$$P(z) = a_0 + a_{p_1} z^{p_1} + ... + a_{p_m} z^{p_m} + a_{N(r)} z^{N(r)}, \left|a_{N(r)}\right| \neq 0. \tag{3.12}$$

Also for $\rho^L > 0$, $|a_0| \left(\rho^L\right)^{N(r)} \geq \left|a_{p_1}\right| \left(\rho^L\right)^{N(r)-p_1} \geq ... \geq \left|a_{N(r)}\right|$. Let us consider

$$R(z) = \left(\rho^L\right)^{N(r)} P\left(\frac{z}{\rho^L}\right) = \left(\rho^L\right)^{N(r)} \left\{a_0 + a_{p_1} \frac{z^{p_1}}{\left(\rho^L\right)^{p_1}} + ... + a_{p_m} \frac{z^{p_m}}{\left(\rho^L\right)^{p_m}} + a_{N(r)} \frac{z^{N(r)}}{\left(\rho^L\right)^{N(r)}}\right\}$$
$$= a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m} + a_{N(r)} z^{N(r)}.$$

Therefore

$$|R(z)| \geq \left|a_{N(r)} z^{N(r)}\right| - \left|a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m}\right|. \tag{3.13}$$

Now for $|z| \neq 0$,

$$\begin{aligned}
&\left|a_0 \left(\rho^L\right)^{N(r)} + a_{p_1} \left(\rho^L\right)^{N(r)-p_1} z^{p_1} + ... + a_{p_m} \left(\rho^L\right)^{N(r)-p_m} z^{p_m}\right| \\
&\leq |a_0| \left(\rho^L\right)^{N(r)} + \left|a_{p_1}\right| \left(\rho^L\right)^{N(r)-p_1} |z|^{p_1} + ... + \left|a_{p_m}\right| \left(\rho^L\right)^{N(r)-p_m} |z|^{p_m} \\
&\leq |a_0| \left(\rho^L\right)^{N(r)} \left(1 + |z|^{p_1} + ... + |z|^{p_m}\right) \\
&= |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_2}} + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right).
\end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), we have for $|z| \neq 0$

$$\begin{aligned}
|R(z)| &\geq \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}}\right) \\
&> \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \left(\frac{1}{|z|} + ... + \frac{1}{|z|^{p_m+1-p_1}} + \frac{1}{|z|^{p_m+1}} + ...\right) \\
&= \left|a_{N(r)}\right| |z|^{N(r)} - |a_0| \left(\rho^L\right)^{N(r)} |z|^{p_m+1} \sum_{k=1}^{\infty} \frac{1}{|z|^k}.
\end{aligned} \tag{3.15}$$

The geometric series $\sum\limits_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|}<1$i.e.,for $|z|>1$ and converges to $\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}}=\frac{1}{|z|-1}$.
Therefore $\sum\limits_{k=1}^{\infty}\frac{1}{|z|^k}=\frac{1}{|z|-1}$ for $|z|>1$. So on $|z|>1$,

$$\begin{aligned}
&|R(z)|>0 \text{ if } \left|a_{N(r)}\right||z|^{N(r)}-\frac{|a_0|\,(\rho^L)^{N(r)}\,|z|^{p_m+1}}{|z|-1}\geq 0\\
&\text{i.e., if } \left|a_{N(r)}\right||z|^{N(r)}\geq\frac{|a_0|\,(\rho^L)^{N(r)}\,|z|^{p_m+1}}{|z|-1}\\
&\text{i.e., if } \left|a_{N(r)}\right||z|^{N(r)+1}-\left|a_{N(r)}\right||z|^{N(r)}\geq |a_0|\,(\rho^L)^{N(r)}\,|z|^{p_m+1}\\
&\text{i.e., if } |z|^{p_m+1}\left(\left|a_{N(r)}\right||z|^{N(r)-p_m}-\left|a_{N(r)}\right||z|^{N(r)-p_m-1}-|a_0|\,(\rho^L)^{N(r)}\right)\geq 0.
\end{aligned}$$

Let us consider $g(t)\equiv\left|a_{N(r)}\right||t|^{N(r)-p_m}-\left|a_{N(r)}\right||t|^{N(r)-p_m-1}-|a_0|\,(\rho^L)^{N(r)}=0$. Clearly $g(t)=0$ has one positive root because the maximum number of changes in sign in $g(t)$ is one and $g(0)=-|a_0|\,\rho^{N(r)}$ is $-ve$, $g(\infty)$ is $+ve$.
Let t_0 be the positive root of $g(t)=0$ and $t_0>1$. Clearly for $t>t_0$, $g(t)>0$. If not for some $t_1>t_0$, $g(t_1)<0$.
Then $g(t_1)<0$ and $g(\infty)>0$. Therefore $g(t)=0$ must have another positive root in (t_1,∞) which gives a contradiction .
Hence for $t\geq t_0$, $g(t)\geq 0$ and $t_0>1$. So $|R(z)|>0$ for $|z|\geq t_0$.
Thus $R(z)$ does not vanish in $|z|\geq t_0$.

Hence all the zeros of $R(z)$ lie in $|z|<t_0$.

Let $z=z_0$ be any zero of $P(z)$. So $P(z_0)=0$. Clearly $z_0\neq 0$ as $a_0\neq 0$. Putting $z=\rho^L z_0$ in $R(z)$ we have $R(\rho^L z_0)=(\rho^L)^{N(r)}.P(z_0)=(\rho^L)^{N(r)}.0=0$. Therefore $R(\rho^L z_0)=0$ and so $z=\rho^L z_0$ is a zero of $R(z)$ and consequently $\left|\rho^L z_0\right|<t_0$ which implies $|z_0|<\frac{t_0}{\rho^L}$. As z_0 is an arbitrary zero of $P(z)$,

$$\text{all the zeros of } P(z) \text{ lie in } |z|<\frac{t_0}{\rho^L}. \tag{3.16}$$

Again let us consider $F(z)=(\rho^L)^{N(r)}z^{N(r)}P\left(\frac{1}{\rho^L z}\right)$. Now

$$\begin{aligned}
F(z)&=(\rho^L)^{N(r)}z^{N(r)}.\left\{a_0+a_{p_1}\frac{1}{(\rho^L)^{p_1}z^{p_1}}+.....+a_{p_m}\frac{1}{(\rho^L)^{p_m}z^{p_m}}+a_{N(r)}\frac{1}{(\rho^L)^{N(r)}z^{N(r)}}\right\}\\
&=a_0(\rho^L)^{N(r)}z^{N(r)}+a_{p_1}(\rho^L)^{N(r)-p_1}z^{N(r)-p_1}+...+a_{p_m}(\rho^L)^{N(r)-p_m}z^{N(r)-p_m}+a_{N(r)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\left|a_{p_1}(\rho^L)^{N(r)-p_1}z^{N(r)-p_1}+...+a_{p_m}(\rho^L)^{N(r)-p_m}z^{N(r)-p_m}+a_{N(r)}\right|\\
&\leq\left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}|z|^{N(r)-p_1}+...+\left|a_{p_m}\right|(\rho^L)^{N(r)-p_m}|z|^{N(r)-p_m}+\left|a_{N(r)}\right|\\
&\leq\left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\left(|z|^{N(r)-p_1}+|z|^{N(r)-p_2}+...+|z|^{N(r)-p_m}+1\right).
\end{aligned}$$

So for $|z| \neq 0$,

$$\begin{aligned}|F(z)| &\geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_{p_1}(\rho^L)^{N(r)-p_1}z^{N(r)-p_1} + \ldots + a_{p_m}(\rho^L)^{N(r)-p_m}z^{N(r)-p_m} + a_{N(r)}\right| \\ &\geq |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + |z|^{N(r)-p_2} + \ldots + |z|^{N(r)-p_m} + 1\right) \\ &= |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\frac{1}{|z|} + \frac{1}{|z|^{p_2-p_1+1}} + \ldots + \frac{1}{|z|^{N(r)-p_1+1}}\right)\end{aligned}$$

i.e., on $|z| \neq 0$, $|F(z)| > |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right)$. The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for $\frac{1}{|z|} < 1$i.e., for $|z| > 1$ and converges to $\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}$. Therefore $\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1}$ for $|z| > 1$. Therefore for $|z| > 1$

$$|F(z)| > |a_0|(\rho^L)^{N(r)}|z|^{N(r)} - \left|a_{p_1}\right|(\rho^L)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(\frac{1}{|z|-1}\right)$$

$$\begin{aligned}&= (\rho^L)^{N(r)-p_1}\left((\rho^L)^{p_1}|a_0||z|^{N(r)} - \left|a_{p_1}\right|\frac{|z|^{N(r)-p_1+1}}{|z|-1}\right) \\ &= (\rho^L)^{N(r)-p_1}|z|^{N(r)-p_1+1}\left(|a_0|\left(\rho^L\right)^{p_1}|z|^{p_1-1} - \frac{\left|a_{p_1}\right|}{|z|-1}\right)\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}|F(z)| > 0 \text{ if } &|a_0|(\rho^L)^{p_1}|z|^{p_1-1} - \frac{\left|a_{p_1}\right|}{|z|-1} \geq 0 \\ \text{i.e., if } &|a_0|(\rho^L)^{p_1}|z|^{p_1-1} \geq \frac{\left|a_{p_1}\right|}{|z|-1} \\ \text{i.e., if } &|a_0|(\rho^L)^{p_1}|z|^{p_1} - |a_0|(\rho^L)^{p_1}|z|^{p_1-1} - \left|a_{p_1}\right| \geq 0. \qquad (3.17)\end{aligned}$$

Therefore on $|z| > 1, |F(z)| > 0$ if (3.17) holds. Let us consider $f(t) = |a_0|(\rho^L)^{p_1}t^{p_1} - |a_0|(\rho^L)^{p_1}t^{p_1-1} - \left|a_{p_1}\right| = 0$. Clearly $f(t) = 0$ has exactly one positive root and is greater than one. Let t_0' be the positive root of $f(t) = 0$. Therefore $t_0' > 1$. Obviously if $t \geq t_0'$ then $f(t) \geq 0$. So for $|F(z)| > 0$, $|z| \geq t_0'$. Therefore $F(z)$ does not vanish in $|z| \geq t_0'$.

Hence all the zeros of $F(z)$ lie in $|z| < t_0'$.

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.

Now putting $z = \frac{1}{\rho^L z_0}$ in $F(z)$ we obtain that $F\left(\frac{1}{\rho^L z_0}\right) = (\rho^L)^{N(r)}\left(\frac{1}{\rho^L z_0}\right)^{N(r)}.P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)}.P(z_0)$ $= \left(\frac{1}{z_0}\right)^{N(r)}.0 = 0$. Therefore $z = \frac{1}{\rho^L z_0}$ is a zero of $F(z)$. Now $\left|\frac{1}{\rho^L z_0}\right| < t_0'$ *i.e.,* $\left|\frac{1}{z_0}\right| < \rho^L t_0'$ *i.e.,* $|z_0| > \frac{1}{\rho^L t_0'}$. As z_0 is an arbitrary zero of $P(z)$ therefore we obtain that

$$\text{all the zeros of } P(z) \text{ lie in } |z| > \frac{1}{\rho^L t_0'}. \qquad (3.18)$$

Using (3.16) and (3.18) we get that all the zeros of $P(z)$ lie in the ring shaped region $\frac{1}{\rho^{L}t_0'} < |z| < \frac{t_0}{\rho^{L}}$ where t_0, t_0' are the unique positive roots of the equations $g(t) = 0$ and $f(t) = 0$ respectively whose forms are given in the statement of Theorem 3.3. This proves the theorem. □

In the line of Theorem 3.5, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.6. *Let $P(z)$ be an entire function with L^*-order ρ^{L^*}. For sufficiently large values of r in the disk $|z| \le [re^{L(r)}]$, the Taylor's series expansion of $P(z)$*

$$P(z) = a_0 + a_{p_1}z^{p_1} + a_{p_2}z^{p_2} + ... + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}, a_0 \neq 0$$

be such that $1 \le p_1 < p_2 ... < p_m \le N(r) - 1$, p_i's are integers and for $\rho^{L^*} > 0$,

$$|a_0| (\rho^{L^*})^{N(r)} \ge \left|a_{p_1}\right| (\rho^{L^*})^{N(r)-p_1} \ge ... \ge \left|a_{p_m}\right| (\rho^{L^*})^{N(r)-p_m}.$$

Then all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^{L^*}t_0'} < |z| < \frac{1}{\rho^{L^*}}t_0$$

where t_0 and t_0' are the unique positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)-p_m} - \left|a_{N(r)}\right| t^{N(r)-p_m-1} - |a_0| (\rho^{L^*})^{N(r)} = 0 \textit{ and}$$
$$f(t) \equiv |a_0| (\rho^{L^*})^{p_1} t^{p_1} - |a_0| (\rho^{L^*})^{p_1} t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively.

The proof is omitted.

Corollary 3.3. *In view of Theorem 3.5 we may state that all the zeros of the polynomial $P(z) = a_0 + a_{p_1}z^{p_1} + ... + a_{p_m}z^{p_m} + a_nz^n$ of degree n with $1 \le p_1 < p_2 < ... < p_m \le n-1$, p_i's are integers such that*

$$|a_0| \ge \left|a_{p_1}\right| \ge .... \ge |a_n|$$

lie in ring shaped region

$$\frac{1}{t_0'} < |z| < t_0$$

where t_0, t_0' are the unique positive roots of the equations

$$g(t) \equiv |a_n| t^{n-p_m} - |a_n| t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| t^{p_1} - |a_0| t^{p_1-1} - \left|a_{p_1}\right| = 0$$

respectively just substituting $\rho^{L} = 1$.

Corollary 3.4. *In view of Theorem 3.6 we may state that all the zeros of the polynomial* $P(z) = a_0 + a_{p_1}z^{p_1} + ... + a_{p_m}z^{p_m} + a_n z^n$ *of degree n with* $1 \leq p_1 < p_2 < ... < p_m \leq n-1$, p_i*'s are integers such that*

$$|a_0| \geq |a_{p_1}| \geq .... \geq |a_n|$$

lie in ring shaped region

$$\frac{1}{t_0'} < |z| < t_0$$

where t_0, t_0' *are the unique positive roots of the equations*

$$g(t) \equiv |a_n| \, t^{n-p_m} - |a_n| \, t^{n-p_m-1} - |a_0| = 0$$

and

$$f(t) \equiv |a_0| \, t^{p_1} - |a_0| \, t^{p_1-1} - |a_{p_1}| = 0$$

respectively just substituting $\rho^{L^*} = 1$.

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Primality Testing and Integer Factorization by using Fourier Transform of a Correlation Function Generated from the Riemann Zeta Function

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Abstract

In this article, the author tries to make primarity testing and factorization of integers by using Fourier transform of a correlation function generated from the Riemann zeta function.

Keywords: Primarity testing, factorization, Fourier transform, Riemann zeta function.
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1. Introduction

In number theory, integer factorization or prime factorization is the decomposition of a composite number into smaller non-trivial divisors, which when multiplied together equal the original integer. When the numbers are very large, no efficient, non-quantum integer factorization algorithm is known; an effort by several researchers concluded in 2009, factoring a 232-digit number (RSA-768), utilizing hundreds of machines over a span of 2 years. The presumed difficulty of this problem is at the heart of widely used algorithms in cryptography such as RSA (Rivest *et al.*, 1978). Many areas of mathematics and computer science have been brought to bear on the problem, including elliptic curves, algebraic number theory, and quantum computing.

In this article, the author tries to make primarity testing and factorization of integers by using Fourier transform of a correlation function generated from the Riemann zeta function.

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2. Frequency Spectrum of a Correlation Function generated from the Riemann Zeta Function

Riemann zeta function is an analytic function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which can also be given by (Hardy & Riesz, 2005).

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad (\mathrm{Re}[\,s\,] > 1)\,, \tag{2.1}$$

where $\Gamma(s)$ is a Gamma function.

We define the Fourier transform of $z_\sigma(t, \tau)$ shown as

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \int_{-T}^{+T} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau\,, \tag{2.2}$$

where $z_\sigma(t, \tau)$ is a time-dependent autocorrelation function (Yen, 1987) defined by

$$z_\sigma(t, \tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2))\,.$$

In this formula, $\zeta^*(s)$ is a conjugate of $\zeta(s)$.

From the infinite sum of the Riemann zeta function given by $\zeta(\sigma - it) = \sum_{n=1}^{\infty} \frac{\exp(it \log n)}{n^\sigma}$, we have

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \int_{-T}^{+T} \sum_{k=1}^{\infty} \frac{1}{k^\sigma} \exp[\,i\,(t + \tau/2) \log k\,] \cdot \sum_{l=1}^{\infty} \frac{1}{l^\sigma} \exp[\,-i\,(t - \tau/2) \log l\,]\, e^{-i\omega\tau} d\tau$$

$$= \lim_{T\to\infty} \int_{-T}^{+T} \sum_{k,l=1}^{\infty} \frac{1}{(kl)^\sigma} \exp[\,i \log(k/l)t\,] \exp[\,i \log(kl)\tau/2\,]\, e^{-i\omega\tau} d\tau.$$

For the integer n , put $n = kl$, then we can write

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \sum_{k,l=1}^{\infty} \frac{1}{n^\sigma} \exp[\,i \log(k/l)\, t\,] \int_{-T}^{+T} \exp(i\tau \log n/2)\, e^{-i\omega\tau} d\tau\,,$$

where $\int_{-T}^{+T} \exp(i\tau \log n/2)\, e^{-i\omega\tau} d\tau = \frac{2T \sin\left(\omega - \frac{1}{2}\log n\right)}{(\omega - \frac{1}{2}\log n)}$.

When we let $a(n, t) = \sum_{n=kl} \exp[i \log(k/l)\, t]$, Eq.(2) can be rewritten as

$$Z_\sigma(t, \omega) = \lim_{T\to\infty} \sum_{n=1}^{\infty} \frac{a(n, t)}{n^\sigma} \frac{2T \sin\left(\omega - \frac{1}{2}\log n\right)}{\left(\omega - \frac{1}{2}\log n\right)} = \sum_{n=1}^{\infty} \frac{a(n, t)}{n^\sigma} 2\pi\delta\left(\omega - \frac{1}{2}\log n\right)\,,$$

where $a(n,t)$ is a real valued function given by

$$a(n,t) = \frac{1}{2}\sum_{n=kl}\left\{\exp\left[i\log\left(k/l\right)t\right] + \exp\left[i\log\left(l/k\right)t\right]\right\} = \sum_{n=kl}\cos\left[\,\log\left(k/l\right)t\,\right]$$

and $\delta(\omega)$ is a Dirac's delta function.

Lemma 2.1. *$a(n,t)$ is a multiplicative on n .*

Proof. As we can write $a(n,t) = \sum_{n=kl}\exp[i\log(k/l)t]$, the multiplicative property of which can be shown from

$$a(n,t) = \sum_{k|n}\exp\left(it\log(k^2/n)\right) = \frac{1}{n^{it}}\sum_{k|n}k^{2it},$$

where the subscript $k|n$ indicates integers k which divide n .

If $f(n)$ is multiplicative, then $F(n) = \sum_{d|n} f(d)$ is multiplicative. From which, we have $a(mn,t) = a(m,t)a(n,t)$ for the case when satisfying $(m,n) = 1$, because k^{2it} is multiplicative. □

From the definition of $a(n,t)$, we can obtain the following recurrence formula given by (Musha, 2012).

$$a(p^r,t) = a(p^{r-1},t)\cos(t\log p) + \cos(rt\log p) \quad (r = 1,2,3,\cdots). \tag{2.3}$$

From which, it can be proved that $a(p^r,t) = \dfrac{\sin[(r+1)t\log p]}{\sin(t\log p)}$. (2.4)

From Eq.(3), we have $Z_\sigma\left(t, \frac{1}{2}\log n\right) = \dfrac{2\pi\delta(0)}{n^\sigma}a(n,t)$.

For the integer n given by $n = p^a q^b r^c\cdots$, we have

$$Z_\sigma\left(t, \tfrac{1}{2}\log n\right) = \frac{2\pi\delta(0)}{n^\sigma}\frac{\sin[(a+1)t\log p]}{\sin(t\log p)}\frac{\sin[(b+1)t\log q]}{\sin(t\log q)}\frac{\sin[(c+1)t\log r]}{\sin(t\log r)}\cdots,$$

from Lemma.1 and Eq.(5).

From the Fourier transform of $Z_a\left(t, \frac{1}{2}\log n\right)$ given by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \frac{1}{2}\log n\right)e^{-i\omega t}dt$, we can obtain the following Lemma.

Lemma 2.2. *If $n = p_1p_2p_3\cdots p_k$, where $p_1, p_2, p_3, \cdots, p_k$ are different primes, $F_n(\omega)$ is consisted of 2^{k-1} discrete spectrum.*

Proof. From Eq. (4), we have

$$a(n,t) = 2\cos(t\log p_1)\cdot 2\cos(t\log p_2)\cdot 2\cos(t\log p_3)\cdots 2\cos(t\log p_k).$$

By the trigonometrical formula shown as $\cos\alpha \cdot \cos\beta = \frac{1}{2}\{\cos(\alpha-\beta) + \cos(\alpha+\beta)\}$, we have

$$a(n,t) = 2^2 \times \frac{1}{2}\left\{\cos[t(\log p_1 - \log p_2)] + \cos[t(\log p_1 + \log p_2)]\right\} \cdot 2\cos(t\log p_3)\cdots 2\cos(t\log p_k)$$
$$= 2^2\left\{\cos[t(\log p_1 - \log p_2)]\cos(t\log p_3) \quad + \cos[t(\log p_1 + \log p_2)]\cos(t\log p_3)\right\}\cdots 2\cos(t\log p_k)$$
$$= 2^2 \times \frac{1}{2}\left\{\cos[t(\log p_1 - \log p_2 - \log p_3)] \quad + \cos[t(\log p_1 - \log p_2 + \log p_3)]\right.$$
$$\left. + \cos[t(\log p_1 + \log p_2 - \log p_3)] + \cos[t(\log p_1 + \log p_2 + \log p_3)]\right\} 2\cos(t\log p_4)\cdots 2\cos(t\log p_k)$$

By repeating the above computations, we have

$$a(n,t) = 2\sum_{i=1}^{2^{k-1}} \cos[\, t(\lambda_{i1}\log p_1 + \lambda_{i2}\log p_2 + \cdots + \lambda_{ik}\log p_k)],$$

where $\lambda_{i1} = +1$ and $\lambda_{ij} = +1$ or -1 for $j > 1$.

As $\log p_1$, $\log p_2$, $\log p_3$, $\cdots$, $\log p_k$ are linearly independent over **Z** (Kac, 1959), thus $F_n(\omega)$ is consisted of 2^{k-1} different spectrum. □

Then we obtain following Theorems.

Theorem 2.1. *If and only $F_n(\omega)$ is consisted of a single spectra for $\omega \geq 0$, then n is a prime.*

Proof. The Fourier transform of $\cos(t\log p)$ can be given by $\pi[\delta(\omega - \log p) + \delta(\omega + \log p)]$, and thus it is clear from Lemma 2.2. □

Theorem 2.2. *If and only $F_n(\omega)$ is consisted of two spectrum for $\omega \geq 0$, then n has either form of $n = p \cdot q\ (p \neq q)$, $n = p^2$ or $n = p^3$.*

Proof. From Theorem I, there is only a case for the integer $n = p_1 p_2 \cdots p_k$, when $F_n(\omega)$ is consisted of two spectrum, that is $n = p \cdot q \quad (p \neq q)$.

From Eq.(4), we have following equations for $a(p^r, t)$;

$$r = 1,\ a(p,t) \ = 2\cos(t\log p)$$
$$r = 2,\ a(p^2,t) = 1 + 2\cos(2t\log p)$$
$$r = 3,\ a(p^3,t) = 2\cos(t\log p) + 2\cos(3t\log p)$$
$$r = 4,\ a(p^4,t) = 1 + 2\cos(2t\log p) + 2\cos(4t\log p)$$
$$r = 5,\ a(p^5,t) = 2\cos(t\log p) + 2\cos(3t\log p) + 2\cos(5t\log p)$$
$$r = 6,\ a(p^6,t) = 1 + 2\cos(2t\log p) + 2\cos(4t\log p) + 2\cos(6t\log p)$$
$$r = 7,\ a(p^7,t) = 2\cos(t\log p) + 2\cos(3t\log p) + 2\cos(5t\log p) + 2\cos(7t\log p)$$
$$\vdots$$

Including the spectra at $\omega = 0$, there are cases for $r = 2$ and $r = 3$ when $a(n,t)$ has two spectrum. □

Theorem 2.3. *If $F_n(\omega)$ is consisted of two spectrums at frequencies ω_1 and ω_2 and $n = p \cdot q$, we can obtain factors of an integer n given by $p = \exp\left(\frac{\omega_2 - \omega_1}{2}\right)$ and $q = \exp\left(\frac{\omega_1 + \omega_2}{2}\right)$.*

Proof. If $n = p \cdot q$, then we obtain $Z_\sigma\left(t, \frac{1}{2}\log n\right) = \frac{4\pi\delta(0)}{n^\sigma} \times \cos(t \log p) \cdot \cos(t \log q) = \frac{2\pi\delta(0)}{n^\sigma}\left\{\cos[(\log q - \log p)t] + \cos[(\log q + \log p)t]\right\}$.

When we let $\omega_1 = \log q - \log p$, $\omega_2 = \log q + \log p$, we have $p = \exp\left(\frac{\omega_2 - \omega_1}{2}\right)$, $q = \exp\left(\frac{\omega_1 + \omega_2}{2}\right)$. □

3. Primarity Testing and Factorization from Fourier spectrum

From Theorems 2.1, 2.2 and 2.3, we can make primality testing and factorization of the integer n consisted of two primes from the Fourier spectrum $F_n(\omega)$ $(\omega \geq 0)$ by following procedures;

At first, compute the Fourier transform $Z_\sigma(t, \omega) = \int_{-\infty}^{+\infty} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau$, where $z_\sigma(t, \tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2))$, from which we can obtain the Fourier spectrum by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \frac{1}{2}\log n\right) e^{-i\omega t} dt$. Then we can make primality testing and integer factorization of an integer n, the process of which is shown in Figure 1.

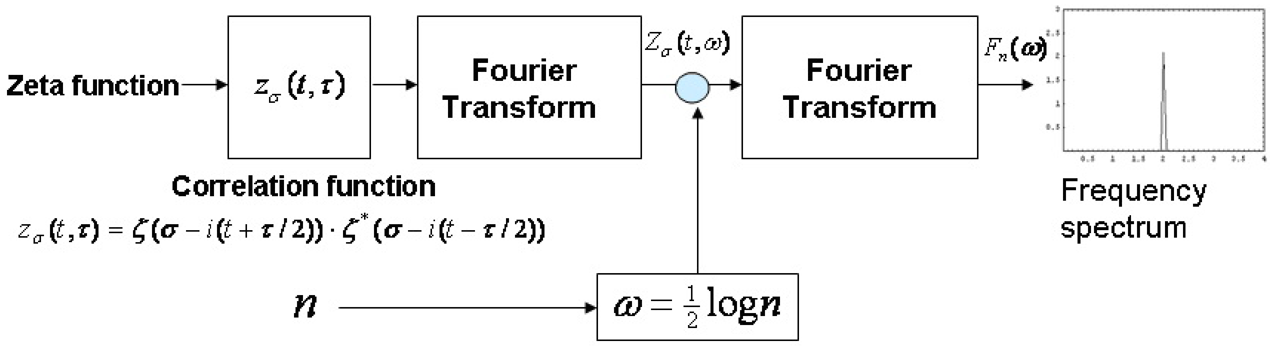

Figure 1. Process to conduct primality testing for the integer n.

From this process, we can recognize the prime as a single spectra from the frequency analysis result. If there are two spectrum observed from the calculation result, n has either form of $n = p \cdot q$ $(p \neq q)$, $n = p^2$ or $n = p^3$.

In this case, we can obtain factors of an integer n from Theorem 2.3.

As the Fourier transform $Z_\sigma(t, \omega) = \int_{-\infty}^{+\infty} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau$ can be computed by using discrete FFT (fast Fourier transform) algorithm for the calculation of Wigner distribution function (Boashash & Black, 1987), (Dellomo & Jacyna, 1991) because $Z_\sigma(t, \omega)$ can be regarded as a Wigner distribution of the Riemann's zeta function, we can obtain the Fourier spectrum of $F_n(\omega)$ by conducting FFT calculations.

By using this method, we can propose some possible applications which use the theory presented in this paper.

- Primary testing of large numbers such as Mersenne numbers $2^m - 1$ can be conducted by using the algorithm shown in Figure 1 from the approximation, $\omega = \frac{1}{2}\log(2^m - 1) = \frac{m}{2}\log 2 - 1/2^{m+1} - 1/2^{2m+2} - \cdots$.
- Factorization of an integer n consisted of two primes can be conducted by using this method. By using FFT algorithm, there is a possibility to complete the computation within a polynomial time, whereas there is no known efficient algorithm that runs in polynomial time (Ribenboim, 1991).
- Breaking the public-key crypto system, which is considered to be hard by using the conventional computer systems, because the RSA crypto-system depends on the factorization of an integer composed of two large primes.

It is also known that Fourier transform can be conducted by the quantum computer, the schematic diagram for the quantum Fourier transform is shown in Figure 2 (Nielsen & Chuang, 2000).

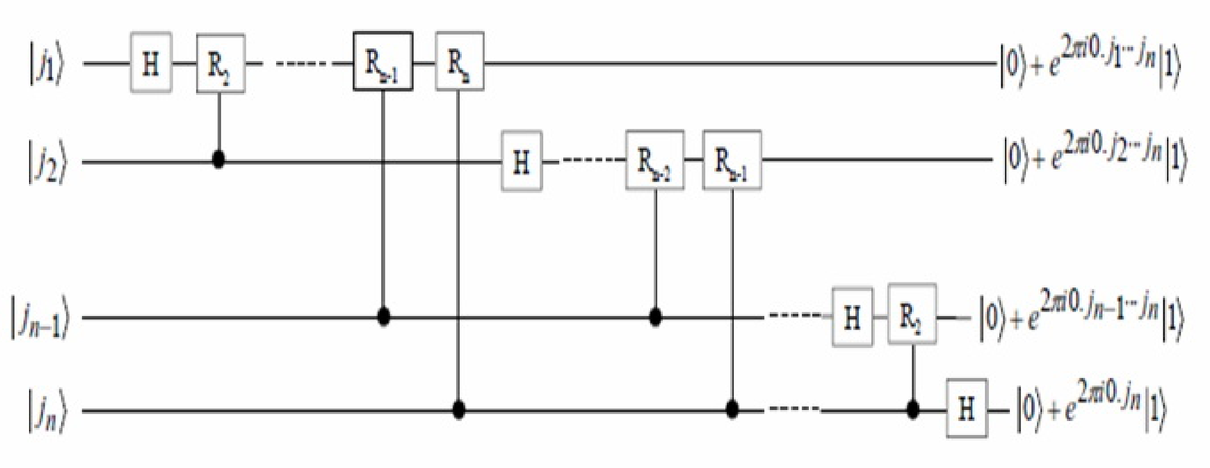

Figure 2. Schematic diagram for the quantum Fourier transform.

In this figure, H is a Hadamard gate and R_k is a unitary transformation given by

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}.$$

Hence it can be seen that primality testing and integer factorization of an integer n consisted of two primes can be conducted efficiently by using quantum computation besides the notably Shor's integer factorization algorithm (Yang, 2002), which gives us the possibility to break the RSA cryptosystem.

4. Conclusion

From the spectrum obtained by the Fourier transform of a correlation function generated from the Riemann zeta function given by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \frac{1}{2}\log n\right) e^{-i\omega t} dt$, we can see the primarity of an integer n if and only the $F_n(\omega)$ has a single spectra. Moreover we can factorize the integer n consisted of two primes by using this method.

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Sixth Order Multiple Coarse Grid Computation for Solving 1D Partial Differential Equation

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Abstract

We present a new method using multiple coarse grid computation technique to solve one dimensional (1D) partial differential equation (PDE). Our method is based on a fourth order discretization scheme on two scale grids and the Richardson extrapolation. For a particular implementation, we use multiple coarse grid computation to compute the fourth order solutions on the fine grid and all the coarse grids. Since every fine grid point has a corresponding coarse grid point with fourth order solution, the Richardson extrapolation procedure is applied for every fine grid point to increase the order of solution accuracy from fourth order to sixth order. We compare the maximum absolute error and the order of solution accuracy for our new method, the standard fourth order compact (FOC) scheme and Wang-Zhang's sixth order multiscale multigrid method. Two convection-diffusion problems are solved numerically to validate our proposed method.

Keywords: partial differential equation, multiple coarse grid computation, multigrid method.
2010 MSC: 65N06, 65N55, 65F10.

1. Introduction

Numerical solutions of partial differential equations (PDEs) play a crucial role in many simulation and engineering modeling applications, such as airplane manufacturing (Gamet *et al.*, 1999), auto manufacturing (Gerlinger *et al.*, 1998), medical imaging (Kang *et al.*, 2004), oil exploration and production (Li *et al.*, 2005), semiconductor (Carey *et al.*, 1996), communications (Kim & Kim, 2004), etc. Over the past several decades, computational mathematicians and engineers have developed many efficient fast algorithms to reduce the computation time. However, the increasing

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demand for higher resolution simulations in less computer time has continuously challenged the computational scientists to come up with more efficient, scalable numerical algorithms to solve PDEs.

In many scientific and engineering applications, such as the global ocean modeling and wide area weather forecasting, the computational domains are huge and the grid spaces are not small. In the context of the finite difference methods, the standard second order discretization scheme or the first order upwind difference scheme yield unsatisfactory results because they may need fine mesh griddings to compute approximate solutions of acceptable accuracy. In addition, the second order scheme may also produce numerical solutions with nonphysical oscillations for the convection dominated problems (Spotz, 1995).

Higher order (more than two) discretization methods are considered to be useful to reduce computational cost in very large scale modelings and simulations, which use relatively coarser mesh griddings to yield approximate solutions of comparable accuracy, compared with lower order discretization. Generally, higher order discretization schemes need more complicated procedures and more preprocessing costs to construct the coefficient matrix. However, they usually yield linear systems of much smaller size, compared with those from the lower order methods.

For the development of fourth order compact difference schemes, Gupta *et al*. proposed a fourth order nine-point compact (FOC) scheme to discretize the two dimensional (2D) convection-diffusion equation with variable coefficients (Gupta *et al.*, 1984). There are also some other similar fourth order compact schemes that have been developed for the convection-diffusion equations. Readers are referred to (Li *et al.*, 1995; Spotz, 1995; Spotz & Carey, 1995) and the references therein for more details.

For the sixth order schemes, Chu and Fan (Chu & Fan, 1998, 1999) proposed a three point combined compact difference (CCD) scheme for solving 2D Stommel Ocean model, which is a convection-diffusion equation. Their scheme can achieve sixth order accuracy for the inner grid points and fifth order accuracy for the boundary grid points. CCD scheme is considered as an *implicit* scheme because it does not compute the solution of the variables of interest directly. It also has a stability problem that for certain problems, if a large meshsize is used, the computed solution may be oscillatory (Zhang & Zhao, 2005).

In contrary, the *explicit* compact schemes compute the solutions of the variables directly. In addition, the explicit schemes have an additional advantage that they can avoid the oscillations in computed solutions. However, the higher order explicit compact schemes are more complicated to develop in higher dimensions, compared with the implicit schemes. As far as we know, there is no existing explicit compact scheme on a single scale grid that is higher than the fourth order.

By using the idea of multiscale computation, Sun and Zhang (Sun & Zhang, 2004) first proposed a sixth order explicit finite difference discretization strategy, which is based on the Richardson extrapolation technique and an operator interpolation scheme. Recently, Wang and Zhang developed an efficient and scalable sixth order explicit compact scheme for 2D/3D Poisson and convection-diffusion equations by using multiscale mutigrid method and an operator based interpolation combined with extrapolation technique (Wang & Zhang, 2009, 2011, 2010). The However, for the operator based interpolation, if the coefficient matrix A is not diagonally dominant like the convection-diffusion equation with very large cell Reynolds number, it may take a large number of iterations to converge. In this paper, we present another technique called the multiple coarse grid computation technique. This approach can be used to compute the fourth order solutions on the fine grid and every coarse grid, which means that we can directly apply Richardson extrapolation for every grid point on the find grid and no operator based interpolation is needed.

An outline of the paper is as follows. In Section 2, we illustrate our sixth order strategy by using multiple coarse grid computation technique. Numerical results will be provided in Section 3. Section 4 contains the concluding remarks.

2. Sixth Order Multiple Coarse Grid Computation

Our motivation is to build an efficient and scalable method for solving PDEs like the convection-diffusion equations with high order of solution accuracy. In addition, we want the new method to have good potential to be modified to work on parallel computers. In (Wang & Zhang, 2009), Wang and Zhang successfully increase the order of solution accuracy from fourth order to sixth order by using multiscale multigrid method, Richardson extrapolation and an operator based interpolation. Important properties of the Richardson extrapolation has been studied by Zlatev *et al*. Readers are referred to (Zlatev *et al.*, 2010) and the references therein for more details. The interpolation strategy is an mesh-refinement type of iterative method and it is very efficient for some PDEs like the Poisson equation. Since their discretization scheme is based on the standard explicit fourth order compact scheme, so their is no nonphysical oscillation in the computed solutions. The proof and numerical analysis of this property can be found in (Spotz, 1995). However, it is not efficient and scalable for some problems like the convection-diffusion equation with high Reynolds numbers (Wang & Zhang, 2011). For some cases, the interpolation procedure may take thousands of iterations to converge. In addition, this method does not have a good potential for parallel implementation.

The idea of using multiple coarse grid computation is from the parallel superconvergent multigrid method. In addition to splitting the original grid and filtering residual vector to exploit parallelism, one can use the concurrent relaxation method on multiple grids (Zhu, 1993). The multigrid superconvergent method uses multiple coarse grids to generate better correction for the fine grid solution than that from a single coarse grid. The reason is that for standard multigrid method of 1D problem as in figure 1, the residual of the fine grid is projected to only *even* coarse grid. But we can also project the residual to *odd* coarse grids. Therefore, a combination of error correction from all the coarse grids may make the fine grid converge faster than that from a single coarse grid. In general, for a d dimensional problem, the fine grid can be easily coarsened into 2^d coarse grids. If the computation work for each coarse grid can be loaded to a separate processor and computed simultaneously, we can develop an parallel solver for solving PDEs.

2.1. 1D multiple coarse grid computation

Let's consider the multiple coarse grid computation technique for the one dimensional (1D) convection diffusion equation, which can be written as

$$u_x + b(x)u_x + c(x)u = f(x), \qquad 0 \le x \le l, \tag{2.1}$$

where the known functions $b(x)$, $c(x)$ and $f(x)$ are assumed to have the necessary derivatives up to certain orders. Eq. (2.1) can be discretized by some finite difference scheme to result in a system of linear equations

$$A^h u^h = f^h, \tag{2.2}$$

where h is the uniform grid spacing of the discretized domain Ω^h.

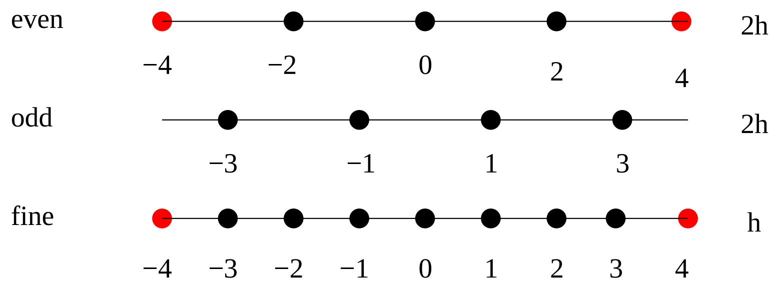

Figure 1. Illustration of the multiple coarse grid for 1D problem.

In order to achieve sixth order solution accuracy, we need to compute the fourth order solutions for the fine grid and two coarse grids like figure 1. Then we can apply the Richardson extrapolation. The fourth order compact (FOC) scheme we use is from (Wang & Zhang, 2011).

From figure 1, we can find out that two coarse grids are generated in such a way that all the even-numbered grid points from Ω_h belong to coarse grid Ω_{even} and all the odd-numbered grid points belong to coarse grid Ω_{odd}. So we have

$$\Omega_{even} = \{x_j | x_j \in \Omega_h \textit{ and } (j = even)\},$$
$$\Omega_{odd} = \{x_j | x_j \in \Omega_h \textit{ and } (j = odd)\}.$$

We note that the even indexed coarse grid is easy to be solved by double the mesh size from h to $2h$. However, the coarse grid Ω_{odd} only contains the *black* color grid points from fine grid but no *red* color boundary grid points. It is very difficult to develop the finite difference schemes for coarse grid Ω_{odd} if we only have the inner grid points. One possible approach is to add these red color boundaries to Ω_{odd} and develop special computational stencil for grid point u_{-3} and u_3 as shown in figure 2. For the 1D problem in figure 2, the computational stencil for the grid points near the boundaries are different with other inner grid points. For the inner grid points like u_{-1} and u_1, their finite difference schemes are based on $2h$ meshsize. However, if we take grid point u_{-3} in Ω_{odd} as an example, its compact finite difference scheme needs the boundary grid point u_{-4} and inner grid point u_{-1}. The meshsize between u_{-4} and u_{-1} are h and $2h$.

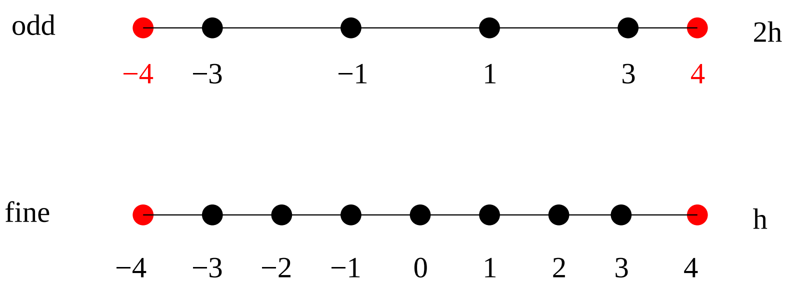

Figure 2. Ω_{odd} with two added red color boundary grid points.

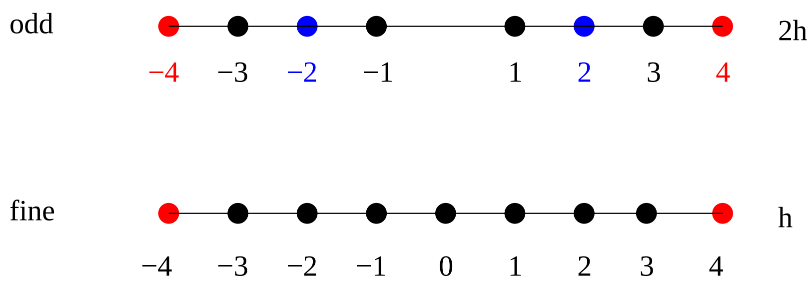

Figure 3. Ω_{odd} with two red color boundary and two blue color inner grid points .

Lemma 2.1. *For coarse grid as shown in figure 2, the solution accuracy for the central difference operator becomes first order.*

Proof. It can be easily verified by using Taylor series expansion.

□

Since the second order central difference operator is degraded to the first order, the FOC scheme which is based on the approximation for the second order terms will be degraded to the second order for these near boundary grid points. In order to compute fourth order solution for every coarse grid point, we add two more grid points to the Ω_{odd} like the *blue* color grid points in figure 3.

By adding these four grid points, now we can discretize every grid point in Ω_{odd} with fourth order accuracy using FOC scheme. Let's assume the Ω_{odd} contains Nx grid points

$$u_{odd}(0), u_{odd}(1), \ \dots , u_{odd}(Nx)$$

as in figure 4. Then the Ω_{even} will contains $Nx - 3$ grid points and fine grid will contains $2Nx - 7$ grid points. The grid points on Ω_{odd} are approximated as follows:

- For $j \in \{1, 2, Nx - 2, Nx - 1\}$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-1)$ and $u_{odd}(j+1)$ with meshsize h. The truncation error is $O(h^4)$.

- For $j = 3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j-2)$ and $u_{odd}(j+1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.

- For $j \in [4, Nx - 4]$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j - 1)$ and $u_{odd}(j + 1)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.

- For $j = Nx - 3$, $u_{odd}(j)$ is approximated by three-point computational stencil from FOC scheme using grid points $u_{odd}(j - 1)$ and $u_{odd}(j + 2)$ with meshsize $2h$. The truncation error is $O((2h)^4)$.

Figure 4. Representation of modified Ω_{odd} for 1D problem.

By using above discretization strategy, we can approximate the fourth order solution for every grid point on Ω_{odd}. After we get fourth order solutions for the fine grid and two coarse grids, each grid point on the fine grid will have a corresponding grid point on either Ω_{even} or Ω_{odd}. Then we apply Richardson extrapolation (Cheney & Kincard, 1999) for every fine grid point to approximate the sixth order solution like

$$\tilde{u}_j^h = \frac{16u_j^h - u_j^{2h}}{15}, \tag{2.3}$$

where u_j^h is jth grid point from fine grid and u_j^{2h} is the corresponding coarse grid point.

3. Numerical Results

Two 1D convection-diffusion equations are solved using the multiple coarse grid computation strategy discussed in the previous sections. We compared the truncated error and the order of accuracy by using our multiple coarse grid computation technique (MCG), standard fourth order scheme (FOC), and the sixth order operator based interpolation scheme (SOC) in (Wang & Zhang, 2011).

The codes are written in Fortran 77 programming language and run on one node of the Lipscomb HPC Cluster at the University of Kentucky. Each node has 36GB of local memory and runs at 2.66GHz. The initial guess for our test cases is the zero vector. The stopping criteria for the iterative methods we tested and the operator based interpolation procedure is 10^{-10}. The errors reported are the maximum absolute errors over the discrete grid of the finest level.

For the order of solution accuracy, we denote $E(h)$ and $E(H)$ to be the solution error with meshsize h and H, respectively. The order of accuracy m is calculated from the formula

$$\frac{E(h)}{E(H)} = \frac{h^m}{H^m}$$
$$\Longrightarrow m = \log_{(h/H)}(E(h)/E(H)).$$

The order of accuracy is formally defined when the meshsize approaches zero. Therefore, when the meshsize is relatively large, the discretization scheme may not achieve its formal order of accuracy.

Problem 1. Let's consider the examples from Sun's previous work (Sun & Zhang, 2004), which is a 1D convection-diffusion equation like

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} - u = -\cos x - 2\sin x, \qquad 0 \le x \le \pi. \tag{3.1}$$

Eq. (3.1) has the Dirichlet boundary conditions as $u(0) = u(\pi) = 0$. The analytic solution for this problem is $u(x) = \sin x$.

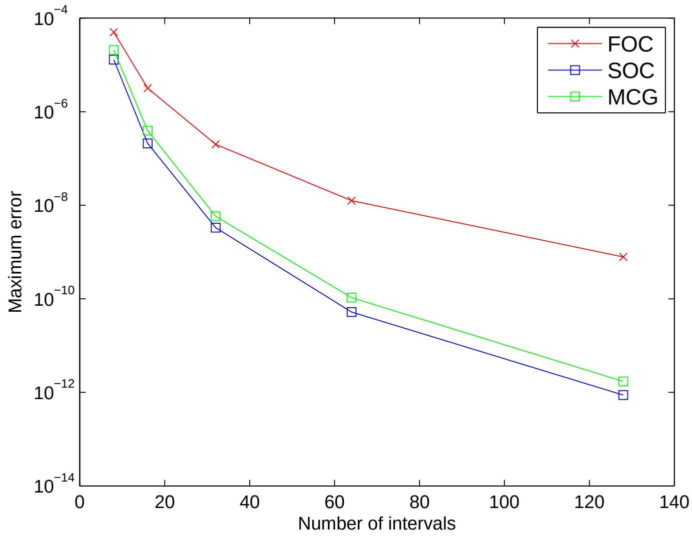

Figure 5. Comparison of maximum errors of FOC, SOC and MCG methods for Problem 1.

The computational results are listed in Table 1 and figure 5. From Table 1, we can see that the multiple coarse grid method (MCG) is more accurate than the fourth order scheme (FOC). Although the MCG method is not as accurate as the SOC but it can achieve the sixth order solution accuracy when the number of intervals is bigger than 8. The reason why MCG is less accurate than SOC is that there are two near boundary grid point using meshsize h to approximate instead of $2h$ in Ω_{odd}.

Table 1. Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.1).

	FOC		SOC		MCG	
h	Error	Order	Error	Order	Error	Order
$\pi/8$	5.02e-5	4.0	1.30e-5	5.9	2.08e-5	5.7
$\pi/16$	3.18e-6	4.0	2.10e-7	6.0	3.94e-7	6.1
$\pi/32$	2.00e-7	4.0	3.32e-9	6.0	5.81e-9	5.8
$\pi/64$	1.25e-8	4.1	5.20e-11	6.0	1.06e-10	6.0
$\pi/128$	7.83e-10	4.1	8.73e-13	6.0	1.71e-12	6.0

Problem 2. We solve another classical 1D convection-diffusion equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = 0, \qquad 0 \le x \le 1. \tag{3.2}$$

The boundary condition for Eq. (3.2) is $u_0 = 0$ and $u_1 = 1$. The analytic solution is $u(x) = (e^x - 1)/(e - 1)$.

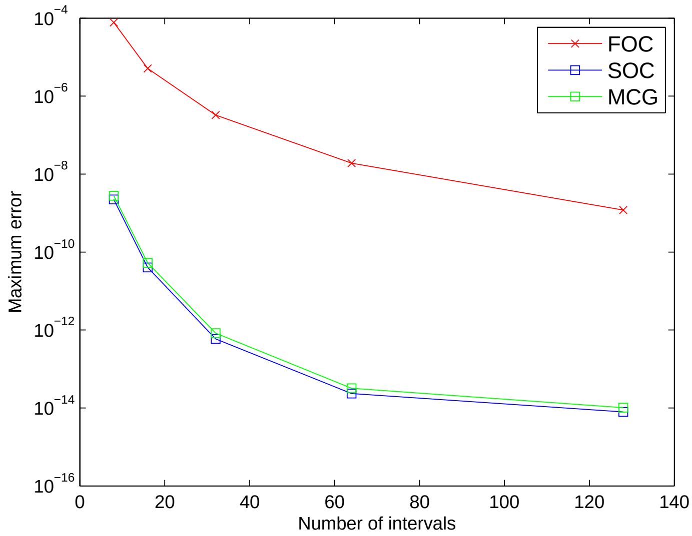

Figure 6. Comparison of maximum errors of FOC, SOC and MCG methods for Problem 2.

The numerical results of Problem 2 are listed in Table 2 and figure 6. We note that when $n > 32$, the order of solution accuracy is not high enough as we hope. The reason is that the computed solutions with $h = 1/64$ and $h = 1/128$ are not as accurate as they should be, due to the stopping criteria we set. Once again, the solutions from our MCG method are more accurate than the FOC method and can achieve the sixth order when $n < 64$.

Table 2. Comparison of maximum errors and the order of accuracy by using FOC, SOC, and MCG methods for Eq. (3.2).

	FOC		SOC		MCG	
h	Error	Order	Error	Order	Error	Order
1/8	7.76 e-5	3.9	2.24e-9	5.9	2.78-9	5.7
1/16	5.12e-6	4.0	4.01e-11	6.0	5.29-11	6.0
1/32	3.27e-7	4.0	5.91e-13	6.0	8.27-13	5.9
1/64	1.91e-8	4.0	2.34e-14	4.9	3.21-14	4.8
1/128	1.19e-9	4.0	7.93e-15	1.6	1.02-14	1.6

We want to mention here that the SOC method for both test cases is slightly more accurate than the MCG method, but the MCG method has a very good potential for parallel implementation. The computing tasks for MCG procedure can be divided to three independent processors (one for find grid and two for coarse grids). In addition, since the MCG method does not need the operator based interpolation procedure to approximate the sixth order fine grid solution, it will save a large amount of CPU costs for some high Reynolds number problems (Wang & Zhang, 2011).

4. Concluding Remarks and Future Work

We presented a new sixth order solution method based on the fourth order discretization and multiple coarse grid computation for solving 1D convection-diffusion equation. Our numerical experiments show that the new sixth order strategy is more accurate than the standard fourth order scheme and can achieve the sixth order solution accuracy.

It is worth pointing out that our solution strategy can be applied to solve many other types of PDEs, because it does not require the additional work to redesign the discretization schemes. The advantage of using multiple coarse grids is that we can use it to increase the order of accuracy without using operator based interpolation scheme. However, there is still a lot of work that needs to be done to develop a useful multiple coarse grid computation method that can be applied to real-world problems. In this paper, we just use the standard Gauss-Seidel iterative method for MCG strategy, because our goal is to test whether the MCG method can achieve the sixth order accuracy or not. For some real applications, we should use multigrid method and implement the multiple coarse grid computation in the multigrid cycle.

For the future research work, we will extend our 1D multiple coarse grid computation method to higher dimensional problems. For 2D problems, we will generate four course grids by the index of x and y direction as $(even, even)$, $(even, odd)$, $(odd, even)$ and (odd, odd). Here, $(even, even)$ is the course grid in standard multigrid method. Like 1D strategy, only the $(even, even)$ course grid has the full boundary conditions. We need to find a way to add artificial boundary grid points for other three course grids. Another possible solution is to use algebraic multigrid method instead of geometric multigrid method, this is also one of our research interest in the future.

For the parallelization, the parallel multiscale multigrid (MCG) method has been discussed in (Xiao, 1994; Zhu, 1993). However, these parallel MCG methods are only used to speed up the convergence. As we mentioned in previous section, the computation of each course grid and the fine grid is independent. If we want to solve a 3D problem, we can use nine processors to solve the fourth order solutions on the fine grid and eight coarse grids. Then an Richardson extrapolation, which can also been parallelized, can increase the order of accuracy to sixth order.

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On Nonuniform Polynomial Stability for Evolution Operators on the Half-line

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Abstract

The main aim of this paper is to study a concept of nonuniform polynomial stability for evolution operators on the half-line. The obtained results are variants for nonuniform polynomial stability of some well-known theorems due to Barbashin, Datko, Rolewicz and Zabczyk in the case of uniform exponential stability. This paper generalizes well-known results for the nonuniform exponential stability (Lupa & Megan, 2012) and the uniform polynomial stability (Megan & Ceausu, 2012).

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1. Introduction and preliminaries

The notion of exponential stability plays an important role in the theory of differential equations in Banach spaces, particularly in the study of asymptotical behaviors. It has gained prominence since appearance of two fundamental monographs of J. L. Massera, J. J. Sch*ä*ffer (Massera & Schäffer, 1966) and J. L. Daleckii, M. G. Krein (Daleckii & Krein, 1974). These were followed by the important books of C. Chicone and Yu. Latushkin (Chicone & Latushkin, 1999) and L. Barreira and C. Valls (Barreira & Valls, 2008).

The most important stability concept used in the qualitative theory of differential equations is the uniform exponential stability. In some situations, particulary in the nonautonomous setting, the concept of uniform exponential stability is too restrictive and it is important to look for a more general behavior.

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Two different perspectives can be identified to generalize the concept of uniform exponential stability, on the one hand one can define exponential stabilities that depends on the initial time (and therefore are nonuniform) and, on the other hand, one can consider grow rates that are not necessarily exponential.

The first approach leads to the concepts of nonuniform exponential stabilities and can be found in the works (Barreira & Valls, 2008), (Lupa & Megan, 2012), (Megan, 1995), (Minda & Megan, 2011), (Pinto, 1988) and the second approach is presented in the papers (Barreira & Valls, 2009), (Bento & Silva, 2009), (Bento & Silva, 2012), (Megan & Ramneantu, 2011), (Megan & Minda, 2011).

A natural generalization is to consider stability concepts that are both nonuniform and not necessarily exponential. This was the approach followed by Barreira and Valls in (Barreira & Valls, 2009) and A. Bento and C. Silva in (Bento & Silva, 2009), (Bento & Silva, 2012), who studied a nonuniform polynomial dichotomy concept. A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all variational equations in a finite dimensional space admit a nonuniform exponential dichotomy.

In this paper we consider a concept of nonuniform polynomial stability for evolution operators in Banach spaces. This concept has been considered in the case of invertible evolution operators in the papers (Barreira & Valls, 2009) due to L. Barreira and C. Valls, respectively in (Bento & Silva, 2009), (Bento & Silva, 2012) due to A. Bento and C. Silva.

Some results concerning polynomial stability for evolution operators were published in our papers (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012), (Megan & Minda, 2011). We remark that the results obtained in (Megan & Ramneantu, 2011) are for the case of evolution operators with uniform exponential growth. In this paper we consider the case of evolution operators with nonuniform polynomial growth.

The obtained results in this paper can be considered as variants for nonuniform polynomial stability of some well-known theorems due to Barbashin ((Barbashin, 1967)), Datko ((Datko, 1972)) and Rolewicz ((Rolewicz, 1986)) in the case of uniform exponential stability. We remark that our proofs are not adaptations for polynomial stability of the proofs presented in (Barbashin, 1967), (Datko, 1972) and (Rolewicz, 1986). The case of nonuniform exponential stability has been studied in (Lupa & Megan, 2012), (Minda & Megan, 2011), respectively (Megan & Ramneantu, 2011), (Megan & Ceausu, 2012).

Moreover, we note that we consider evolution operators which are not supposed to be invertible and the polynomial stability concept studied in this paper uses the evolution operators in forward time. Thus the stability results obtained in this paper hold for a much larger class of differential equations than in the classical theory of uniform exponential stability.

Let X be a real or complex Banach space and let I be the identity operator on X. The norm on X and on $\mathcal{B}(X)$, the algebra of all bounded linear operators acting on X, will be denoted by $\|\cdot\|$.

Let

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : \ t \geq s\}.$$

We recall that a mapping $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is called an *evolution operator* on X if

(e_1) $\Phi(t,t) = I$, for all $t \geqslant 0$;

(e_2) $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, for all $(t,s),(s,r) \in \Delta$.

Definition 1.1. An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

(i) *with polynomial growth* (and denote p.g) if there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s+1)^{\omega} \parallel \Phi(t,s) \parallel \leq M(t+1)^{\omega}(s+1)^{\varepsilon}, \quad for\ \ all\ \ (t,s) \in \Delta;$$

(ii) *polynomially stable* (and denote p.s) if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^{\alpha} \parallel \Phi(t,s) \parallel \leq N(s+1)^{\alpha+\beta}, \quad for\ \ all\ \ (t,s) \in \Delta;$$

(iii) *exponentially stable* (and denote e.s) if there exist $N_1 \geq 1$, $\alpha_1 > 0$ and $\beta_1 \geq 0$ such that

$$e^{\alpha_1 t} \parallel \Phi(t,s) \parallel \leq N_1 e^{(\alpha_1+\beta_1)s}, \quad for\ \ all\ \ (t,s) \in \Delta.$$

Definition 1.2. An evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is said to be

(i) *measurable*, if for all $(s,x) \in \mathbb{R}_+ \times X$ the mapping $t \longmapsto \parallel \Phi(t,s)x \parallel$ is measurable on $[s,\infty)$.

(ii) $*$*-measurable*, if for all $(s,x^*) \in \mathbb{R}_+ \times X^*$ the mapping $s \longmapsto \parallel \Phi(t,s)^* x^* \parallel$ is measurable on $[0,t]$.

2. Results

Theorem 2.1. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator. If Φ is p.s then there exist $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \leq D(s+1)^{d+c}\|x\|, \tag{2.1}$$

for all $s \geq 0$ and $x \in X$.

Proof. If Φ is p.s, then according to Definition 1.1 (ii) there exist the constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that, for all $d \in (0,\alpha)$ and $c = \beta$ we have

$$\int_s^{\infty} (\tau+1)^{d-1} \|\Phi(\tau,s)x\| d\tau \leq N(s+1)^{\alpha+\beta}\|x\| \int_s^{\infty} (\tau+1)^{d-\alpha-1} d\tau \leq D(s+1)^{d+c}\|x\|,$$

for all $(s,x) \in \mathbb{R}_+ \times X$, where $D = \frac{N+\alpha-d}{\alpha-d}$.

□

Theorem 2.2. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a measurable evolution operator with p.g and with the property that there exist $D \geq 1$, $c \geq 0$ and $d > \varepsilon$ such that (2.1) holds, where ε is given by Definition 1.1(i). Then Φ is p.s.*

Proof. Let $x \in X$ and $t \geq 2s + 1$. Because

$$\int_{\frac{t-1}{2}}^{t} (\tau + 1)^{a-1} d\tau = (t + 1)^a \frac{2^a - 1}{a2^a},$$

for all $t \geq 0$ and $a > 0$ we have

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| = N \int_{\frac{t-1}{2}}^{t} (\tau + 1)^{d-\varepsilon-1} \|\Phi(t, s)x\| d\tau$$

$$= N \int_{\frac{t-1}{2}}^{t} (\tau + 1)^{d-\varepsilon-1} \|\Phi(\tau, s)x\| M \left(\frac{t + 1}{\tau + 1}\right)^{\omega} (\tau + 1)^{\varepsilon} d\tau$$

$$\leq 2^{\omega} NM \int_{s}^{\infty} (\tau + 1)^{d-1} \|\Phi(\tau, s)x\| d\tau \leq 2^{\omega} NMD(s + 1)^{d+c} \|x\|.$$

Hence, we have that

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq 2^{\omega} NMD(s + 1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $t \geq 2s + 1$ and $x \in X$, where $N = \frac{(d-\varepsilon)2^{d-\varepsilon}}{2^{d-\varepsilon}-1}$.

For $t \in [s, 2s + 1)$ we have

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq 2^{d+\omega-\varepsilon} M(s + 1)^{d} \|x\|$$

and hence,

$$(t + 1)^{d-\varepsilon} \|\Phi(t, s)x\| \leq K(s + 1)^{d-\varepsilon+c+\varepsilon} \|x\|,$$

for all $(t, s, x) \in \Delta \times X$, where $K = \max\{2^{\omega} NMD, 2^{d-\varepsilon+\omega} M\}$.
Finally, we obtain that Φ is p.s.

□

A discrete variant of the Theorem 2.2 is

Theorem 2.3. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator with p.g and with the property that there exist the constants $D \geq 1$, $d > 0$ and $c \geq 0$ such that*

$$\sum_{k=n}^{\infty} (k + 1)^d \|\Phi(k, n)x\| \leq D(n + 1)^{d+c} \|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$. Then Φ is p.s.

Proof. According the hypothesis, if we consider $k = m$ then we have

$$(m + 1)^d \|\Phi(m, n)x\| \leq D(n + 1)^{d+c} \|x\|,$$

for all $m, n \in \mathbb{N}$, $m \geq n$ and $x \in X$, which proves that Φ is p.s. □

Remark. Theorem 2.3 can be considered a Zabczyk's (Zabczyk, 1974) type theorem for polynomial stability.

Theorem 2.4. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator. Then Φ is p.s if and only if there exist the constants $B \geq 1$ and $b > c \geq 0$ such that*

$$\sum_{k=0}^{n} (k+1)^{-b-1} \|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$.

Proof. Necessity. If we consider $b \in (\beta, \alpha+\beta)$, $c = \beta$ and $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$ we have

$$\sum_{k=0}^{n} (k+1)^{-b-1} \|\Phi(n,k)x\| \leq N(n+1)^{-\alpha}\|x\| \sum_{k=0}^{n} (k+1)^{\alpha+\beta-b-1}$$

$$\leq N(n+1)^{-\alpha}\|x\| \left(1 + \int_0^n (\tau+1)^{\alpha+\beta-b-1} d\tau\right) \leq B(n+1)^{c-b}\|x\|,$$

for all $n \in \mathbb{N}$ and $x \in X$.
Sufficiency. Let $n \geq k \geq 0$ with $n, k \in \mathbb{N}$. According to the hypothesis we have that

$$(k+1)^{-b-1}\|\Phi(n,k)x\| \leq B(n+1)^{c-b}\|x\|$$

which implies

$$(n+1)^{b-c}\|\Phi(n,k)x\| \leq B(k+1)^{b-c+1+c}\|x\|,$$

for all $x \in X$. Hence, Φ is p.s. □

Remark. Theorem 2.4 can be considered a Barbashin's type theorem for polynomial stability (see (Barbashin, 1967)).

We consider the set

$$\mathcal{R} = \{R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ | R \ \ nondecreasing, \ \ R(t) > 0, \ \ \forall \ \ t > 0\}.$$

Theorem 2.5. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-measurable evolution operator with p.g. Then Φ is p.s if and only if there exist $B \geq 1$, $b > c \geq 0$ and a function $R \in \mathcal{R}$ such that*

$$\int_0^t R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^* x^*\|\right) d\tau \leq BR\left((t+1)^{c-b}\|x^*\|\right),$$

for all $(t, s, x^*) \in \Delta \times X^*$.

Proof. Necessity. Let us consider $R(t) = t$, $t \geq 0$. If Φ is p.s, then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for all $b \in (\beta, \alpha+\beta)$ and $c = \beta$ we have

$$\int_0^t (\tau+1)^{-b-1}\|\Phi(t,\tau)^* x^*\| d\tau \leq N(t+1)^{-\alpha}\|x^*\| \int_0^t (\tau+1)^{\alpha+\beta-b-1} d\tau = B(t+1)^{c-b}\|x^*\|,$$

where $B = \frac{N+\alpha+\beta-b}{\alpha+\beta-b}$.
Sufficiency. Let $x \in X$ with $\|x\| \leq 1$ and $a - 1 > B$. For $t \geq as + a - 1$ we have

$$(a-1)R\left(M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle x^*, \Phi(t,s)x\rangle|\right)$$

$$= \int_s^{s+a-1} R\left(M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle \Phi(t,\tau)^*x^*, \Phi(\tau,s)x\rangle|\right)d\tau$$

$$\leq \int_s^{as+a-1} R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|a^{-b-\omega-1}\left(\frac{\tau+1}{s+1}\right)^{b+\omega+1}\right)d\tau$$

$$\leq \int_0^t R\left((\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|\right)d\tau < (a-1)R\left((t+1)^{c-b}\|x^*\|\right).$$

Since R is nondecreasing, we obtain that

$$M^{-1}a^{-b-\omega-1}(s+1)^{-b-c-1}|\langle x^*, \Phi(t,s)x\rangle| \leq (t+1)^{c-b}\|x^*\|\|x\|.$$

By taking supremum relative to $\|x^*\| \leq 1$, we have that

$$(t+1)^{b-c}\|\Phi(t,s)\| \leq Ma^{b+\omega+1}(s+1)^{b+c+1}.$$

If $t \in [s, as + a - 1)$ we have

$$(t+1)^{b-c}\|\Phi(t,s)\| \leq M\left(\frac{t+1}{s+1}\right)^{b-c+\omega}(s+1)^b \leq Ma^{b+\omega+1}(s+1)^b,$$

and, further,

$$(t+1)^{b-c}\|\Phi(t,s)\| \leq Ma^{b+\omega+1}(s+1)^{b+c+1},$$

for all $(t,s) \in \Delta$, which proves that Φ is p.s. □

Remark. Theorem 2.5 can be considered a Rolewicz's type theorem for polynomial stability (see (Rolewicz, 1986)).

Corollary 2.6. Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a $*$-measurable evolution operator with p.g. Then Φ is p.s if and only if there exist $B \geq 1$ and $b > c \geq 0$ such that

$$\int_0^t (\tau+1)^{-b-1}\|\Phi(t,\tau)^*x^*\|d\tau \leq B(t+1)^{c-b}\|x^*\|,$$

for all $(t, s, x^*) \in \Delta \times X^*$.
Proof. It follows from Theorem 2.5 for $R(t) = t$.

Remark. A similar result was obtained by N. Lupa and M. Megan in (Lupa & Megan, 2012) for the case of nonuniform exponential stability.

3. Examples

In this section we will give some examples that illustrate the connection between the exponential stability and the polynomial stability, as well as the connection between polynomial growth and polynomial stability. Furthermore, we will present some examples of evolution operators which are not p.s and the integral from (2.1) is convergent, respectively divergent.

In contrast with uniform case (where uniform exponential stability implies uniform polynomial stability, see (Megan & Ramneantu, 2011)) in the nonuniform case there is no connection between the concepts of exponential stability and polynomial stability, as shown in the following examples.

Example 3.1. We consider the function

$$u : [1, \infty) \longrightarrow \mathbb{R}_+^*, \quad u(t) = (t+1)^3 + 1$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{(s+1)^2 u(s)}{(t+1)^2 u(t)} I.$$

We have that

$$(t+1)^2 \|\Phi(t, s)\| \leq 2(s+1)^5, \quad for \ \ all \ \ (t, s) \in \Delta.$$

It results that Φ is p.s. If we suppose that Φ is e.s, then there exist $N_1 \geq 1$, $\alpha_1 > 0$ and $\beta_1 \geq 0$ such that

$$e^{\alpha_1 t}(s+1)^2[(s+1)^3 + 1] \leq N_1 e^{(\alpha_1+\beta_1)s}(t+1)^2[(t+1)^3 + 1], \quad for \ \ all \ \ (t, s) \in \Delta.$$

For $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction and hence Φ is not e.s.

Example 3.2. The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t, s) = \frac{e^{(2-\cos s)s}}{e^{(2-\cos t)t}} I$$

satisfies the condition

$$e^t \|\Phi(t, s)\| \leq e^{3s}, \quad for \ \ all \ \ (t, s) \in \Delta.$$

Hence Φ is e.s. If we suppose that Φ is p.s then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^\alpha e^{(2-\cos s)s} \leq N(s+1)^{\alpha+\beta} e^{(2-\cos t)t}, \quad for \ \ all \ \ (t, s) \in \Delta.$$

From here, for $t = 2(n+1)\pi$ and $s = (2n+1)\pi$ we obtain

$$(2n\pi + 2\pi + 1)^\alpha \, e^{4n\pi+\pi} \leq N \, (2n\pi + \pi)^{\alpha+\beta},$$

which for $n \longrightarrow \infty$ yields a contradiction.

It is obvious that if an evolution operator is p.s then it has p.g. The next example presents an evolution operator with p.g, which is not p.s and the integral from (2.1) is divergent.

Example 3.3. The evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{(s+1)^{1-\cos(s+1)}}{(t+1)^{1-\cos(t+1)}} I$$

satisfies the relation

$$(s+1)^{\omega}\|\Phi(t,s)\| \leq (t+1)^{\omega}(s+1)^{\varepsilon}, \quad for \;\; all \;\; (t,s) \in \Delta.$$

It results that Φ has p.g for all $\omega > 0$ and $\varepsilon = 2$.
If we suppose that Φ is p.s then there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(t+1)^{\alpha}\frac{(s+1)^{1-\cos(s+1)}}{(t+1)^{1-\cos(t+1)}} \leq N(s+1)^{\alpha+\beta},$$

for all $(t,s) \in \Delta$. For $s = \frac{\pi}{2} - 1$ and $t = 2\pi + 2n\pi - 1$, we obtain

$$(2\pi + 2n\pi)^{\alpha}\frac{\pi}{2} \leq N\left(\frac{\pi}{2}\right)^{\alpha+\beta},$$

which if $n \longrightarrow \infty$, leads to a contradiction. We obtain thus that Φ is not p.s.
Let $d \geq 2$ and $s \geq 0$. Then we have

$$\int_s^{\infty}(\tau+1)^{d-1}\|\Phi(\tau,s)x\|d\tau \geq (s+1)^{1-\cos(s+1)}\|x\|\int_s^{\infty}(\tau+1)^{d-3}d\tau = \infty.$$

The next evolution operator does is not p.s and the integral from (2.1) is divergent.

Example 3.4. We consider the set

$$A = \{n + \frac{1}{n+1} \; : \; n \in \mathbb{N}\}$$

and a function $u : [0,\infty) \longrightarrow [1,\infty)$

$$u(t) = \begin{cases} e^{t+1}, & t \notin A \\ e^2, & t \in A. \end{cases}$$

and the evolution operator

$$\Phi : \Delta \longrightarrow \mathcal{B}(X), \quad \Phi(t,s) = \frac{u(s)}{u(t)} I$$

Let $d > 0$ and $s \geq 0$. Then we have

$$\int_s^{\infty}(\tau+1)^{d-1}\|\Phi(\tau,s)x\|d\tau = u(s)\|x\|\int_s^{\infty}(\tau+1)^{d-1}e^{-(\tau+1)}d\tau$$

$$\leq u(s)\|x\|\int_{s+1}^{\infty}y^{d-1}e^{-y}dy = \|x\|u(s)\Gamma(d) < \infty.$$

If we suppose that Φ has p.g then there exist $M \geq 1$, $\omega > 0$ and $\varepsilon \geq 0$ such that

$$(s+1)^{\omega}u(s) \leq M(t+1)^{\omega}(s+1)^{\varepsilon}u(t), \quad for \;\; all \;\; (t,s) \in \Delta.$$

For $s = n$ and $t = n + \frac{1}{n+1}$ we obtain

$$e^{n+1}(n+1)^{2\omega} \leq Me^2(n+1)^{\varepsilon}(n^2+2n+2)^{\omega},$$

which for $n \longrightarrow \infty$ yields a contradiction. Hence, Φ does not have p.g and so Φ is not p.s.

4. Open Problem

Finally, we put the following open problems:

1 There exist evolution operators which are not p.s and the relation (2.1) is satisfied?

2 There are evolution operators with p.g with $\varepsilon > 0$, which are not p.s and the relation (2.1) is satisfied for $d \in (0, \varepsilon)$?

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Zweier I-convergent Sequence Spaces Defined by a Sequence of Moduli

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Abstract

In this article we introduce the sequence spaces $\mathcal{Z}^I(F)$, $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_\infty^I(F)$ for a sequence of moduli $F = (f_k)$ and study some of the topological and algebraic properties on these spaces.

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1. Introduction

Let $\mathbb{R}$, and $\mathbb{C}$ be the sets of all real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \; or \; \mathbb{C}\}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $||x||_\infty = \sup_k |x_k|$. Each linear subspace of ω, for example $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines

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a matrix mapping from λ to μ and we denote it by writing $A : \lambda \longrightarrow \mu$. If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen (Altay *et al.*, 2006), Başar and Altay (Altay & Başar, 2003), Malkowsky (Malkowsky, 1997), Ng and Lee (Ng & Lee, 1978) and Wang (Wang, 1978). Şengönül (Şengönül, 2007) defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e, $y_i = px_i + (1-p)x_{i-1}$ where $x_{-1} = 0, p \neq 0, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar and Altay (Altay & Başar, 2003), Şengönül (Şengönül, 2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows $\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$, $\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$. Here we quote below some of the results due to Şengönül (Şengönül, 2007) which we will need in order to establish the results of this article.

Theorem 1.1 ((Şengönül, 2007), Theorem 2.1)**.** *The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm*

$$\|x\|_{\mathcal{Z}} = \|x\|_{\mathcal{Z}_0} = \|Z^p x\|_c.$$

Theorem 1.2 ((Şengönül, 2007), Theorem 2.2)**.** *The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$.*

Theorem 1.3 ((Şengönül, 2007), Theorem 2.3)**.** *The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.*

Theorem 1.4 ((Şengönül, 2007), Theorem 2.6)**.** *$\mathcal{Z}_0$ is solid.*

Theorem 1.5 ((Şengönül, 2007), Theorem 3.6)**.** *$\mathcal{Z}$ is not a solid sequence space.*

The concept of statistical convergence was first introduced by Fast (Fast, 1951) and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Connor (Connor, 1988, 1989; Connor & Kline, 1996), Connor, Fridy and Kline (Fridy & Kline, 1994) and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

Later on it was studied by Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński (Kostyrko *et al.*, 2000). Later on it was studied by Šalát, Tripathy, Ziman (Šalát *et al.*, 2004; Šalát *et al.*, 2005) and Demirci (Connor *et al.*, 2001). Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. A set I$\subseteq 2^X$ (2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I$, $B \subseteq A \Rightarrow B \in I$. A non empty family of sets $£(I) \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin £(\mathrm{I})$, for $A, B \in £(I)$ we have $A \cap B \in £(I)$ and for each $A \in £(I)$ and $A \subseteq B$ implies $B \in £(I)$. An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $£(I)$ corresponding to I. i.e $£(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

Definition 1.2. A sequence space E is said to be monotone if it contains the canonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $(x_k) \in E$ where $\pi(k)$ is a permutation on $\mathbb{N}$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write I-lim $x_k = L$.
The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if $L = 0$. In this case we write I-lim $x_k = 0$.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 1.10. A modulus function f is said to satisfy $\triangle_2$-condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 1.11. Take for I the class I_f of all finite subsets of $\mathbb{N}$. Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

Definition 1.12. For I= I_δ and $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical covergence (see (Khan & Ebadullah, 2011; Kostyrko *et al.*, 2000)).

Definition 1.13. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$ (see (Khan & Ebadullah, 2011)).

Definition 1.14. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in \ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $F(I) = \ell_\infty \cap c^I$ (see (Khan & Ebadullah, 2011; Tripathy & Hazarika, 2011)).

Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function (see (Khan & Ebadullah, 2011)). The following Lemmas will be used for establishing some results of this article.

Lemma 1.1. *Let E be a sequence space. If E is solid then E is monotone (see (Kamthan & Gupta, 1981), page 53).*

Lemma 1.2. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

Lemma 1.3. *If $I \subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I (see (Tripathy & Hazarika, 2009, 2011)).*

The idea of modulus was structured in 1953 by Nakano (See (Nakano, 1953)). A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if (1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is nondecreasing, and
(4) f is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967, 1973) used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle (Ruckle, 1973) proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (Ruckle, 1968, 1967, 1973) proved that, for any modulus f, $X(f) \subset \ell_1$ and $X(f)^\alpha = \ell_\infty$, where $X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$. The space $X(f)$ is a Banach space with respect to the norm $||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$.(See[31]).

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in (Gramsch, 1971). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling (Garling, 1966, 1968), Köthe (Köthe, 1970) and Ruckle (Ruckle, 1968, 1967, 1973). After then Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space $X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$.(See[22-23]).

(c.f (Dems, 2005; Gurdal, 2004; Khan *et al.*, 2012*b*,*a*, 2013; Šalát, 1980; Tripathy & Hazarika, 2009, 2011)).

Recently Khan and Ebadullah in (Khan *et al.*, 2013) introduced the following classes of sequences $\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\} \in I\}$, $\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\}$, $\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}$.

We also denote by $m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$ and $m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f)$.

In this article we introduce the following class of sequence spaces:

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\} \in I\},$$
$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$ and $m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}_0^I(F)$.

2. Main Results

Theorem 2.1. *For a sequence of moduli $F = (f_k)$, the classes of sequences $\mathcal{Z}^I(F), \mathcal{Z}_0^I(F), m_{\mathcal{Z}}^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are linear spaces.*

Proof. We shall prove the result for the space $\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly.
Let $(x_k), (y_k) \in \mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C};$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.1}$$

$$A_2 = \{k \in N : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.2}$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|) \leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|).$$

Now, by (2.1) and (2.2), $\{k \in \text{N}: f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$ Hence $\mathcal{Z}^I(F)$ is a linear space.

□

We state the following result without proof in view of Theorem 2.1.

Theorem 2.2. *The spaces $m^I_{\mathcal{Z}}(F)$ and $m^I_{\mathcal{Z}_0}(F)$ are normed linear spaces, normed by*

$$\|x_k\|_* = \sup_k f_k(|x_k|). \tag{2.3}$$

Theorem 2.3. *A sequence $x = (x_k) \in m^I_{\mathcal{Z}}(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F) \tag{2.4}$$

Proof. Suppose that $L = I - \lim x$. Then $B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I_{\mathcal{Z}}(F)$. For all $\epsilon > 0$. Fix an $N_\epsilon \in B_\epsilon$.Then we have $|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I_{\mathcal{Z}}(F)$ for all $\epsilon > 0$. Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I_{\mathcal{Z}}(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$.

Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I_{\mathcal{Z}}(F)$. This implies that $J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{k \in \mathbb{N} : x_k \in J\} \in m^I_{\mathcal{Z}}(F)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_k \supseteq ...$ with the property that $diamI_k \leq \frac{1}{2} diamI_{k-1}$ for (k=2,3,4.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I_{\mathcal{Z}}(F)$ for $(k = 1, 2, 3, ...)$. Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$. □

Theorem 2.4. *Let (f_k) and (g_k) be modulus functions for some fixed k that satisfy the $\triangle_2$-condition. If X is any of the spaces $\mathcal{Z}^I, \mathcal{Z}^I_0, m^I_{\mathcal{Z}}$ and $m^I_{\mathcal{Z}_0}$ etc, then the following assertions hold.*
(a) $X(g_k) \subseteq X(f_k.g_k)$,
(b) $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$.

Proof. (a) Let $(x_n) \in \mathcal{Z}^I_0(g_k)$. Then

$$I - \lim_n g_k(|x_n|) = 0. \tag{2.5}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 < t < \delta$.
Write $y_n = g_k(|x_n|)$ and consider $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n<\delta} + \lim_n f_k(y_n)_{y_n>\delta}$. We have

$$\lim_n f_k(y_n) \leq f_k(2) \lim_n(y_n). \tag{2.6}$$

For $y_n > \delta$, we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since f_k is non-decreasing, it follows that $f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2y_n}{\delta})$ Since f_k satisfies the $\triangle_2$-condition, we have $f_k(y_n) < \frac{1}{2} K \frac{y_n}{\delta} f_k(2) + \frac{1}{2} K \frac{y_n}{\delta} f_k(2) = K \frac{y_n}{\delta} f_k(2)$.

Hence

$$\lim_n f_k(y_n) \leq \max(1,K) \delta^{-1} f_k(2) \lim_n(y_n). \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $(x_n) \in \mathcal{Z}_0^I(f_k.g_k)$. Thus $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k.g_k)$. The other cases can be proved similarly.

(b) Let $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$. Then $I - \lim_n f_k(|x_n|) = 0$ and $I - \lim_n g_k(|x_n|) = 0$.

The rest of the proof follows from the following equality $\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|)$. □

Corollary 2.1. *$X \subseteq X(f_k)$ for some fixed k and $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.*

Theorem 2.5. *The spaces $\mathcal{Z}_0^I(F)$ and $m_{\mathcal{Z}_0}^I(F)$ are solid and monotone.*

Proof. We shall prove the result for $\mathcal{Z}_0^I(F)$. Let $(x_k) \in \mathcal{Z}_0^I(F)$. Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{2.8}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
Then the result follows from [9] and the following inequality $f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|)$ for all $k \in \mathbb{N}$. That the space $\mathcal{Z}_0^I(F)$ is monotone follows from the Lemma 1.20. For $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly. □

Theorem 2.6. *The spaces $\mathcal{Z}^I(F)$ and $m_{\mathcal{Z}}^I(F)$ are neither solid nor monotone in general.*

Proof. Here we give a counter example. Let $I = I_\delta$ and $f_k(x) = x^2$ for some fixed k and for all $x \in [0, \infty)$. Consider the K-step space $X_K(f_k)$ of X defined as follows. Let $(x_n) \in X$ and let $(y_n) \in X_K$ be such that

$$(y_n) = \begin{cases} (x_n), \text{if n is even}, \\ 0, \textit{otherwise}. \end{cases}$$

Consider the sequence (x_n) defined by $(x_n) = 1$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I(F)$. Thus $\mathcal{Z}^I(F)$ is not monotone. Hence $\mathcal{Z}^I(F)$ is not solid. □

Theorem 2.7. *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are sequence algebras.*

Proof. We prove that $\mathcal{Z}_0^I(F)$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I(F)$. Then $I - \lim f_k(|x_k|) = 0$ and $I - \lim f_k(|y_k|) = 0$. Then we have $I - \lim f_k(|(x_k.y_k)|) = 0$. Thus $(x_k.y_k) \in \mathcal{Z}_0^I(F)$ is a sequence algebra. For the space $\mathcal{Z}^I(F)$, the result can be proved similarly. □

Theorem 2.8. *The spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free in general.*

Proof. Here we give a counter example. Let $I = I_f$ and $f_k(x) = x^3$ for some fixed k and for all $x \in [0, \infty)$. Consider the sequence (x_n) and (y_n) defined by $x_n = \frac{1}{n}$ *and* $y_n = n$ for all $n \in \mathbb{N}$. Then $(x_n) \in \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$, but $(y_n) \notin \mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$. Hence the spaces $\mathcal{Z}_0^I(F)$ and $\mathcal{Z}_0^I(F)$ are not convergence free. □

Theorem 2.9. *If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric.*

Proof. Let $A \in I$ be infinite and $f_k(x) = x$ for some fixed k and for all $x \in [0, \infty)$. If

$$x_n = \begin{cases} 1, \text{for } n \in A, \\ 0, otherwise, \end{cases}$$

then by lemma 1.22 $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(n) = \begin{cases} \phi(n), \text{for } n \in K, \\ \psi(n), otherwise, \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(n)} \notin \mathcal{Z}^I(F)$ and $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$. Hence $\mathcal{Z}^I(F)$ and $\mathcal{Z}_0^I(F)$ are not symmetric. □

Theorem 2.10. $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_k) \in \mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_k(|x_k - L|) = 0$. We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I(F)$. The inclusion $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ is obvious. □

Theorem 2.11. *The function $\hbar : m_{\mathcal{Z}}^I(F) \to \mathbb{R}$ is the Lipschitz function, where $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$, and hence uniformly continuous.*

Proof. Let $x, y \in m_{\mathcal{Z}}^I(F)$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq \|x - y\|_*\} \in I,$$
$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F).$$

Hence also $B = B_x \cap B_y \in m_{\mathcal{Z}}^I(F)$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

Theorem 2.12. *If $x, y \in m_{\mathcal{Z}}^I(F)$, then $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.*

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m_{\mathcal{Z}}^I(F),$$
$$B_x = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m_{\mathcal{Z}}^I(F).$$

Now,

$$|x_k y_k - \hbar(x)\hbar(y)| = |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \quad (2.9)$$

As $m_{\mathcal{Z}}^I(F) \subseteq \mathcal{Z}_\infty^I(F)$, there exists an $M \in \mathbb{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.

Using eqn[10] we get $|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$ For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m_{\mathcal{Z}}^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For the space $m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarity. □

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Exponential Stability versus Polynomial Stability for Skew-Evolution Semiflows in Infinite Dimensional Spaces

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Abstract

As the dynamical systems that model processes issued from engineering, economics or physics are extremely complex, of great interest is to study the solutions of differential equations by means of associated evolution families. In this paper we emphasize some notions of asymptotic stability for skew-evolution semiflows on Banach spaces, such as exponential and polynomial stability, in a nonuniform setting. Examples for every concept and connections between them are also presented, as well as some characterizations.

Keywords: Skew-evolution semiflow, exponential stability, Barreira-Valls exponential stability, polynomial stability.
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1. Preliminaries

The theory of asymptotic properties for evolution equations has witnessed lately an explosive development. We intend to emphasize in our paper a framework which enables us to obtain characterizations in a unitary approach for the asymptotic stability on Banach spaces. The notion of skew-evolution semiflow, introduced in (Megan & Stoica, 2008), is more appropriate for the study in the nonuniform case. They depend on three variables, making thus possible the generalization for skew-product semiflows and evolution operators, which depend only on two. Hence, the study of asymptotic behaviors for skew-evolution semiflows in the nonuniform setting arises as natural, relative to the third variable. The notion has proved itself of interest in the development of the stability theory, in a uniform as well as in a nonuniform setting, being already adopted by some researchers, as, for example, A.J.G. Bento and C.M. Silva (see (Bento & Silva, 2012)), P. Viet Hai (see (Hai, 2010) and (Hai, 2011)) and T. Yue, X.Q. Song and D.Q. Li (see (Yue *et al.*, 2014)), which have contributed to the expansion of the concept of skew-evolution semiflows and deepened the study of their asymptotic behaviors and applications. Some properties for skew-evolution semiflows are defined and characterized in (Stoica, 2010).

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The definitions of various types of stability are illustrated by examples and the connections between them are emphasized. Our aim is also to give some integral characterizations for them. We present a concept of nonuniform exponential stability, given and studied by L. Barreira and C. Valls in (Barreira & Valls, 2008), which we call *"Barreira-Valls exponential stability"*. In this paper, some generalizations for the results obtained in the uniform setting in (Stoica & Megan, 2010) are proved in the nonuniform case.

2. Skew-evolution semiflows

This section gives the notion of skew-evolution semiflow on a Banach space, defined by means of an evolution semiflow and of an evolution cocycle.

Let (X, d) be a metric space, V a Banach space and V^* its topological dual. Let $\mathcal{B}(V)$ be the space of all V-valued bounded operators defined on V. The norm of vectors on V and on V^* and of operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. I is the identity operator. Let us denote $Y = X \times V$ and $T = \left\{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\right\}$.

Definition 2.1. A mapping $\varphi : T \times X \to X$ is said to be *evolution semiflow* on X if following properties are satisfied:

(es_1) $\varphi(t, t, x) = x,\ \forall (t, x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x),\ \forall (t, s), (s, t_0) \in T,\ \forall x \in X$.

Definition 2.2. A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if it satisfies following properties:

(ec_1) $\Phi(t, t, x) = I,\ \forall t \geq 0,\ \forall x \in X$;

(ec_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X$.

Let Φ be an evolution cocycle over an evolution semiflow φ. The mapping

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v) \tag{2.1}$$

is called *skew-evolution semiflow* on Y.

Example 2.1. Let us denote $C = C(\mathbb{R}, \mathbb{R})$ the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. For every $x, y \in C$, we define

$$d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

The set C is metrizable with respect to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}.$$

We consider for every $n \in \mathbb{N}^*$ a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \text{ with the property } \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geq 0.$$

Let X be the closure in C of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The mapping

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{where } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geq 0,$$

is an evolution semiflow on X. Let us consider the Banach space $V = \mathbb{R}^p$, $p \geq 1$, with the norm $\left\|(v_1, ..., v_p)\right\| = |v_1| + ... + |v_p|$. Then the mapping

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, ..., e^{\alpha_p \int_s^t x(\tau - s)d\tau} v_p\right),$$

where $(\alpha_1, ..., \alpha_p) \in \mathbb{R}^p$ is fixed, is an evolution cocycle over the evolution semiflow φ and $C = (\varphi, \Phi)$ is a skew-evolution semiflow on Y.

Example 2.2. For $X = \mathbb{R}_+$, the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow. For every evolution cocycle Φ over φ, we obtain that the mapping $E_\Phi : T \to \mathcal{B}(V)$, $E_\Phi(t, s) = \Phi(t, s, 0)$ is an evolution operator on V.

Example 2.3. If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$\Phi_\alpha : T \times X \to \mathcal{B}(V),\ \Phi_\alpha(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x), \tag{2.2}$$

is also a skew-evolution semiflow, called the *α-shifted skew-evolution semiflow*.

3. Exponential stability

In this section we consider several concepts of exponential stability for skew-evolution semiflows. Some connections between these concepts are established. We will emphasize that they are not equivalent.

The nonuniform exponential stability is given by

Definition 3.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *exponentially stable* ($e.s.$) if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s) \in T$, following relation takes place:

$$||\Phi(t, t_0, x)v|| \leq N(s)e^{-\alpha t} ||\Phi(s, t_0, x)v||,\ \forall (x, v) \in Y. \tag{3.1}$$

A concept of nonuniform exponential stability, which we will name *"Barreira-Valls exponential stability"*, is given by L. Barreira and C. Valls in (Barreira & Valls, 2008) for evolution equations.

Definition 3.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *Barreira-Valls exponentially stable* ($BV.e.s.$) if there exist some constants $N \geq 1$, $\alpha > 0$ and β such that, for all $(t, s), (s, t_0) \in T$, the relation holds:

$$||\Phi(t, t_0, x)v|| \leq Ne^{-\alpha t}e^{\beta s} ||\Phi(s, t_0, x)v||,\ \forall\ (x, v) \in Y. \tag{3.2}$$

The asymptotic property of nonuniform stability is considered in

Definition 3.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *stable* ($s.$) if there exists a mapping $N : \mathbb{R}_+ \to [1, \infty)$ such that, for all $(t, s), (s, t_0) \in T$, the relation is true:

$$||\Phi(t, t_0, x)v|| \leq N(s) ||\Phi(s, t_0, x)v||,\ \forall\ (x, v) \in Y. \tag{3.3}$$

Let us remind the property of exponential growth for skew-evolution semiflows, given by

Definition 3.4. A skew-evolution semiflow $C = (\varphi, \Phi)$ has *exponential growth* (*e.g.*) if there exist two mappings $M, \omega : \mathbb{R}_+ \to [1, \infty)$, ω nondecreasing, such that, for all $(t, s), (s, t_0) \in T$, we have:

$$\|\Phi(t, t_0, x)v\| \leq M(s)e^{\omega(s)(t-s)} \|\Phi(s, t_0, x)v\|, \ \forall\ (x, v) \in Y. \tag{3.4}$$

Remark. The relations concerning the previously defined asymptotic properties for skew-evolution semiflows are given by

$$(BV.e.s.) \Longrightarrow (e.s.) \Longrightarrow (s.) \Longrightarrow (e.g.) \tag{3.5}$$

The reciprocal statements are not true, as shown in what follows.

The following example presents a skew-evolution semiflow which is exponentially stable but not Barreira-Valls exponentially stable.

Example 3.1. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$.
Let us consider a continuous function $u : \mathbb{R}_+ \to [1, \infty)$ with

$$u(n) = e^{n \cdot 2^{2n}} \text{ and } u\left(n + \frac{1}{2^{2n}}\right) = e^4.$$

We define

$$\Phi_u(t, s, x)v = \frac{u(s)e^s}{u(t)e^t}v, \text{ where } (t, s) \in T,\ (x, v) \in Y.$$

As following relation

$$\|\Phi_u(t, s, x)v\| \leq u(s)e^s e^{-t} \|v\|$$

holds for all $(t, s, x, v) \in T \times Y$, it results that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is exponentially stable.

Let us now suppose that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is Barreira-Valls exponentially stable. Then, according to Definition 3.2, there exist $N \geq 1$, $\alpha > 0, \beta > 0$ and $t_1 > 0$ such that

$$\frac{u(s)e^s}{u(t)e^t} \leq Ne^{-\alpha t}e^{\beta s},\ \forall t \geq s \geq t_1.$$

For $t = n + \frac{1}{2^{2n}}$ and $s = n$ it follows that

$$e^{n\left(2^{2n}+1\right)} \leq Ne^{n+\frac{1}{2^{2n}}+4}e^{-\alpha\left(n+\frac{1}{2^{2n}}\right)}e^{\beta n},$$

which is equivalent with

$$e^{n\left(2^{2n}-\beta\right)} \leq Ne^{\frac{1}{2^{2n}}+4-\alpha\left(n+\frac{1}{2^{2n}}\right)}.$$

For $n \to \infty$, a contradiction is obtained, which proves that C_u is not Barreira-Valls exponentially stable.

There exist skew-evolution semiflows that are stable but not exponentially stable, as results from the following

Example 3.2. Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1, \infty) \text{ with the property } \lim_{t\to\infty} \frac{u(t)}{e^t} = 0.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \ \Phi_u(t,s,x)v = \frac{u(s)}{u(t)}v$$

is an evolution cocycle. As $|\Phi_u(t,s,x)v| \leq u(s)|v|$, $\forall (t,s,x,v) \in T \times Y$, it follows that $C_u = (\varphi, \Phi_u)$ is a stable skew-evolution semiflow, for every evolution semiflow φ.

On the other hand, if we suppose that C_u is exponentially stable, according to Definition 3.1, there exist a mapping $N : \mathbb{R}_+ \to [1,\infty)$ and a constant $\alpha > 0$ such that, for all $(t,s),(s,t_0) \in T$, we have

$$\|\Phi_u(t,t_0,x)v\| \leq N(s)e^{-\alpha t}\,\|\Phi_u(s,t_0,x)v\|\,,\ \forall\,(x,v) \in Y.$$

It follows that

$$\frac{u(s)}{N(s)} \leq \frac{u(t)}{e^{\alpha t}}.$$

For $t \to \infty$ we obtain a contradiction, and, hence, C_u is not exponentially stable.

Following example gives a skew-evolution semiflow that has exponential growth but is not stable.

Example 3.3. We consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1,\infty) \text{ with the property } \lim_{t\to\infty} \frac{e^t}{u(t)} = \infty.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \ \Phi_u(t,s,x)v = \frac{u(s)e^t}{u(t)e^s}v$$

is an evolution cocycle. We have $|\Phi(t,s,x)v| \leq u(s)e^{t-s}|v|$, $\forall (t,s,x,v) \in T \times Y$. Hence, $C_u = (\varphi, \Phi_u)$ is a skew-evolution semiflow, over every evolution semiflow φ, and has exponential growth.

Let us suppose that C_u is stable. According to Definition 3.3, there exists a mapping $N : \mathbb{R}_+ \to [1,\infty)$ such that $u(s)e^t \leq N(s)u(t)e^s$, for all $(t,s) \in T$. If $t \to \infty$, a contradiction is obtained. Hence, C_u is not stable.

4. Polynomial stability

In this section, we introduce a new concept of nonuniform stability for skew-evolution semiflows, given by the next

Definition 4.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *polynomially stable* (*p.s.*) if there exist a mapping $N : \mathbb{R}_+ \to [1,\infty)$ and a constant $\gamma > 0$ such that:

$$\|\Phi(t,s,x)v\|\,ds \leq N(s)(t-s)^{-\gamma}\,\|v\|\,, \tag{4.1}$$

for all $t > s \geq 0$ and all $(x,v) \in Y$.

Remark. If a skew-evolution semiflow C is exponentially stable, then it is polynomially stable.

$$(e.s.) \Longrightarrow (p.s.)$$

The reciprocal statement is not true, as shown in

Example 4.1. Let $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$. We consider

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as we have

$$|\Phi_u(t, s, x)v| \le \frac{s^2}{t}|v| = s\frac{s}{t}|v|,\ \forall t \ge s \ge 1 = t_0,\ \forall (x, v) \in Y,$$

it follows that $C = (\varphi, \Phi)$ is a Barreira-Valls polynomially stable skew-evolution semiflow.

If we suppose that C is exponentially stable, according to Definition 3.1, there exist $N : \mathbb{R}_+ \to [1, \infty)$ and $\alpha > 0$ such that

$$\frac{s+1}{t+1} \le N(s)e^{-\alpha t},\ \forall t \ge s \ge t_0,$$

which is equivalent with

$$\frac{e^{\alpha t}}{t+1} \le \frac{N(t_0)}{t_0+1},\ \forall t \ge t_0,$$

and which, for $t \to \infty$, leads to a contradiction. Hence, C is not exponentially stable.

Remark. For $\alpha \ge \beta$ in Definition 3.2, a Barreira-Valls exponentially stable skew-evolution semiflow C is polynomially stable.

$$(B.V.e.s.) \Longrightarrow (p.s.)$$

Example 4.2. Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t^2 + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. We define

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as the relation

$$|\Phi_u(t, s, x)v| \le (s^2 + 1)(t - s)^{-2}|v|,\ \forall t > s \ge 0,\ \forall (x, v) \in Y$$

holds, it follows that $C = (\varphi, \Phi)$ is a polynomially stable skew-evolution semiflow. On the other hand, C is not Barreira-Valls exponentially stable.

A similar concept to the nonuniform exponential growth can be considered the following nonuniform asymptotic property, given by

Definition 4.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ has *polynomial growth* ($p.g.$) if there exist two mappings $M, \gamma : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:

$$\|\Phi(t, s, x)v\| \le M(s)(t - s)^{\gamma(s)}\, \|v\|\,, \tag{4.2}$$

for all $t > s \ge 0$ and all $(x, v) \in Y$.

Remark. If a skew-evolution semiflow C has polynomial growth, then it has exponential growth.

$$(p.g.) \Longrightarrow (e.g.)$$

In order to obtain an integral characterization for the property of nonuniform polynomial stability for skew-evolution semiflows, we introduce the following concept, given by

Definition 4.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be $*$-*strongly measurable* ($* - s.m.$) if for every $(t, t_0, x, v^*) \in T \times X \times V^*$ the mapping defined by $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\|$ is measurable on $[t_0, t]$.

A particular class of $*$-strongly measurable skew-evolution semiflows is given by the next

Definition 4.4. A $*$-strongly measurable skew-evolution semiflow $C = (\varphi, \Phi)$ is called $*$-*integrally stable* ($* - i.s.$) if there exists a nondecreasing mapping $B : \mathbb{R}_+ \to [1, \infty)$ such that:

$$\int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \le B(s) \left\|v^*\right\|, \tag{4.3}$$

for all $(t, s) \in T$, all $x \in X$ and all $v^* \in V^*$ with $\|v^*\| \le 1$.

Theorem 4.3. *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If C is $*$-integrally stable, then C is stable.*

Proof. Let us consider the function

$$\gamma_1 : \mathbb{R}_+ \to \mathbb{R}_+,\ \gamma_1(t) = \frac{1}{1 + \gamma(t)},$$

where the mapping γ is given by Definition 4.2. We remark that for $t \ge s + 1$ we have

$$\int_s^t (\tau - s)^{-\gamma(s)} d\tau = \int_0^{t-s} u^{-\gamma(s)} du \ge \int_0^1 u^{-\gamma(s)} du = \gamma_1(s).$$

Hence, it follows that

$$\gamma_1(s)| < v^*, \Phi(t, s, x)v > | \le$$

$$\le \int_s^t (\tau - s)^{-\gamma(s)} \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| \|\Phi(\tau, s, x)v\| d\tau \le$$

$$\le M(s) \|v\| \int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \le M(s)B(s) \|v\| \left\|v^*\right\|,$$

where the existence of function M is assured by Definition 4.2. We obtain

$$\|\Phi(t, s, x)v\| \le M_1(s) \|v\|,\ \forall t \ge s + 1 > s \ge 0,\ \forall (x, v) \in Y,$$

where we have denoted

$$M_1(s) = \frac{M(s)B(s)}{\gamma(s)},\ s \ge 0.$$

On the other hand, for $t \in [s, s + 1)$, we have

$$\|\Phi(t, s, x)v\| \le M(s)(t - s)^{\gamma(s)} \|v\| \le M(s) \|v\|,$$

and, hence, it follows that

$$\|\Phi(t, s, x)v\| \le [M(s) + M_1(s)] \|v\|,\ \forall (t, s) \in T,\ \forall (x, v) \in Y,$$

which proves that the skew-evolution semiflow C is stable. □

The main result of this section is the following

Theorem 4.4. *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If C is $*$-integrally stable, then C is polynomially stable.*

Proof. As the skew-evolution semiflow $C = (\varphi, \Phi)$ is $*$-integrally stable, according to Theorem 4.3, it follows that there exists a mapping $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|<v^*, \Phi(t,s,x)v>| = |<\Phi(t,\tau,\varphi(\tau,s,x))^* v^*, \Phi(\tau,s,x)v>| \leq$$

$$\leq \|\Phi(\tau,s,x)v\| \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\| \leq M_2(s)\|v\| \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\|.$$

By integrating on $[s,t]$ we obtain for $(x,v) \in Y$ and $v^* \in V^*$ with $\|v^*\| \leq 1$

$$(t-s)|<v^*, \Phi(t,s,x)v>| \leq M_2(s)\|v\| \int_s^t \left\|\Phi(t,\tau,\varphi(\tau,s,x))^* v^*\right\| d\tau \leq$$

$$\leq M_2(s)B(s)\|v\| \left\|v^*\right\|,$$

which implies

$$(t-s)\|\Phi(t,s,x)v\| \leq M_2(s)B(s)\|v\|.$$

Hence, following relation

$$\|\Phi(t,s,x)v\| \leq M_2(s)B(s)(t-s)^{-1}\|v\|$$

holds for all $(t,s) \in T$ and all $(x,v) \in Y$.

Finally, it results that the skew-evolution semiflow $C = (\varphi, \Phi)$ is polynomially stable. □

Remark. In (Stoica & Megan, 2010), a variant of Theorem 4.4 for the case of uniform exponential stability is proved, as a generalization of a well known theorem of E.A. Barbashin, given in (Barbashin, 1967) for differential systems and of a result obtained in (Buşe *et al.*, 2007) by C. Buşe, M. Megan, M. Prajea and P. Preda for evolution operators. We remark that, in the nonuniform setting, the property of $*$-integral stability only implies the polynomial stability.

Remark. The reciprocal of Theorem 4.4 is not true. The skew-evolution semiflow given in Example 4.2 is polynomially stable but not $*$-strongly measurable. If we suppose that C is $*$-strongly measurable, we have

$$\int_s^t \frac{\tau^2+1}{t^2+1} d\tau = \frac{t-s}{t^2+1}\left(1 + \frac{t^2+ts+s^2}{3}\right) \leq N(s).$$

For $t \to \infty$, a contradiction is obtained.

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Common Fixed Points of Fuzzy Mappings in Quasi-Pseudo Metric and Quasi-Metric Spaces

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Abstract

In this paper, we prove common fixed point theorems for fuzzy mappings satisfying a new inequality initiated by Constantin (1991) in complete quasi-pseudo metric space and we also obtain some new common fixed point theorems for a pair of fuzzy mappings on complete quasi-metric space under a generalized contractive condition. Our results generalized many recent fixed point theorems.

Keywords: fuzzy sets, fuzzy mappings, common fixed points, quasi-pseudo metric space, quasi-metric space, fuzzy contraction mappings.
2010 MSC: 47H10, 54H25, 54A40.

1. Introduction

It is a well known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models. Over the period of last forty years the theory of fixed points has been developed regarding the results which are related to finding the fixed points of self and non-self nonlinear mappings. In 1922, Banach proved a contraction principle which states that for a complete metric space (X, d), the mapping $T: X \to X$ satisfying the following contraction condition

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 < \alpha < 1$$

has a unique fixed point in X. Banach contraction principle plays a fundamental role in the emergence of modern fixed point theory and it gains more attention because it is based on iteration, so it can be easily applied using computer. Initially Zadeh (1965) introduced the concept of Fuzzy Sets in 1965, has been an attempt to develop a mathematical framework in which two system or phenomena which due to intrinsic indefiniteness-as distinguished from mere statistical variation can't themselves be characterized precisely. The classical work of Zadeh (1965) stimulated a great interest among mathematicians, engineers, biologists, economists, psychologists and experts in other areas who use mathematical method in their research.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (1975) and Butnariu (1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadler (1969) theorem for multi-valued mappings. Park & Jeong (1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in Beg & A. (1992); Park & Jeong (1997).

Motivated and inspired by the works of Arora & V. (2000), Constantin (1991) and Park & Jeong (1997) the purpose of this paper is to prove some common fixed point theorems for fuzzy mappings satisfying new contractive-type condition of Constantin (1991) in complete quasi-pseudo metric space. Our results are the fuzzy extensions of some theorems in Beg & A. (1992); Iseki (1995); Popa (1985); Singh & Whitfield (1982) . Also, our results generalize the results of Arora & V. (2000), Heilpern (1981), and Park & Jeong (1997).

Recently Chen (2011, 2012) considered a new type contraction ψ contractive mapping in complete quasi metric space. The aim of this paper is to introduced a new class of fuzzy contraction mappings, which will be call fuzzy ψ contractive mappings in complete quasi metric space and to prove the existence of common fixed point for these contractions.

2. Basic concepts

For this purpose we need the following definitions and Lemmas.

Definition 2.1. Sahin *et al.* (2005) A quasi-pseudo metric on a non-empty set X is a non-negative real valued function d on $X \times X$ such that, for all $x, y, z \in X$:

(i) $d(x, x) = 0$, and
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X, d) is called a quasi-pseudo metric space, if d is a quasi-pseudo metric on X. A quasi-pseudo metric d such that $x = y$ whenever $d(x, y) = 0$ is a quasi metric so that a quasi pseudo metric space we do not assume that $d(x, y) = d(y, x)$ for every x and y. Each quasi-pseudo metric d on X induces a topology $\tau(d)$ which has base the family of all d balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ If d is a quasi-pseudo metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi-pseudo metric on X. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$ respectively.

Definition 2.2. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space and let A and B be non-empty subsets of X. Then the Hausdroff distance between subsets of A and B is defined by

$$H(A, B) = max\{sup_{a \in A} d(a, B), sup_{b \in B} d(b, A)\}$$

where $d(a, B) = inf\{d(a, x) : x \in B\}$.

Note that: $H(A, B) \geq 0$ with $H(A, B) = 0$ if and only if closure of A is equal to closure of B, $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$ for any non-empty subset A, B and C of X when d is a metric on X, clearly H is the usual Hausdroff distance.

Definition 2.3. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on (X, d) are defined by

$$W^*(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d \text{ - closed and } d^{-1}\text{-}compact\},$$

$$W'(X) = \{A \text{ in } I^X : A_1 \text{ is non-empty, } d \text{ - closed and } d\text{-compact}\}.$$

As per Heilpern (1981), the family W(X) of fuzzy sets on metric linear space (X, d) is defined as follows: $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $supA(x) = 1$ for $x \in X$.
If (X, d) is a metric linear space, then we have

$$W(X) \subset W^*(X) = W'(X) = \{A \in I^X : A_1 \text{ is non-empty and } d\text{-compact } \} \subset I^X.$$

Definition 2.4. Gregori. & Pastor (1999) Let (X, d) be a quasi-pseudo metric space and let $A, B \in W^*(X)$ or $A, B \in W'(X)$ and $\alpha \in [0, 1]$. Then we define

$$p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$\delta_\alpha(A, B) = sup\{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the Hausdroff distance deduced from the quasi-pseudo metric d on X, $p(A, B) = Sup\{p_\alpha(A, B) : \alpha \in [0, 1]\}$, $\delta(A, B) = Sup\{\delta_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = Sup\{ D_\alpha(A, B) : \alpha \in [0, 1]\}$. It is noted that p_α is non-decreasing function of α.

Definition 2.5. Gregori. & Pastor (1999) Let X be an arbitrary set and Y be any quasi-pseudo metric space. G is said to be a fuzzy mapping if G is a mapping from the set X into $W^*(Y)$ or $W'(Y)$. This definition is more general than the one given in Heilpern (1981). A fuzzy mapping G is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$.

Definition 2.6. Sahin *et al.* (2005) A point x is a fixed point of the mapping $G : X \to I^X$, if $\{x\} \subseteq G(x)$.

Note that : If $A, B \in I^X$, then $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.
The following Lemmas were proved by Gregori. & Pastor (1999).

Lemma 2.1. *Let (X, d) be a quasi-pseudo metric space and let $x \in X$ and $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ iff $p_\alpha(x, A) = 0$, for each $\alpha \in [0, 1]$.*

Lemma 2.2. *Let (X, d) be a quasi-pseudo metric space and let $A \in W^*(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

Lemma 2.3. *Let (X, d) be a quasi-pseudo metric space and let $\{x_0\} \subset A$. Then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W^*(X)$ and $\alpha \in [0, 1]$.*

Above Lemmas were proved by Heilpern (1981) for the family $W(X)$ in a metric linear space.

Proposition 1. *Let (X, d) be a complete quasi-pseudo metric space and $G : X \to W^*(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Proposition 2. *Let (X, d) be a quasi-pseudo metric space and $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Now we shall use the notations as in Isufati & Hoxha (2010).
In the following, the letter Γ denotes the set of positive integers.
If A is a subset of a topological space (X, τ), we will denote by $cl_\tau A$ the closure of A in (X, τ).
A quasi-metric on a non-empty set X is a non-negative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X,d) is called a quasi-metric space, if d is a quasi-metric on X.
Each quasi-metric d on X induces a T_0 topology $\mathcal{T}(d)$ on X, which has a base, the family of all $d-$ balls $\{B_d(x, r) : x \in X, r > 0\}$, where, $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

If d is a quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also quasi metric on X. By $d \wedge d^{-1}$ we denote min $\{d, d^{-1}\}$ and also we denote d^s the metric on X by $d^s(x, y) = max\{d(x, y), d(y, x)\}$ for all $x, y \in X$.

A sequence $(x_n)_{n\in\Gamma}$ in a quasi metric space (X, d) is called left $k-$ Cauchy Reilly *et al.* (1982) if for each $\varepsilon > 0$ there is a $n_\varepsilon \in \Gamma$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \Gamma$ with $m \geq n \geq n_\varepsilon$. Let (X, d) be a quasi-metric space and let $\mathcal{K}_0^s(X)$ be the collection of all non-empty compact subset of the metric space (X, d^s). Then the Hausdroff distance H_d on $\mathcal{K}_0^s(X)$ is defined by

$$H_d(A, B) = \max\{\ \sup d(a, B) : a \in A,\ \ \sup d(A, b) : b \in B\}\ \text{ whenever } A, B \in \mathcal{K}_0^s(X).$$

A fuzzy set on X is an element of I^X where $I = [0, 1]$. If A is a fuzzy set in X, then the number $A(x)$ is called the grade of membership of x in A. The α- level set of A, denoted by A_α, and defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$ and $A_0 = \overline{\{x : A(x) > 0\}}$ where the closure is taken in (X, d^s).

Definition 2.7. Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space. A fuzzy set A in quasi-metric space (X, d) will be called an approximate quantity. The family $\mathcal{A}(X)$ of all fuzzy sets on (X, d) is defined by $\mathcal{A}(X) = \{A \in I^X : A_\alpha$ is d^s -compact for each $\alpha \in [0, 1]$ and $supA(x) = 1 : x \in X\}$.

Definition 2.8. Gregori & Romaguera (2000) Let $A, B \in \mathcal{A}(X)$ then A is said to be more accurate than B, denoted by $A \subset B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

Definition 2.9. Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and α in $[0, 1]$. Then we define $p_\alpha(A, B) = Inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha)$, $D_\alpha(A, B) = H_d(A_\alpha, B_\alpha)$, $p(A, B) = sup\{P_\alpha(A, B) : \alpha \in [0, 1]\}$, $D(A, B) = sup\{D_\alpha(A, B) : \alpha \in [0, 1]\}$, for $x \in X$, we write $p_\alpha(x, A)$ instead of $p_\alpha(\{x\}, A)$. We denote that p_α is a non-decreasing function of α and D is metric on $\mathcal{A}(x)$.

Definition 2.10. Gregori & Romaguera (2000) A fuzzy mapping on a quasi-metric space (X, d) is a function F defined on X, which satisfies the following two conditions

(i) $F(x) \in \mathcal{A}(X)$ for all $x \in X$,

(ii) If $a, z \in X$ such that $(F(z))(a) = 1$ and $p(a, F(a)) = 0$ then $(F(a))(a) = 1$.

We need the following lemmas for our main result which was given by Gregori & Romaguera (2000).

Lemma 2.4. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A, B \in \mathcal{A}(X)$ and $x \in A_1$. There exist $y \in B_1$ such that $d(x, y) \leq D_1(A, B)$.*

Lemma 2.5. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $y \in A$. Then $p(x, A) \leq d(x, y)$ for each $x \in X$.*

Lemma 2.6. *Gregori & Romaguera (2000) Let $x \in X$, $A \in \mathcal{A}(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$, then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Lemma 2.7. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and $A \in \mathcal{A}(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$.*

Lemma 2.8. *Gregori & Romaguera (2000) Let (X, d) be a quasi-metric space and let $A \in \mathcal{A}(X)$ and $x \in A$. Then $p_\alpha(x, B) \leq D_\alpha(A, B)$ for each $B \in \mathcal{A}(X)$ and each $\alpha \in [0, 1]$.*

Lemma 2.9. *Gregori & Romaguera (2000) Let A and B be non-empty compact subset of a quasi-metric space (X, d) if $a \in A$, then there exists $b \in B$, such that $d(a, b) \leq H(A, B)$.*

Lemma 2.10. *Gregori & Romaguera (2000) Let (X, d) be a complete quasi metric space and let F be a fuzzy mapping from X into $\mathcal{A}(X)$ and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

We consider the set of function $\Psi = \{\psi \colon R^{+5} \to R^+\}$ satisfying the following conditions

(i) ψ strictly increasing, continuous function in each coordinate and
(ii) for all $g \in R^+$ such that $\psi(g, g, g, 0, 2g) < g$, $\psi(g, g, g, 2g, 0) < g$, $\psi(0, 0, g, g, 0) < g$ and $\psi(g, 0, 0, g, g) < g$.

Example 2.11. *Let $\psi \colon R^{+5} \to R^{+5}$ denote by $\psi(g_1, g_2, g_3, g_4, g_5) = k\ max(g_1, g_2, g_3, \frac{g_4}{2}, \frac{g_5}{2})$ for $k \in (0, 1)$ then ψ satisfies above conditions (i) and (ii).*

3. Main Result

Following Constantin (1991) we consider the set $\mathcal{G}$ of all continuous functions $g \colon [0, \infty)^5 \to [0, \infty)$ with the following properties:

(1) g is non-decreasing in the $2^{nd}, 3^{th}, 4^{th}$ and 5^{th} variable,
(2) if $u, v \in [0, \infty)$ are such that $u \leq g(v, v, u, u + v, 0)$ or $u \leq g(v, u, v, 0, u + v)$ then $u \leq qv$ where $0 < q < 1$ is a given constant,
(3) if $u \in [0, \infty)$ is such that $u \leq g(u, 0, 0, u, u)$ then $u = 0$.

Now we are ready to prove our main theorems.

Theorem 3.1. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for $x, y \in X$*

$$D(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x)))$$

then there exists $z \in X$ such that $\{z\} \subset F_1(z)$ and $\{z\} \subset F_2(z)$.

Proof. Let $x_0 \in X$. Then by Proposition 2.1 there exists an $x_1 \in X$ such that $\{x_1\} \subset G_1(X_0)$. From Proposition 2.1 there exists $x_2 \in (G_2(x_1))_1$. Since $(G_1(x_0))_1, (G_2(x_1))_1 \in CP(X)$ then by Proposition 2.2 we obtain,

$$\begin{aligned} d(x_1, x_2) \leq\ & D_1(G_1(x_0), G_2(x_1)) \leq D(G_1(x_0), G_2(x_1)) \leq g(d(x_0, x_1), p(x_0, G_1(x_0)), p(x_1, G_2(x_1)), \\ & p(x_0, G_2(x_1)), p(x_1, G_1(x_0))) \leq g(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

therefore, $d(x_1, x_2) \leq qd(x_0, x_1)$. Following similar process we obtain, $d(x_2, x_3) \leq qd(x_1, x_2)$. By induction, we produce a sequence (x_n) of points of X such that for each $k \geq 0$ $\{x_{2k+1}\} \subset G_1(x_{2k})$, and $\{x_{2k+2}\} \subset G_2(x_{2k+1})$, $d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq \cdots \leq q^n d(x_0, x_1)$. Furthermore, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{q^n + q^{n+1} + \cdots + q^{m-1}\} d(x_0, x_1) \leq \frac{q^n}{(1 - q)} d(x_0, x_1). \end{aligned}$$

It follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset G_i(z), i = 1, 2$. Now by Lemma 2.2 $p_0(z, G_2(z)) \le d(z, x_{2n+1}) + p_0(x_{2n+1}, G_2(z))$. Then by Lemma 2.3,

$$\begin{aligned} p(z, G_2(z)) &\le d(z, x_{2n+1}) + D(G_1(x_{2n}), G_2(z)) \le d(z, x_{2n+1}) + f(d(x_{2n}, z), p(x_{2n}, G_1(x_{2n})), \\ &\quad p(z, G_2(z)), p(x_{2n}, G_2(z)), p(z, G_1(x_{2n}))) \\ &\le d(z, x_{2n+1}) + g(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, G_2(z)), p(x_{2n}, G_2(z)), d(z, x_{2n+1})). \end{aligned}$$

As $n \to \infty$, we obtain from above inequality that $p(z, G_2(z)) \le g(0, 0, p(z, G_2(z)), p(z, G_2(z)), 0)$, so by properties of g we have $p(z, G_2(z)) = 0$. by (2). So by Lemma 2.1, we get $\{z\} \subset G_2(z)$. Similarly, it can be shown that $\{z\} \subset G_1(z)$. □

As corollaries of Theorem 3.1, we have the following:

Corollary 3.2 (Park & Jeong (1997); Theorem 3.1)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $\alpha, 0 \le \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \le \alpha \cdot \max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \max\{x_1, x_2, x_3, \frac{(x_4+x_5)}{2}\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.1. □

Corollary 3.3 (Park & Jeong (1997); Theorem 3.2)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. satisfying $D(G_1(x), G_2(y)) \le k[p(x, G_1(x)) \cdot p(y, G_2(y))]^{\frac{1}{2}}$, for all $x, y \in X$ and $0 < k < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = k[x_2 \cdot x_3]^{\frac{1}{2}}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.2. □

Corollary 3.4 (Park & Jeong (1997); Theorem 3.4)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$, such that*

$$D(G_1(x), G_2(y)) \le \alpha \cdot \frac{p(y, G_1(y))[1 + p(x, G_2(x))]}{1 + d(x, y)} + \beta d(x, y)$$

for all $x \ne y, \alpha, \beta > 0$ and $\alpha + \beta < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \frac{x_3(1+x_2)}{(1+x_1)} + \beta x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.3. □

Corollary 3.5 (Arora & V. (2000); Theorem 3.2)**.** *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $r, 0 \le r < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \le r \cdot \max\{d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = r \cdot \max\{x_1, x_2, x_3, x_4, x_5\}$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.4. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Iseki (1995).

Corollary 3.6. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If for each $x, y \in X$, such that $D(G_1(x), G_2(y)) \leq \alpha[p(x, G_1(x)) + p(y, G_2(y))] + \beta[p(x, G_2(y)) + p(y, G_1(x))] + \gamma d(x, y)$ where α, β, γ are non-negative and $2\alpha + 2\beta + \gamma < 1$. Then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha[x_2 + x_3] + \beta[x_4 + x_5] + \gamma x_1$. Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain corollary 3.5. □

The following Corollary is a fuzzy version of the fixed point theorem for multi-valued mappings of Singh & Whitfield (1982).

Corollary 3.7. *Let X be a complete quasi-pseudo metric space and let G_1 and G_2 be fuzzy mappings from X into $W^*(X)$. If there exists a constant $\alpha, 0 \leq \alpha < 1$, such that for each $x, y \in X$, $D(G_1(x), G_2(y)) \leq \alpha \cdot \max\{d(x, y), \frac{[p(x,G_1(x))+p(y,G_2(y))]}{2}, \frac{[p(x,G_2(y))+p(y,G_1(x))]}{2}\}$ then there exists $z \in X$ such that $\{z\} \subset G_1(z)$ and $\{z\} \subset G_2(z)$.*

Proof. We consider the function $g : [0, \infty)^5 \to [0, \infty)$ defined by $g(x_1, x_2, x_3, x_4, x_5) = \alpha \cdot \max\{x_1, \frac{[x_2+x_3]}{2}, \frac{[x_4+x_5]}{2}\}$ Since $g \in \mathcal{G}$ we can apply Theorem 3.1 and obtain Corollary 3.6. □

Remark. If there exists a function $g \in \mathcal{G}$ such that for all $x, y \in X$

$$\delta(G_1(x), G_2(y)) \leq g(d(x, y), p(x, G_1(x)), p(y, G_2(y)), p(x, G_2(y)), p(y, G_1(x))),$$

then the conclusion of Theorem 3.1 remains valid. This result is considered as special case of Theorem 3.1 because (see, Hicks (1997); page 414) $D(G_1(x), G_2(y)) \leq \delta(G_1(x), G_2(y))$. Moreover, this result generalize Theorem 3.3 of Park & Jeong (1997).

The following theorem extends Theorem 3.1 to a sequence of fuzzy mappings:

Theorem 3.8. *Let X be a complete quasi-pseudo metric space and let $\{G_n : n \in Z^+\}$ be fuzzy mappings from X into $W^*(X)$. If there is a $g \in \mathcal{G}$ such that for all $x, y \in X$*

$$D(G_0(x), G_n(y)) \leq g(d(x, y), p(x, G_0(x)), p(y, G_n(y)), p(x, G_n(y)), p(y, G_0(x)))$$

then there exists a common fixed point of the family $\{G_n : n \in Z^+\}$.

Proof. From Theorem 3.1, we get a common fixed point $x_i, i = 1, 2, \ldots,$ for each pair $(G_0, G_i), i = 1, 2, \ldots.$ Applying Lemma 2.2, one can have that $p_\alpha(x_i, G_0 x_i) = P_\alpha(x_i, G_i(x_i)) = 0$, for all $i = 1, 2, \ldots.$ Thus one can deduce form Lemma 2.3, for $i \neq j$, that

$$\begin{aligned} d(x_i, x_j) &= p_\alpha(x_i, G_j(x_j)) \leq D_\alpha(G_i(x_i), G_j(x_j)) \leq D(G_i(x_i), G_j(x_j)) \\ &\leq g(d(x_i, x_j), p(x_i, G_i(x_i)), p(x_j, G_j(x_j)), p(x_i, G_j(x_j)), p(x_j, G_i(x_i))) \\ &= g(d(x_i, x_j), 0, 0, d(x_i, x_j), d(x_i, x_j)). \end{aligned}$$

Therefore $d(x_i, x_j) = 0$, i.e., $x_i = x_j$ for all $i, j \in \mathbb{N}$. □

Corollary 3.9. *(Arora & V. (2000);Theorem (3.4)) Let X be a complete quasi-pseudo metric space and let $\{G_n : n \in \mathbb{N}^+\}$ be fuzzy mappings from X into $W^*(X)$. If for each $x, y \in X$, and $r \in (0, \frac{1}{2}), n = 1, 2, , ...,$ such that $D(G_0(x), G_i(y)) \leq r\max\{d(x, y), p(x, G_o(x)), p(y, G_i(y)), p(x, G_i(y)), p(y, G_0(x))\}$. Then there exists a common fixed point of the family $\{G_n : n \in \mathbb{N}^+\}$.*

Theorem 3.10. *Let (X, d) be a complete quasi-metric space, let $T_1, T_2 \colon X \to \mathcal{A}(X)$ be fuzzy ψ contractive mappings satisfies $D(T_1x, T_2y) \leq \psi\{(d(x, y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x)\}$ then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.*

Proof. Let $x_0 \in X$ then by Lemma 2.10 there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$ for $x_1 \in T_2(x_1))_1$ is non-empty compact subset of X. Since $(T_1(x_0))_1, (T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0)_1)$, then by lemma 2.9 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1, x_2) \leq D_1(T_1(x_0), T_2(x_1))$ so, from Lemma 2.6 and properties of ψ function, we have

$$\begin{aligned} d(x_1, x_2) &\leq D_1(T_1(x_0), T_2(x_1)) \leq D(T_1(x_0), T_2(x_1)) \\ &\leq \psi(d(x_0, x_1), p(x_0, T_1x_0), p(x_1, T_2x_1), p(x_0, T_2x_1), p(x_1, T_1x_0)) \\ &\leq \psi(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \end{aligned}$$

and

$$\begin{aligned} d(x_2, x_1) &\leq D_1(T_2(x_1), T_1(x_0)) \leq D(T_2(x_1), T_1(x_0)) \\ &\leq \psi(d(x_1, x_0), p(x_1, T_2x_1), p(x_0, T_1x_0), p(x_1, T_1x_0), p(x_0, T_2x_1)) \\ &\leq \psi(d(x_1, x_0), d(x_1, x_2), d(x_0, x_1), 0, d(x_0, x_1) + d(x_1, x_2)) \end{aligned}$$

by induction, we have a sequence (x_n) of points such that for all $n \in R^+ \cup \{0\}$ we have $\{x_{2n+1}\} \subset T_1(x_{2n})$ and $\{x_{2n+2}\} \subset T_2(x_{2n+1})$ then

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0) \tag{3.1}$$

and

$$d(x_{n+1}, x_n) \leq \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \tag{3.2}$$

so, by the properties of the ψ function we have that for each $n \in R^+$ $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ and $d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})$. The sequence $(b_m)_{m \in R^+}$, such that $b_m = d(x_m, x_{m+1})$ is a non-increasing sequence and bounded below. Thus it must converges to some $b \geq 0$. By the inequality 3.1 and 3.2 we have

$$b \leq b_m \leq \psi(b_{m-1}, b_{m-1}, b_m, b_{m-1} + b_m, 0) < b \tag{3.3}$$

passing to the limit, as $m \longrightarrow \infty$, and by properties of the ψ function we have $b \leq b \leq \psi(b, b, b, 2b, 0) < b$ which is contradiction. Hence $b = 0$. Thus, the sequence $(x_n)_{n \in R^+}$ must be a Cauchy sequence.

Similarly, the sequence $(c_n)_{n \in R^+}$ such that $c_n = d(x_{n+1}, x_n)$ is a non-increasing sequence and bounded below. Thus, it must converges to some $c \geq 0$.
By the inequality 3.1 and 3.2 we have

$$c \leq c_n \leq \psi(c_{n-1}, c_{n-1}, c_n, c_{n-1} + c_n, 0) < b \tag{3.4}$$

passing to the limit, as $n \longrightarrow \infty$, and by properties of the ψ function we have $c \leq c \leq \psi(c, c, c, 2c, 0) < c$ which is possible if and only if $c = 0$.

We next claim that to prove that for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in R^+$, such that for all $m > n > n_0(\varepsilon)$

$$d(x_m, x_n) < \varepsilon. \tag{3.5}$$

Suppose that 3.5 is false then, there exists some $\varepsilon > 0$ such that for all $k \in R^+$, there exists the smallest number m_k, such that $m_k, n_k \in R^+$ with $m_k > n_k \leq k$ satisfying $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ so,

$$\begin{aligned}
\varepsilon &\leq d(x_{m_k}, x_{n_k}) \leq D(Tx_{m_k-1}, Tx_{n_k-1}) \\
&\leq \psi(d(x_{m_k-1}, x_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}), p(x_{n_k-1}, Tx_{n_k-1}), p(x_{m_k-1}, Tx_{n_k-1}), p(x_{n_k-1}, Tx_{m_k-1}) \\
&\leq \psi(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k}) \\
&\leq \psi(c_{m_k-1} + d(x_{m_k}, x_{n_k}) + c_{n_k-1}, c_{m_k-1}, c_{n_k-1}, c_{m_k-1} + d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k}) + c_{n_k-1}).
\end{aligned}$$

Letting $k \to \infty$ we have $\varepsilon \leq \psi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \varepsilon$ which is a contradiction. It follows from 3.5 that (x_n) is a Cauchy sequence since (X, d) is a complete quasi-metric space,then there exists $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Next we show that $\{z\} \subset T_2(z)$.

By Lemmas 2.7 and 2.8 we get $p_\alpha(z, T_2z) \leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_n, T_2z)$ for each $\alpha \in [0, 1]$. Taking supremum on α in the last inequality, we obtain from the properties of ψ that

$$\begin{aligned}
p_\alpha(z, T_2z) &\leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, T_2z) \leq d(z, x_{2n+1}) + D_\alpha(T_1x_{2n}, T_2z) \\
&\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), p(x_{2n}, T_1x_{2n}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})) \\
&\leq d(z, x_{2n+1}) + \psi(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, T_2z), p(x_{2n}, T_2z), d(z, x_{2n+1})).
\end{aligned}$$

As $n \to \infty$, we have $p(z, T_2z) \leq \psi(0, 0, p(z, T_2z), p(z, T_2z), 0) < p(z, T_2z)$. It yields that $p(z, T_2z) = 0$. So, we get from Lemma 2.10 that $\{z\} \subset T_2z$. Similarly we prove that $\{z\} \subset T_1z$. □

Corollary 3.11. *Let (X, d) be a complete quasi metric space and let $T: X \to \mathcal{A}(X)$ be a fuzzy ψ contraction mapping then there exists $z \in X$ such that $\{z\} \subset T(z)$.*

Proof. If put $T_1 = T_2 = T$ in theorem 3.3 we get the conclusion of corollary 3.8. □

Corollary 3.12. *Let (X, d) be a complete quasi metric space and let $T: X \to \mathcal{A}(X)$ be a fuzzy ψ contraction mapping, such that for all $x, y \in X$ $D(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), \frac{p(x,T_2y)}{2}, \frac{p(y,T_1x)}{2})$ then there exists $z \in X$ such that $\{z\} \subset T_1z$ and $\{z\} \subset T_2z$.*

Proof. We consider the function $\psi: R^{+5} \to R^{+5}$ denoted by $\psi(t_1, t_2, t_3, t_4, t_5) = k \max\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\}$ for $k \in (o, 1)$. Since $\psi \in \Psi$ we can apply theorem 3.3 and obtain Corollary 3.9. □

Remark. As examples of the main results we can taking theorems in which the contractions conditions are compatible with the condition (i) and (ii).

Remark. If there is a $\psi \in \Psi$ such that for all $x, y \in X$

$$\delta(T_1x, T_2y) \leq \psi(d(x, y), p(x, T_1x), p(y, T_2y), p(x, T_2y), p(y, T_1x))$$

then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D_1(T_1x, T_2y) \leq \delta(T_1x, T_2y)$ for all $x, y \in X$. The following theorem generalizes Theorem 3.3 to a sequence of fuzzy contractive mappings.

Theorem 3.13. *Let $(T_n : n \in (0, \infty) \cup \{0\})$ be a sequence of fuzzy mappings from a complete quasi metric space X into $\mathcal{A}(X)$. If there is a $\psi \in \Psi$ such that for all $x, y \in X$*

$$D(T_0x, T_ny) \leq \psi(d(x, y), p(x, T_0x), p(y, T_ny), p(x, T_ny), p(y, T_0x))$$

for all $n \in (0, \infty) \cup \{0\}$, then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$.

Proof. Putting $T_1 = T_0$ and $T_2 = T_n$ for all $n \in N$ in Theorem 3.3 then there exists a common fixed point of the family $(T_n : n \in (0, \infty) \cup \{0\})$. □

4. Conclusion and future work

Fuzzy sets and mappings play an important role in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings has been developed largely. We generalize, extend and unify several known results of metric spaces, into a weaker and generalize setting of quasi-pseudo metric space and quasi metric space for fuzzy mappings. We use a more generalize contractive condition than the existing ones, also we prove our results in quasi-pseudo metric space, quasi metric space and so as to obtain better results under weaker conditions. We conclude this paper with an open problem: Is it possible to prove the results of this paper in the setting of b-metric and partial metric spaces?

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