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A Mixed Integer Linear Programming Formulation for Restrained Roman Domination Problem

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Abstract

This paper deals with a subgroup of Roman domination problems (RDP) named Restrained Roman domination problem (RRDP). It introduces a new mixed integer linear programming (MILP) formulation for the RRDP. The presented model uses relatively small number of the variables and constraints and could be of use both in theoretical and practical purposes. Proof of its correctness is given, i.e. it was shown that optimal solution to the RRDP formulation is equal to the optimal solution of the original problem.

Keywords: Restrained Roman domination in graphs, combinatorial optimization, integer linear programming.
2010 MSC: 90C11, 05C69.

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1. Introduction

With contiguous territories throughout Europe, North Africa, and the Middle East, the Roman Empire was one of the largest in history (Kelly, 2006). The idea of building ”empire without end” (Nicolet, 1991) expressed the ideology that neither time nor space limited the Empire. During the fourth century A.D., Emperor of Rome, Constantine the Great, intended to accomplish that idea. In order to expand the Roman Empire, he dealt with the next problem: How to organize legions such that entire Empire of Rome stayed defended? Since legions were highly trained, it was assumed that they could move fast from one city to another. City was considered to be defended if at least one legion was stationed in it or it was adjacent to a city with two legions within. The second condition was made because legion could move from a stationed city only if such an act won't leave it undefended.

Inspired by this historical problem, a new subgroup of the domination problems, named Roman domination problem (RDP), was proposed by Stewart (1999). RDP can be described as a problem of finding the minimal number of legions such that entire Empire of Rome is defended.

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More details about the RDP can be found in (ReVelle & Rosing, 2000), (Currò, 2014), (Liedloff *et al.*, 2005) and (Xing *et al.*, 2006).

Restrained Roman domination problem (RRDP), previously introduced by Pushpam & Sampath (2015), is defined also as a problem of finding the minimal number of legions such that entire Empire of Rome is defended but the conditions are slightly changed. Again, a city is considered to be defended if at least one legion is stationed within. But, a city without legion within is consider to be defended if it is adjacent to at least one city with two legions within and to at least one undefended city.

The Roman domination problem and the Restrained Roman domination problem can be illustrated by a graph such that each city of the Empire of Rome is represented by a vertex and, for two connected cities, the corresponding vertices are set to be adjacent.

Assuming that five cities, marked by numbers 1 - 5, are constructed such that a city marked by 1 is only adjacent to a city marked by 2 and that all other cities are adjacent to each other, a small illustration of the RDP and RRDP solutions are given in the figure below.

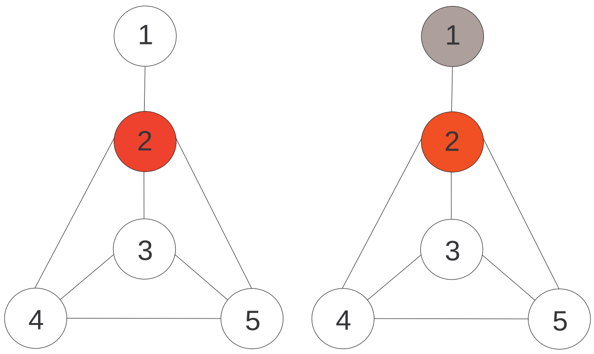

Figure 1. Illustrations of the the RDP (left) and RRDP (right) solutions

Vertex colored in red indicates that corresponding city is defended by two legions, vertex colored in gray indicates that corresponding city is defended by one legion, while vertices colored in white stands for the cities without legions within. For the Roman domination problem (shown on the figure on the left) by using only two legions, all five cities could be defended, i.e. assigning two legions to a city marked by 2, corresponding city and all adjacent cities are consider to be defended. Note that minimal number of legions for the RRDP (shown on the figure on the right) is three, i.e. assigning two legions to a city marked by 2, corresponding city and cities marked by 3, 4 and 5 are set to be defended because they are adjacent to a city with two legions within and adjacent to two cities with no assigned legions; city marked by 1 is set to be defended by one legion, since it can't be adjacent to at least one city without legions within and to at least one city with two legions within, at the same time. Given solution for RRDP is not unique since the same result could be obtained by assigning two legions to a city marked by 3 instead of the city marked by 2.

In the next sections, MILP formulation for the RRDP together with the proof of its validity, are proposed.

2. Problem definition

Let $G = (V, E)$ be an undirected graph with a vertex set V such that each vertex $u \in V$ represents a city of Roman Empire and each edge, $e \in E$, represents an existing road between two adjacent cities. A neighborhood set N_u ($N_u \subset V$), of a vertex $u \in V$, is defined as a set of vertices v adjacent to a vertex u. For a function f

$$f : V \to \{0, 1, 2\} \tag{2.1}$$

let a number of legions assigned to a city represented by a vertex u to be equal to a value $f(u)$. Additionally, let a function f satisfy the condition that for every vertex $u \in V$such that $f(u) = 0$ there exists vertices $v, w \in V$ such that $f(v) = 2$ and $f(w) = 0$. In other words, if there is an undefended city u, then there exist at least one city $v, v \in N_u$ with two legions within and at least one undefended city $w, w \in N_u$. Function f is called a restrained Roman domination function.

Mathematically, a proposed problem can be formulated as:

$$\min_f F_1(f) \tag{2.2}$$

subject to:

$$F_1(f) = \sum_{u \in V} f(u) \tag{2.3}$$

$$(\forall u \in V) f(u) = 0 \Rightarrow (\exists v, w \in N_u)(f(v) = 2 \land f(w) = 0). \tag{2.4}$$

Now, using a proposed notations, a solution to the illustrated RRDP can be written as: $F_1(f) = 3$ for $f(2) = 2$, $f(1) = 1$ and $f(3) = f(4) = f(5) = 0$ and it is not unique ($F_1(f) = 3$ for $f(3) = 2$, $f(1) = 1$ and $f(2) = f(4) = f(5) = 0$).

3. A mixed integer linear programming formulation for the RRDP

For a function f, defined by (2.1), let a continuous decision variable x_i, $x_i \in [0, \infty)$, indicate a number of legions assigned to a corresponding city $i \in V$. Although, $f \in \{0, 1, 2\}$ and $x_i \in [0, \infty)$, x_i and $f(i)$ are with equal values in the optimal solution, and not necessary with equal values for every feasible solution. Let binary decision variables y_i and z_i indicate if there are two or none legions assigned to a corresponding city $i \in V$,

$$y_i = \begin{cases} 1, & f(i) = 2 \\ 0, & otherwise \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1, & f(i) = 0 \\ 0, & otherwise \end{cases}.$$

A mixed integer linear programming (MILP) formulation for the RRDP can now be formulated as follows:

$$\min \sum_{i \in V} x_i \tag{3.1}$$

subject to

$$x_i + \sum_{j \in N_i} y_j \geq 1, \qquad i \in V \tag{3.2}$$

$$x_i + \sum_{j \in N_i} z_j \geq 1, \qquad i \in V \tag{3.3}$$

$$x_i \geq 2y_i, \qquad i \in V \tag{3.4}$$

$$x_i + 2z_i \leq 2, \qquad i \in V \tag{3.5}$$

$$x_i \in [0, +\infty); \qquad y_i, z_i \in \{0, 1\}, \quad i \in V. \tag{3.6}$$

Further, for vector values $x = [x_i], y = [y_i]$ and $z = [z_i]$, which satisfies constraints (3.2) - (3.6), notation $F_2(x, y, z) = \sum_{i \in V} x_i$ will be used. Now, condition (3.1) which minimizes the number of legions, can be written as $\min\limits_{(x,y,z)} F_2(x, y, z)$. By the constraints (3.2) it is ensured that each undefended vertex i is adjacent to at least one vertex with two legions within. Similarly, by the constraints (3.3) it is ensured that each undefended vertex i is adjacent to at least one vertex which is also undefended. From the inequalities (3.4) and (3.5) it follows that for each city $i \in V$ with at most 1 legion within, corresponding value y_i is set to be equal to zero and that for each city $i \in V$ with at least one legions within, corresponding value z_i is set to be equal to zero. Finally, decision variables x are set to be continuous, while y and z are set to be binary by the constraints (3.6).

A given MILP formulation consists of $2|V|$ variables which are binary and $|V|$ continuous variables. Number of constraints is equal to $4|V|$.

A proof of the validity of the MILP formulation for the RRDP is given in the next proposition.

Proposition 1. *The optimal objective function value $F_1(f)$ of the Restrained Roman domination problem (2.1) - (2.4) is equal to the optimal objective function value $F_2(x, y, z)$ of the MILP formulation (3.1) - (3.6).*

Proof. ($\Rightarrow$) In this part will be proven that the optimal objective function value of the Restrained Roman domination problem (2.1) - (2.4) is greater or equal to the optimal objective function value of the MILP formulation (3.1) - (3.6), i.e. $F_1(f) \geq F_2(x, y, z)$.

For a fixed city $i \in V$ and a function f given by (2.1), let decision variables x_i, y_i and z_i be defined as

$$x_i = f(i), \qquad y_i = \begin{cases} 1, & f(i) = 2 \\ 0, & otherwise \end{cases} \qquad \text{and} \qquad z_i = \begin{cases} 1, & f(i) = 0 \\ 0, & otherwise \end{cases}.$$

Since $x_i \in [0, +\infty)$, $y_i, z_i \in \{0, 1\}$, conditions (3.4) - (3.6) are satisfied by the definition. For example, a condition (3.5) is satisfied because $z_i = 1$ for $x_i = f(i) = 0$ ($x_i + 2z_i = 2$) and $z_i = 0$ for

$x_i = f(i) = 1$ ($x_i + 2z_i = 1 < 2$). Similarly $z_i = 0$ for $x_i = f(i) = 2$, which again implies that $x_i + 2z_i = 2$.

Assuming that conditions (3.2) and (3.3) holds for a fixed vertex $i \in V$, there are two cases:
Case 1. Let values $f(i)$ are set to be greater or equal to one. Since $x_i = f(i)$, relation $x_i \geq 1$ implies. From the last relation and by the binary notations of the variables y_i and z_i it implies that $x_i + \sum_{j \in N_i} y_j \geq 1$ and $x_i + \sum_{j \in N_i} z_j \geq 1$.
Case 2. Let values $f(i)$ are set to be equal to zero. Satisfying relation (2.4) $(\exists v, w \in N_u)(f(v) = 2 \wedge f(w) = 0)$, it follows that $y_v = 1$ and $z_w = 1$. Therefore, $x_i + \sum_{j \in N_i} y_j = \sum_{j \in N_i} y_j \geq 1$ and $x_i + \sum_{j \in N_i} z_j = \sum_{j \in N_i} z_j \geq 1$.

Finally, since decision variables satisfies the conditions (3.1) - (3.6) for a fixed vertex i, it follows that $F_2(x, y, z) = \sum_{i \in V} x_i = \sum_{i \in V} f(i) = F_1(f)$.

($\Leftarrow$) In this part it will be proven that optimal objective function value of the Restrained Roman domination problem (2.1) - (2.4) is less or equal to the optimal objective function value of the MILP formulation (3.1) - (3.6), i.e. $F_1(f) \leq F_2(x, y, z)$.

For a given set of decision variables x_i, y_i and z_i which satisfy conditions (3.1) - (3.6), let a function f be defined as

$$f(i) = \begin{cases} 0, & x_i \in [0, 1) \\ 1, & x_i \in [1, 2) \\ 2, & x_i \in [2, +\infty) \end{cases} . \tag{3.7}$$

By the definition of the function f, condition (2.1) holds. Since the condition (2.1) holds, for a fixed vertex $u \in V$ there are two cases:
Case 1. Let $x_u \in [1, +\infty)$. By the definition of the function f, it follows that $f(u) = 1$ or $f(u) = 2$. Now, condition (2.4) holds, since $\perp \Rightarrow p$ is tautology for any logical statement p.
Case 2. Let $x_u \in [0, 1)$. By the definition of the function f, $f(u) = 0$. Because of the condition (3.2), $x_u + \sum_{j \in N_u} y_j \geq 1$, it follows that $\sum_{j \in N_u} y_j \geq 1 - x_u > 0$. Since the decision variables y_j are binary, $\sum_{j \in N_u} y_j$ has to be integer, which implies that $\sum_{j \in N_u} y_j \geq 1$. Therefore, there exists a vertex $v \in N_u$, $y_v = 1$. From the constraints (3.4), and because of the $x_v \geq 2y_v = 2$, it follows that $f(v) = 2$. Similarly, from the constraints (3.3) it follows that $\sum_{j \in N_u} z_j \geq 1 - x_u > 0$. Because of the binary type of the decision variables z_j, $\sum_{j \in N_u} z_j$ has an integer value. Now, since $\sum_{j \in N_u} z_j \geq 1$, there exists a vertex $w \in N_u$ such that $z_w = 1$. Finally, by the constraints (3.5), $x_w \leq 2 - 2z_w = 0$, it follows that $x_w = 0$ and that $f(w) = 0$ which means that condition (2.4) holds also.

By the definition of the function f, it is clear that $f(i) \leq x_i$, for $i \in V$. Therefore, $F_1(f) = \sum_{i \in V} f(i) \leq \sum_{i \in V} x_i = F_2(x, y, z)$.

So, for each feasible solution to the problem (2.1) - (2.4) there exists a feasible solution to the problem (3.1) - (3.6), satisfying the relation $F_2(x, y, z) \leq F_1(f)$, and for each feasible solution to the (3.1) - (3.6) there exists a feasible solution to the (2.1) - (2.4) satisfying the relation $F_1(f) \leq$

$F_2(x, y, z)$. Therefore, it follows that $\min_f F_1(f) = \min_{(x,y,z)} F_2(x, y, z)$. □

Applying the given MILP formulation to the illustrated RRDP, solution $\min_{(x,y,z)} F_2(x, y, z)$ to the proposed problem is equal to 3, and it can be obtained for $x = [1, 0, 2, 0, 0]$, $y = [0, 1, 0, 0, 0]$ and $z = [0, 0, 1, 1, 1]$.

4. Conclusions

This paper is devoted to the Restrained Roman domination problem. A mixed integer linear programming formulation is introduced and the correctness of the corresponding formulation is proved. The presented model uses relatively small number of the variables and constraints, which indicates that presented model can be used both in theoretical and practical considerations. As a future study, it is planned to construct an exact method for solving the corresponding mathematical model. Construction of the metaheuristics for solving the proposed problem can also be a part of a possible future study.

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New Čebyšev Type Inequalities for Functions whose Second Derivatives are (s_1, m_1)-(s_2, m_2)-convex on the Co-ordinates

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Abstract

In this paper, we establish some new Čebyšev type inequalities for functions whose second derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

Keywords: Čebyšev type inequalities, co-ordinates (s_1, m_1)-(s_2, m_2)-convex, integral inequality.
2010 MSC: 26D15, 26D20, 39A12.

1. Introduction

In 1882, Čebyšev (Chebyshev, 1882) gave the following inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1.1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and $g'$are bounded, where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|$.

During the past few years, many researchers established various generalizations, extensions and variants of Čebyšev type inequalities, we can mention the works (Ahmad *et al.*, 2009; Boukerrioua & Guezane-Lakoud, 2007; Guazene-Lakoud & Aissaoui, 2011; Latif & Alomari, 2009;

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Pachpatte & Talkies, 2006; Pachpatte, 2006; Sarikaya *et al.*, 2014). Recently the authors of (Guazene-Lakoud & Aissaoui, 2011), established a new Čebyšev type inequality for functions of two independent variables whose second derivatives are bounded. Also in (Sarikaya *et al.*, 2014), the authors obtained some new Čebyšev type inequalities involving functions whose mixed partial derivatives are s-convex on the co-ordinates. The main purpose of this work is to obtain new Čebyšev type inequalities for functions whose mixed partial derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

This paper is organized as follows: In section 2, we present some preliminaries. In the third section, we prove a new identity for functions of two independent variables then we used it to establish new Čebyšev type inequalities for functions whose mixed partial derivatives are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates.

2. Preliminaries

Throughout this paper we denote by Δ the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b]\times[c,d]$ with $a < b$ and $c < d$, $\Delta_0 =: [0, b^*] \times [0, d^*]$ with $b^* > b$, $d^* > d$, $k =: (b-a)(d-c)$ and $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$ by $f_{\lambda\alpha}$.

Definition 2.1. (Dragomir, 2001) A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda\alpha f(x,y) + \lambda(1-\alpha) f(x,v) \\ & + (1-\lambda)\alpha f(t,y) + (1-\lambda)(1-\alpha) f(t,v), \end{aligned} \tag{2.1}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, it exists a co-ordinated convex function which is not convex.

Definition 2.2. (Alomari & Darus, 2008) A function $f : \Delta \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ, if the following inequality:

$$\begin{aligned} f(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)v) \leq\ & \lambda^s\alpha^s f(x,y) + \lambda^s(1-\alpha)^s f(x,v) \\ & + (1-\lambda)^s\alpha^s f(t,y) + (1-\lambda)^s(1-\alpha)^s f(t,v), \end{aligned} \tag{2.2}$$

holds for all $\lambda, \alpha \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$, for some fixed $s \in (0,1]$.

s-convexity on the co-ordinates does not imply the s-convexity, it exist a functions which are s-convex on the co-ordinates but are not s-convex.

Definition 2.3. (Bai & Qi, 2013; Chun, 2014) A function $f : \Delta_0 \to \mathbb{R}$ is said (s,m)-convex on Δ, if the following inequality

$$f(\lambda x + m(1-\lambda)t, \lambda y + m(1-\lambda)v) \leq \lambda^s f(x,y) + m(1-\lambda^s) f(t,v), \tag{2.3}$$

holds for all $(x,y), (t,v) \in \Delta$ and $\lambda \in [0,1]$ and for some fixed $s, m \in (0,1]$.

Definition 2.4. (Bai & Qi, 2013; Chun, 2014) A function $f : \Delta_0 \to \mathbb{R}$ is said to be (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates on Δ_0, if the following inequality

$$\begin{aligned} f(\lambda x + m_1(1-\lambda)t, \alpha y + m_2(1-\alpha)v) \leq\ & \lambda^{s_1}\alpha^{s_2} f(x,y) + m_2\lambda^{s_1}(1-\alpha^{s_2}) f(x,v) \\ & + m_1(1-\lambda^{s_1})\alpha^{s_2} f(t,y) \\ & + m_1 m_2 (1-\lambda^{s_1})(1-\alpha^{s_2}) f(t,v), \end{aligned} \tag{2.4}$$

holds for all $(x,y), (x,v), (t,y), (t,v) \in \Delta$ with $\lambda, \alpha \in [0,1]$ and $s_1, m_1, s_2, m_2 \in (0,1]$.

3. Main result

Lemma 3.1. *Let $f : \Delta \to \mathbb{R}$ be partially differentiable function on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha} \in L_1(\Delta)$, then for any $(x,y) \in \Delta \subset \Delta_0$, we have the following identity*

$$\begin{aligned} f(x,y) =\ & \frac{1}{(b-a)} \int_a^b f(m_1 t, y) dt + \frac{1}{(d-c)} \int_c^d f(x, m_2 z) dz \\ & - \frac{1}{k} \int_a^b \int_c^d f(m_1 t, m_2 z) dz dt + \frac{1}{k} \int_a^b \int_c^d (x - m_1 t)(y - m_2 z) \\ & \times \left(\int_0^1 \int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z) \, d\alpha d\lambda \right) dz dt, \end{aligned} \tag{3.1}$$

where $k = (b-a)(d-c)$.

Proof. For any $x, t \in [m_1 a, m_1 b]$ and $y, z \in [m_2 c, m_2 d]$ such that $t \neq x, y \neq z$, we have

$$\begin{aligned} \int_{m_1 t}^{x} \int_{m_2 z}^{y} f_{\sigma\tau}(\sigma, \tau) \, d\tau d\sigma &= \int_{m_1 t}^{x} (f_\sigma(\sigma, y) - f_\sigma(\sigma, m_2 z)) \, d\sigma \\ &= f(x,y) - f(x, m_2 z) - f(m_1 t, y) + f(m_1 t, m_2 z), \end{aligned} \tag{3.2}$$

which implies

$$f(x,y) = f(x, m_2 z) + f(m_1 t, y) - f(m_1 t, m_2 z) + \int_{m_1 t}^{x} \int_{m_2 z}^{y} f_{\sigma\tau}(\sigma, \tau) \, d\tau d\sigma. \tag{3.3}$$

For $\sigma = \lambda x + m_1(1-\lambda)t$ and $\tau = \alpha y - m_2(1-\alpha)z$, (3.3) becomes

$$\begin{aligned} f(x,y) =\ & f(x, m_2 z) + f(m_1 t, y) - f(m_1 t, m_2 z) \\ & + (x - m_1 t)(y - m_2 z) \int_0^1 \int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z) \, d\tau d\sigma. \end{aligned} \tag{3.4}$$

Integrating (3.4) over $[a,b] \times [c,d] \subset \Delta_0$, with respect to t, z, multiplying the resultant equality by $\frac{1}{k}$, we obtain the desired equality. □

Theorem 3.1. *Let $f, g : \Delta_0 \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ_0, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, m_1)-(s_2, m_2)-convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{(1+m_1s_1)(1+m_2s_2)}{8(m_1m_2k)^2(1+s_1)(1+s_2)} \int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} \left[M|g(x,y)| + N|f(x,y)|\right] \times \left[(x-m_1a)^2 + (m_1b-x)^2\right]\left[(y-m_2c)^2 + (m_2d-y)^2\right] dydx, \quad (3.5)$$

where

$$\begin{aligned} T(f,g) &= \frac{1}{m_1m_2k}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)g(x,y)dydx - \frac{(d-c)}{m_1^2m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_{m_1a}^{m_1b} f(t,y)dt\right)dydx \\ &- \frac{(b-a)}{m_1m_2^2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_{m_2c}^{m_2d} f(x,z)dz\right)dydx \\ &+ \frac{1}{m_1^2m_2^2k^2}\left(\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)dydx\right)\left(\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(t,z)dzdt\right), \end{aligned} \quad (3.6)$$

$M = \underset{x,t\in[a,b],y,z\in[c,d]}{ess\sup} \left[|f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,z)| + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,z)|\right],$
$N = \underset{x,t\in[a,b],y,z\in[c,d]}{ess\sup} \left[|g_{\lambda\alpha}(x,y)| + |g_{\lambda\alpha}(x,z)| + |g_{\lambda\alpha}(t,y)| + |g_{\lambda\alpha}(t,z)|\right],$
$(s_1, m_1), (s_2, m_2) \in (0,1]^2$ *and* $k = (b-a)(d-c)$.

Proof. By Lemma 3.1, we have

$$f(x,y) - \frac{1}{(b-a)}\int_a^b f(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d f(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d f(m_1t,m_2z)dzdt$$
$$= \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt, \quad (3.7)$$

and

$$g(x,y) - \frac{1}{(b-a)}\int_a^b g(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d g(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d g(m_1t,m_2z)dzdt$$
$$= \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x + m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt. \quad (3.8)$$

Multiplying (3.7) by $\frac{1}{2m_1m_2k}\, g(x,y)$ and (3.8) by $\frac{1}{2m_1m_2k} f(x,y)$, summing the resultant equalities, then integrating on $[m_1a, m_1b] \times [m_2c, m_2d]$ with respect to x, y, we get

$$\begin{aligned}
&\frac{1}{m_1m_2k}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)f(x,y)dydx - \frac{(d-c)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y) \\
&\times\left(\int_a^b f(m_1t,y)dt\right)dydx - \frac{(b-a)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_c^d f(x,m_2z)dz\right)dydx \\
&-\frac{(d-c)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_a^b g(m_1t,y)dt\right)dydx \\
&-\frac{(b-a)}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_c^d g(x,m_2z)dz\right)dydx \\
&+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left(\int_a^b\int_c^d f(m_1t,m_2z)dzdt\right)dydx \\
&+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left(\int_a^b\int_c^d g(m_1t,m_2z)dzdt\right)dydx \\
= \;&\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left[\int_a^b\int_c^d (x-m_1t)(y-m_2z)\right. \\
&\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\right]dydx \\
&+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left[\int_a^b\int_c^d (x-m_1t)(y-m_2z)\right. \\
&\times\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}\left(\lambda x+m_1(1-\lambda)t, \alpha y - m_2(1-\alpha)z\right)d\alpha d\lambda\right)dzdt\right]dydx. \qquad (3.9)
\end{aligned}$$

By Fubini's Theorem, we obtain

$$\begin{aligned} T(f,g) &= \frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} g(x,y)\left[\int_a^b\int_c^d (x-m_1t)(y-m_2z)\right. \\ &\times\left.\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt\right]dydx \\ &+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} f(x,y)\left[\int_a^b\int_c^d (x-m_1t)(y-m_2z)\right. \\ &\times\left.\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt\right]dydx. \end{aligned} \tag{3.10}$$

Using the (s_1,m_1)-(s_2,m_2)-convexity and modulus, (3.10) gives

$$\begin{aligned} |T(f,g)| &\le \frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |g(x,y)|\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\right. \\ &\times\left[\int_0^1\int_0^1\left(\lambda^{s_1}\alpha^{s_2}\left|f_{\lambda\alpha}(x,y)\right|+m_2\lambda^{s_1}(1-\alpha^{s_2})\left|f_{\lambda\alpha}(x,z)\right|+m_1(1-\lambda^{s_1})\alpha^{s_2}\left|f_{\lambda\alpha}(t,y)\right|\right.\right. \\ &\left.\left.\left.+m_1m_2(1-\lambda^{s_1})(1-\alpha^{s_2})\left|f_{\lambda\alpha}(t,z)\right|\right)d\alpha d\lambda\right]dzdt\right]dydx \\ &+\frac{1}{2m_1m_2k^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |f(x,y)|\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\right. \\ &\times\left[\int_0^1\int_0^1\left(\lambda^{s_1}\alpha^{s_2}\left|g_{\lambda\alpha}(x,y)\right|+m_2\lambda^{s_1}(1-\alpha^{s_2})\left|g_{\lambda\alpha}(x,z)\right|+m_1(1-\lambda^{s_1})\alpha^{s_2}\left|g_{\lambda\alpha}(t,y)\right|\right.\right. \\ &\left.\left.\left.+m_1m_2(1-\lambda^{s_1})(1-\alpha^{s_2})\left|g_{\lambda\alpha}(t,z)\right|\right)d\alpha d\lambda\right]dzdt\right]dydx. \end{aligned} \tag{3.11}$$

By a simple calculation, we have

$$\begin{aligned} |T(f,g)| &\le \frac{(1+m_1s_1)(1+m_2s_2)M}{2m_1m_2k^2(1+s_1)(1+s_2)}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |g(x,y)|\times\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\,dzdt\right]dydx \\ &+\frac{(1+m_1s_1)(1+m_2s_2)N}{2m_1m_2k^2(1+s_1)(1+s_2)}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d} |f(x,y)|\times\left[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z|\,dzdt\right]dydx. \end{aligned} \tag{3.12}$$

Noting that

$$\int_a^b |x-m_1t|\ dt = \frac{1}{2m_1}\left[(x-m_1a)^2+(m_1b-x)^2\right], \tag{3.13}$$

$$\int_c^d |y-m_2z|\,dz = \frac{1}{2m_2}\left[(y-m_2c)^2+(m_2d-y)^2\right]. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we obtain the required inequality. □

Corollary 3.1. *Let $f,g:\Delta_0\ \rightarrow\mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$, are integrable on Δ_0. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1,s_2)-convex on the co-ordinates, then we have*

$$|T(f,g)|\leq\frac{1}{8k^2}\int_a^b\int_c^d[M\,|g(x,y)|+N\,|f(x,y)|]\left[(x-a)^2+(b-x)^2\right]\times\left[(y-c)^2+(d-y)^2\right]dydx, \tag{3.15}$$

where

$$\begin{aligned}T(f,g) &= \frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx-\frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)dt\right)dydx\\ &-\frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,z)dz\right)dydx\\ &+\frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)dydx\right)\left(\int_a^b\int_c^d g(t,z)dzdt\right),\end{aligned} \tag{3.16}$$

M, N, k are defined as in Theorem 3.1 and $(s_1,s_2)\in(0,1]^2$.

Proof. Applying Theorem 3.1, for $m_1=m_2=1$, we obtain the desired inequality. □

Corollary 3.2. *Under the same hypothesis of Theorem 3.1, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are m_1-m_2-convex on the co-ordinates, then we have the following inequality*

$$\begin{aligned}|T(f,g)| &\leq \frac{(1+m_1)(1+m_2)}{32(m_1m_2k)^2}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d}[M\,|g(x,y)|+N\,|f(x,y)|]\\ &\times\left[(x-m_1a)^2+(m_1b-x)^2\right]\left[(y-m_2c)^2+(m_2d-y)^2\right]dydx,\end{aligned} \tag{3.17}$$

where $T(f,g)$, M, N, k are defined as in Theorem 3.1 and $(m_1,m_2)\in(0,1]^2$.

Proof. Using Theorem 3.1, for $s_1=s_2=1$, we obtain the desired inequality. □

Theorem 3.2. *Under the same hypothesis of Theorem 3.1, we have the following inequality*

$$|T(f,g)| \leq \frac{49}{3600}\left[\frac{(1+m_1s_1)(1+m_2s_2)}{(1+s_1)(1+s_2)}\right]^2 MNk^2m_1^2m_2^2\,, \tag{3.18}$$

where $T(f,g)$, M, N, $(s_1,m_1),(s_2,m_2)$ and k are defined as in Theorem 3.1.

Proof. Let $F, G, \widetilde{F}$ and $\widetilde{G}$ be defined as follows

$$F = f(x,y) - \frac{1}{(b-a)}\int_a^b f(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d f(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d f(m_1t,m_2z)dzdt,$$

$$G = g(x,y) - \frac{1}{(b-a)}\int_a^b g(m_1t,y)dt - \frac{1}{(d-c)}\int_c^d g(x,m_2z)dz + \frac{1}{k}\int_a^b\int_c^d g(m_1t,m_2z)dzdt,$$

$$\widetilde{F} = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt,$$

$$\widetilde{G} = \frac{1}{k}\int_a^b\int_c^d (x-m_1t)(y-m_2z)\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\,d\alpha d\lambda\right)dzdt.$$

By Lemma 3.1, we have

$$F = \widetilde{F} \text{ and } G = \widetilde{G}, \tag{3.19}$$

which implies

$$F\times G = \widetilde{F}\times\widetilde{G}. \tag{3.20}$$

Integrating both sides of (3.20) over $[m_1a,m_1b]\times[m_2c,m_2d]$ with respect to x,y, multiplying the resultant equality by $\frac{1}{m_1m_2k}$, using Fubini's Theorem and modulus, we get

$$\begin{aligned}|T(f,g)| \leq\; & \frac{1}{m_1m_2k^3}\int_{m_1a}^{m_1b}\int_{m_2c}^{m_2d}\Bigg[\int_a^b\int_c^d |x-m_1t|\,|y-m_2z| \\ & \times\left(\int_0^1\int_0^1 \left|f_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\right| d\alpha d\lambda\right)dzdt\times\int_a^b\int_c^d |x-m_1t|\,|y-m_2z| \\ & \times\left(\int_0^1\int_0^1 \left|g_{\lambda\alpha}(\lambda x+m_1(1-\lambda)t,\alpha y-m_2(1-\alpha)z)\right| d\alpha d\lambda\right)dzdt\Bigg]dydx.\end{aligned} \tag{3.21}$$

Using the (s_1, m_1)-(s_2, m_2)-convexity, we obtain

$$\begin{aligned}
|T(f,g)| \leq\ & \frac{1}{m_1 m_2 k^3} \int_{m_1 a}^{m_1 b}\int_{m_2 c}^{m_2 d} \left[\int_a^b \int_c^d |x - m_1 t| \, |y - m_2 z| \left[\int_0^1 \int_0^1 \left[\lambda^{s_1} \alpha^{s_2} \left| f_{\lambda\alpha}(x, y) \right| \right.\right.\right. \\
& + m_2 \lambda^{s_1} (1 - \alpha^{s_2}) \left| f_{\lambda\alpha}(x, z) \right| + m_1 (1 - \lambda^{s_1}) \alpha^{s_2} \left| f_{\lambda\alpha}(t, y) \right| \\
& \left.\left.\left. + m_1 m_2 (1 - \lambda^{s_1})(1 - \alpha^{s_2}) \left| f_{\lambda\alpha}(t, z) \right| \right] d\alpha d\lambda \right] dz dt \right] \\
& \times \left[\int_a^b \int_c^d |x - m_1 t| \, |y - m_2 z| \left[\int_0^1 \int_0^1 \left[\lambda^{s_1} \alpha^{s_2} \left| g_{\lambda\alpha}(x, y) \right| \right.\right.\right. \\
& + m_2 \lambda^{s_1} (1 - \alpha^{s_2}) \left| g_{\lambda\alpha}(x, z) \right| + m_1 (1 - \lambda^{s_1}) \alpha^{s_2} \left| g_{\lambda\alpha}(t, y) \right| \\
& \left.\left.\left. + m_1 m_2 (1 - \lambda^{s_1})(1 - \alpha^{s_2}) \left| g_{\lambda\alpha}(t, z) \right| \right] d\alpha d\lambda \right] dz dt \right] dy dx \\
\leq\ & \left[\frac{(1 + m_1 s_1)(1 + m_2 s_2)}{(1 + s_1)(1 + s_2)} \right]^2 \frac{M \times N}{m_1 m_2 k^3} \\
& \times \left[\int_{m_1 a}^{m_1 b} \left[\int_a^b |x - m_1 t| \, dt \right]^2 dx \right] \left[\int_{m_2 c}^{m_2 d} \left[\int_c^d |y - m_2 z| \, dz \right]^2 dy \right]. \qquad (3.22)
\end{aligned}$$

Taking into account that

$$\left[\int_{m_1 a}^{m_1 b} \left[\int_a^b |x - m_1 t| \, dt \right]^2 dx \right] = \frac{7}{60} m_1^3 (b - a)^5 \qquad (3.23)$$

and

$$\left[\int_{m_2 c}^{m_2 d} \left[\int_c^d |y - m_2 z| \, dz \right]^2 dy \right] = \frac{7}{60} m_2^3 (d - c)^5 . \qquad (3.24)$$

The desired inequality, will be obtained by combining (3.22), (3.23) and (3.24). □

Corollary 3.3. *Let $f, g : \Delta_0 \rightarrow \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$, are integrable on Δ_0. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are (s_1, s_2)-convex on the co-ordinates, then we have*

$$|T(f,g)| \leq \frac{49}{3600} M \times N \times k^2, \qquad (3.25)$$

where $T(f, g)$, M, N and k are defined as in Theorem 3.1.

Proof. Applying Theorem 3.2, for $m_1 = m_2 = 1$, we obtain the desired inequality. □

Corollary 3.4. *Under the same hypothesis of Theorem 3.1, if $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are m_1-m_2-convex on the co-ordinates, we have the following inequality*

$$|T(f,g)| \leq \frac{49}{57600} [(1 + m_1)(1 + m_2)]^2 M \times N \times k^2 m_1^2 \times m_2^2 , \qquad (3.26)$$

where $T(f,g)$, M, N and k are defined as in Theorem 3.1 and $m_1, m_2 \in (0,1]$.

Proof. Applying Theorem 3.2, for $s_1 = s_2 = 1$, we obtain the desired inequality. □

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Hadamard Product of Certain Harmonic Univalent Meromorphic Functions

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Abstract

In this paper the authors extended certain results concerning the Hadamard product for two classes related to starlike and convex harmonic univalent meromorphic functions, results for each class are obtained. Relevant connections of the results obtained here with various known results for meromorphic univalent functions are indicated.

Keywords: Harmonic functions, meromorphic functions, univalent functions, sense-preserving.
2010 MSC: 30C45, 30C50.

1. Introduction and definitions

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic of f. Also, a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$. There are numerous papers on univalent harmonic functions defined in a domain $U = \{z \in \mathbb{C} : |z| < 1\}$ (see (Jahangiri, 1998, 1999), and (Silverman & Silvia, 1999; Silverman, 1998)). Hengartner and Schober (Henartner & Schober, 1987) investigated functions harmonic in the exterior of the unit disc, that is $U^* = \{z \in \mathbb{C} : |z| > 1\}$. They showed that a complex valued, harmonic, sense-preserving univalent function f, defined on U^* and satisfying $f(\infty) = \infty$ must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \quad (A \in \mathbb{C}), \tag{1.1}$$

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where

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n} \ \ (0 \le |\beta| < |\alpha|), \tag{1.2}$$

and $a = \overline{f_{\bar{z}}}/f_z$ is analytic and satisfy $|a(z)| < 1$ for $z \in U^*$.

After this work, Jahangiri and Silverman (Jahangiri & Silverman, 1999) defined the class H_0^* of harmonic sense-preserving functions $f(z)$ that are starlike with respect to the origin in U^* given by (1.1) and (1.2) and proved that

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) < |\alpha| - |\beta| - |A| \,.$$

Denote by Σ_H the class of meromophic functions f that are harmonic univalent and sense-preserving in the exterior of open unit disc U in the form

$$f(z) = h(z) + \overline{g(z)} \tag{1.3}$$

where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \ g(z) = \sum_{n=1}^{\infty} b_n z^{-n}. \tag{1.4}$$

Also, Jahangiri (Jahangiri, 2002) proved that if $f(z)$ given by (1.3) and (1.4) belongs to $\Sigma_H^*(\gamma)$, then

$$\sum_{n=1}^{\infty} \left(\frac{n+\gamma}{1-\gamma} |a_n| + \frac{n-\gamma}{1-\gamma} |b_n| \right) < 1.$$

Several authors have studies the classes of harmonic univalent meromorphic functions (see (Ahuja & Jahangiri, 2002; El-Ashwah *et al.*, 2014) and (Janteng & Halim, 2007)).

Now, we introduce the subclasses $\Sigma_H^*(c_n, d_n, \delta)$, $\Sigma_H^c(c_n, d_n, \delta)$ and $\Sigma_H^k(c_n, d_n, \delta)$ consisting of functions of the form (1.3) and (1.4) which satisfies the inequalities, respectively

$$\sum_{n=1}^{\infty} (c_n |a_n| + d_n |b_n|) < \delta \ \ (c_n \ge d_n \ge c_2 > 0; \delta > 0), \tag{1.5}$$

$$\sum_{n=1}^{\infty} n\,(c_n |a_n| + d_n |b_n|) < \delta \ \ (c_n \ge d_n \ge c_2 > 0; \delta > 0), \tag{1.6}$$

and

$$\sum_{n=1}^{\infty} n^k\,(c_n |a_n| + d_n |b_n|) < \delta \ \ (c_n \ge d_n \ge c_2 > 0; \delta > 0). \tag{1.7}$$

It is easy to see that various subclasses of Σ_H consisting of functions $f(z)$ of the form (1.3) and (1.4) can be represented as $\Sigma_H^k(c_n, d_n, \delta)$ for suitable choices of c_n, d_n, δ *and* k studies earlier by various authors.

(i) $\Sigma_H^0(n, n, 1) = H_0^*$ (see Jahangiri and Silverman. ((Jahangiri & Silverman, 1999), with $\alpha = 1$ and $\beta = A = 0$));

(ii) $\Sigma_H^0(n+\gamma, n-\gamma, 1-\gamma) = \Sigma_H^*(\gamma)(0 \leq \gamma < 1, n \geq 1)$ (see Jahangiri (Jahangiri, 2002));

(iii)$\Sigma_H^0(n(n+2)^m, n(n-2)^m, 1) = MH^*(m)(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}, n \geq 1)$ (see Bostanci and Ozturk (Bostanci & Ozturk, 2010));

(vi) $\Sigma_H^0((n+\gamma)(n+2)^m, (n-\gamma)(n-2)^m, 1-\gamma) = MH^*(m, \gamma)(0 \leq \gamma < 1, m \in \mathbb{N}_0, n \geq 1)$ (see Bostanci and Ozturk (Bostanci & Ozturk, 2011)).

Evidently, $\Sigma_H^0(c_n, d_n, \delta) = \Sigma_H^*(c_n, d_n, \delta)$, and $\Sigma_H^1(c_n, d_n, \delta) = \Sigma_H^c(c_n, d_n, \delta)$. Further $\Sigma_H^{k_1}(c_n, d_n, \delta) \subset \Sigma_H^{k_2}(c_n, d_n, \delta)$ if $k_1 > k_2 \geq 0$, the containment being proper. Moreover, for any positive integer k, we have the following inclusion relation

$\Sigma_H^k(c_n, d_n, \delta) \subset \Sigma_H^{k-1}(c_n, d_n, \delta) \subset ... \subset \Sigma_H^2(c_n, d_n, \delta) \subset \Sigma_H^c(c_n, d_n, \delta) \subset \Sigma_H^*(c_n, d_n, \delta)$.

We also note that for any nonnegative real number k, the class $\Sigma_H^k(c_n, d_n, \delta)$ is nonempty as the function

$$f(z) = z + \sum_{n=1}^{\infty} n^{-k} \frac{\delta}{c_n} \lambda_n z^{-n} + \sum_{n=1}^{\infty} n^{-k} \frac{\delta}{d_n} \lambda_n \overline{z^{-n}}$$

where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n \leq 1$, satisfy the inequality (1.7).

For harmonic meromorphic functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} |a_n| z^{-n} + \sum_{n=1}^{\infty} |b_n| \overline{z^{-n}}$$

and

$$G(z) = z + \sum_{n=1}^{\infty} A_n z^{-n} + \sum_{n=1}^{\infty} B_n \overline{z^{-n}} \quad (A_n, B_n \geq 0),$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z)$$

$$= z + \sum_{n=1}^{\infty} |a_n| A_n z^{-n} + \sum_{n=1}^{\infty} |b_n| B_n \overline{z^{-n}},$$

and similarly, we can define the convolution of more than two meromorphic functions.

Several authors such as Mogra (Mogra, 1994, 1991), Aouf and Darwish (Aouf & Darwish, 2006), El-Ashwah and Aouf (El-Ashwah & Aouf, 2009, 2011) studied the convolution properties of meromorphic functions only.

The object of this paper is to establish a results concerning the Hadamard product of functions in the classes $\Sigma_H^k(c_n, d_n, \delta), \Sigma_H^c(c_n, d_n, \delta)$ and $\Sigma_H^*(c_n, d_n, \delta)$.

Throughout this paper, we assume $f(z)$, $g(z)$, $f_i(z)$, and $g_j(z)$ having following form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} \overline{b_n z^{-n}}, \tag{1.8}$$

$$g(z) = z + \sum_{n=1}^{\infty} u_n z^{-n} + \sum_{n=1}^{\infty} \overline{v_n z^{-n}}, \tag{1.9}$$

$$f_i(z) = z + \sum_{n=1}^{\infty} a_{n,i} z^{-n} + \sum_{n=1}^{\infty} \overline{b_{n,i} z^{-n}} \quad (i = 1, 2, ..., s), \tag{1.10}$$

$$g_j(z) = z + \sum_{n=1}^{\infty} u_{n,j} z^{-n} + \sum_{n=1}^{\infty} \overline{v_{n,j} z^{-n}} \quad (j = 1, 2, ..., q). \tag{1.11}$$

Since to a certain extent the work in the harmonic univalent meromorphic functions case has paralleled that of the harmonic analytic univalent case, one is tempted to search analogous results to those of Porwal and Dixt (Porwal & Dixit, 2015) for meromorphic harmonic univalent functions in U^*.

2. Main Results

Theorem 1. *Let the functions* $f_i(z)$ *defined by (1.10) belong to the class* $\Sigma_H^c(c_n, d_n, \delta)$ *for every* $i = 1, 2, ..., s$; *and let the functions* $g_j(z)$ *defined by (1.11) belong to the class* $\Sigma_H^*(c_n, d_n, \delta)$ *for every* $j = 1, 2, ..., q$. *If* $c_n, d_n \geq n\delta, (n \geq 1)$, *then the Hadamard product* $f_1 * f_2 * .... * f_m * g_1 * g_2 * .... * g_q(z)$ *belongs to the class* $\Sigma_H^{2s+q-1}(c_n, d_n, \delta)$.

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n \left(\prod_{i=1}^{s} |a_{n,i}| \prod_{j=1}^{q} |u_{n,j}| \right) + d_n \left(\prod_{i=1}^{s} |b_{n,i}| \prod_{j=1}^{q} |v_{n,j}| \right) \right] \leq \delta$$

Since $f_i(z) \in \Sigma_H^c(c_n, d_n, \delta)$, we have

$$\sum_{n=1}^{\infty} n \left(c_n |a_{n,i}| + d_n |b_{n,i}| \right) \leq \delta, \tag{2.1}$$

for every $i = 1, 2, ..., s$, and therefore,

$$nc_n |a_{n,i}| \leq \delta \quad or \quad |a_{n,i}| \leq \left(\frac{\delta}{nc_n} \right)$$

and hence

$$|a_{n,i}| \leq n^{-2}, \tag{2.2}$$

for every $i = 1, 2, ..., s$. Also,

$$nd_n |b_{n,i}| \leq \delta \quad or \quad |b_{n,i}| \leq \left(\frac{\delta}{nc_n} \right)$$

and hence

$$|b_{n,i}| \leq n^{-2}, \tag{2.3}$$

for every $i = 1, 2, ..., s$. Further, since $g_j(z) \in \Sigma_H^*(c_n, d_n, \delta)$, we have

$$\sum_{n=1}^{\infty} \left(c_n \left| u_{n,j} \right| + d_n \left| v_{n,j} \right| \right) \leq \delta, \tag{2.4}$$

for every $j = 1, 2, ..., q$. Hence we obtain

$$\left| u_{n,j} \right| \leq n^{-1} \; and \; \left| v_{n,j} \right| \leq n^{-1} \tag{2.5}$$

for every $j = 1, 2, ..., q$.

Using (2.2) and (2.3) for $i = 1, 2, ..., s$; (2.5) for $j = 1, 2, ..., q-1$ and (2.4) for $j = q$, we have

$$\sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n \left(\prod_{i=1}^{s} \left| a_{n,i} \right| \prod_{j=1}^{q-1} \left| u_{n,j} \right| \right) \left| u_{n,q} \right| + d_n \left(\prod_{i=1}^{s} \left| b_{n,i} \right| \prod_{j=1}^{q-1} \left| v_{n,j} \right| \right) \left| v_{n,q} \right| \right]$$

$$\leq \sum_{n=1}^{\infty} n^{2s+q-1} \left[c_n n^{-2s} n^{-(q-1)} \left| u_{n,q} \right| + d_n n^{-2s} n^{-(q-1)} \left| v_{n,q} \right| \right]$$

$$= \sum_{n=1}^{\infty} c_n \left| u_{n,q} \right| + d_n \left| v_{n,q} \right| \leq \delta.$$

Hence $f_1 * f_2 * .... * f_m * g_1 * g_2 * .... * g_q \in \Sigma_H^{2s+q-1}(c_n, d_n, \delta)$.

We note that the required estimate can also be obtained by using (2.2) and (2.3) for $i = 1, 2, ..., s-1$; (2.5) for $j = 1, 2, ..., q$; and (2.1) for $i = s$. □

Taking into account the convolution of the functions $f_i(z)$ defined by (1.10) for every $i = 1, 2, ..., s$; only in the proof of the above theorem and using (2.2) and (2.3) for $i = 1, 2, ..., s-1$, and the relation (2.1) for $i = s$, we have the following corollary:

Corollary 1. Let the functions $f_i(z)$ defined by (1.10) belong to the class $\Sigma_H^c(c_n, d_n, \delta)$ for every $i = 1, 2, ..., s$. If $c_n, d_n \geq n\delta$ $(n \geq 1)$, then the Hadamard product $f_1 * f_2 * ... * f_s(z)$ belongs to the class $\Sigma_H^{2s-1}(c_n, d_n, \delta)$.

Taking into account the convolution of the functions $g_j(z)$ defined by (1.11) for every $j = 1, 2, ..., q$; only in the proof of the above theorem and using (2.5) for $j = 1, 2, ..., q-1$; and the relation (2.4) for $j = q$, we have the following corollary:

Corollary 2. Let the functions $g_j(z)$ defined by (1.11) belong to the class $\Sigma_H^*(c_n, d_n, \delta)$ for every $j = 1, 2, ..., q$. If $c_n, d_n \geq \delta, (n \geq 1)$, then the Hadamard product $g_1 * g_2 * .... * g_q$ belongs to the class $\Sigma_H^{q-1}(c_n, d_n, \delta)$.

Remarks (i) If the co-analytic parts of $f_i(z)$ and $g_j(z)$ are zero for every $i = 1, 2, ..., s$ and $j = 1, 2, ..., q$, then we obtain the results obtained by El-Ashwah and Aouf (El-Ashwah & Aouf, 2011), with $a_{0,i} = 1, i = 1, 2, ..., s$ and $b_{0,j} = 1, j = 1, 2, ..., q$);

(ii) Taking $c_n = n + 1 + \beta(n + 2\alpha - 1)$ and $\delta = 2\beta(1 - \alpha)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ with the co-analytic parts zero in the above results, we obtain the results obtained by Mogra (Mogra, 1994);

(iii) Taking $c_n = n + \alpha$ and $\delta = 1 - \alpha (0 \leq \alpha < 1)$ with co-analytic parts are zero in the above results, we obtain the result obtained by Mogra (Mogra, 1991);

(iv) Taking $c_n = n^m[(n+1) + \beta(n+2\alpha-1)]$ and $\delta = 2\beta(1-\alpha)(0 \leq \alpha < 1, 0 < \beta \leq 1, m \in \mathbb{N}_0)$ with co-analytic parts are zero in the above results, we obtain the results obtained by El-Ashwah and Aouf (El-Ashwah & Aouf, 2009), with $a_{0,i} = 1, i = 1, 2, ..., s$ and $b_{0,j} = 1, j = 1, 2, ..., q$);

(vi) By specializing the coefficients c_n, d_n and the parameter δ we obtain corresponding results for various subclasses such as $H_0^*,\ \Sigma_H^*(\gamma),\ MH^*(m),\ MH^*(m, \gamma)$.

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Katsaras's Type Fuzzy Norm under Triangular Norms

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Abstract

The aim of this paper is to redefine Katsaras's fuzzy norm using the notion of t-norm.

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1. Introduction and preliminaries

The concept of fuzzy set was introduced by L.A. Zadeh in his famous paper (Zadeh, 1965). A fuzzy set in X is a function $\mu : X \to [0,1]$. We will denote by $\mathcal{F}(X)$ the family of all fuzzy sets in X. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L. Zadeh:

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x) \ : \ i \in I\} \, , \ \left(\bigwedge_{i\in I} \mu_i\right)(x) = \inf\{\mu_i(x) \ : \ i \in I\} \, .$$

If $\mu_1, \mu_2 \in \mathcal{F}(X)$, then the inclusion $\mu_1 \subseteq \mu_2$ is defined by $\mu_1(x) \leq \mu_2(x)$.

Definition 1.1. (Chang, 1968) Let X, Y be arbitrary sets and $f : X \to Y$. If μ is a fuzzy set in Y, then $f^{-1}(\mu)$ is a fuzzy set in X defined by

$$f^{-1}(\mu)(x) = \mu(f(x)), (\forall)x \in X \, .$$

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If μ is a fuzzy set in X then $f(\mu)$ is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad f^{-1}(y) = \emptyset \end{cases} .$$

Remark. Previous definition is a special case of Zadeh's extension principle.

Since then many authors have tried to investigate fuzzy sets and their applications from different points of view. An important problem was finding an adequate definition of a fuzzy normed linear space. In the studying of the fuzzy topological vector spaces, Katsaras (1984) introduced for the first time the notion of fuzzy norm on a linear space.

Definition 1.2. (Katsaras & Liu, 1977) A fuzzy set ρ in X is said to be:

1. convex if $t\rho + (1-t)\rho \subseteq \rho, (\forall)t \in [0,1]$;
2. balanced if $\lambda\rho \subseteq \rho, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$;
3. absorbing if $\bigvee\limits_{t>0} t\rho = 1$;
4. absolutely convex if it is both convex and balanced.

Proposition 1. *(Katsaras & Liu, 1977) Let ρ be a fuzzy set in X. Then:*

1. *ρ is convex if and only if*

$$\rho(tx + (1-t)y) \geq \rho(x) \wedge \rho(y), (\forall)x, y \in X, (\forall)t \in [0,1];$$

2. *ρ is balanced if and only if $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$.*

Definition 1.3. (Katsaras, 1984) A Katsaras fuzzy semi-norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing.

Definition 1.4. (Nădăban & Dzitac, 2014) A fuzzy semi-norm ρ on X will be called Katsaras fuzzy norm if

$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \;\Rightarrow\; x = 0 \,.$$

Remark. a) It is easy to see that

$$\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right] \Leftrightarrow \left[\inf_{t>0} \rho\left(\frac{x}{t}\right) < 1, \text{for } x \neq 0\right] .$$

b) The condition $\left[\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0\right]$ is much weaker than that imposed by Katsaras (1984),

$$\left[\inf_{t>0} \rho\left(\frac{x}{t}\right) = 0, \text{for } x \neq 0\right] .$$

In 1992, Felbin (1992) introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. Following Cheng & Mordeson (1994), Bag & Samanta (2003) introduced another concept of fuzzy norm. In paper (Bag & Samanta, 2008) it is shown that the fuzzy norm defined by Bag and Samanta is similar to that of Katsaras. As the notion of fuzzy norm as defined by Cheng & Mordeson (1994) and Bag & Samanta (2003) can be generalized for arbitrary t-norms (see (Goleţ, 2010), (Alegre & Romaguera, 2010), (Nădăban & Dzitac, 2014)) motivates us to investigate the extension of Katsaras's fuzzy norm under triangular norm.

Definition 1.5. (Schweizer & Sklar, 1960) A binary operation

$$* : [0,1] \times [0,1] \to [0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a, (\forall) a, b \in [0,1]$;
2. $a * 1 = a, (\forall) a \in [0,1]$;
3. $(a * b) * c = a * (b * c), (\forall) a, b, c \in [0,1]$;
4. If $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

Example 1.1. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0,1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Remark. $a * 0 = 0, (\forall) a \in [0,1]$.

Definition 1.6. (Nădăban & Dzitac, 2014) Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a Bag-Samanta's type fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0, (\forall) x \in X$;

(N2) $[N(x,t) = 1, (\forall) t > 0]$ if and only if $x = 0$;

(N3) $N(\lambda x, t) = N\left(x, \frac{t}{|\lambda|}\right), (\forall) x \in X, (\forall) t \geq 0, (\forall) \lambda \in \mathbb{K}^*$;

(N4) $N(x + y, t + s) \geq N(x, t) * N(y, s), (\forall) x, y \in X, (\forall) t, s \geq 0$;

(N5) $(\forall) x \in X$, $N(x, \cdot)$ is left continuous and $\lim\limits_{t \to \infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNL-space).

Remark. Bag & Samanta (2003) gave this definition for $* = \wedge$ and Goleţ (2010), Alegre & Romaguera (2010) gave also this definition in the context of real vector spaces.

2. Fuzzy vector spaces under triangular norms

In paper (Das, 1988) the sum of fuzzy sets, fuzzy subspaces and convex fuzzy sets were re-defined using the notion of a t-norm. In this way, several results are obtained, some of which are generalisation of the results of Katsaras & Liu (1977).

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm.

Definition 2.1. Let μ_1, μ_2 be fuzzy sets in X. The sum of fuzzy sets μ_1, μ_2 is denoted by $\mu_1 + \mu_2$ and it is defined by

$$(\mu_1 + \mu_2)(x) = \sup_{x_1+x_2=x} [\mu_1(x_1) * \mu_2(x_2)] .$$

If μ is a fuzzy set in X and $\lambda \in \mathbb{K}$, then the fuzzy set $\lambda\mu$ is defined by

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if} \quad \lambda \neq 0 \\ 0 & \text{if} \quad \lambda = 0, x \neq 0 \\ \vee\{\mu(y) \ : \ y \in X\} & \text{if} \quad \lambda = 0, x = 0 \end{cases} .$$

Remark. In the particular case in which $* = \wedge$ we obtain the definition introduced by Katsaras & Liu (1977).

Proposition 2. *If $\alpha, \beta \in \mathbb{K}$ and $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$, then*

1. $\alpha(\beta\mu) = \beta(\alpha\mu) = (\alpha\beta)\mu$;
2. $\mu_1 \leq \mu_2 \Rightarrow \alpha\mu_1 \leq \alpha\mu_2$.

Proof. 1) *Case 1.* $\alpha \neq 0, \beta \neq 0$.

$$(\alpha(\beta\mu))(x) = (\beta\mu)\left(\frac{x}{\alpha}\right) = \mu\left(\frac{x}{\alpha\beta}\right) = ((\alpha\beta)\mu)(x) .$$

Similarly,

$$(\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = \mu\left(\frac{x}{\alpha\beta}\right) = ((\alpha\beta)\mu)(x) .$$

Case 2. $\alpha = 0, \beta \neq 0$.
Let $x \neq 0$. Then

$$(\alpha(\beta\mu))(x) = 0; ((\alpha\beta)\mu)(x) = 0; (\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = 0 .$$

For $x = 0$ we have

$$(\alpha(\beta\mu))(x) = \sup_{y\in X}(\beta\mu)(y) = \sup_{y\in X}\mu\left(\frac{y}{\beta}\right) = \sup_{y\in X}\mu(y) ;$$

$$((\alpha\beta)\mu)(x) = \sup_{y\in X}\mu(y) ; \ (\beta(\alpha\mu))(x) = (\alpha\mu)\left(\frac{x}{\beta}\right) = \sup_{y\in X}\mu(y) .$$

Case 3. $\alpha \neq 0, \beta = 0$ is similar.
Case 4. $\alpha = 0, \beta = 0$.
For $x \neq 0$ we have

$$(\alpha(\beta\mu))(x) = 0; ((\alpha\beta)\mu)(x) = 0; (\beta(\alpha\mu))(x) = 0 .$$

If $x = 0$, then

$$(\alpha(\beta\mu))(x) = \sup_{y\in X}(\beta\mu)(y) = \sup_{y\in X}\mu(y) .$$

$$((\alpha\beta)\mu)(x) = \sup_{y\in X}\mu(y) \ ; \ (\beta(\alpha\mu))(x) = \sup_{y\in X}(\alpha\mu)(y) = \sup_{y\in X}\mu(y) .$$

2) Let $x \in X$.
Case 1. $\lambda \neq 0$.

$$(\lambda\mu_1)(x) = \mu_1\left(\frac{x}{\lambda}\right) \leq \mu_2\left(\frac{x}{\lambda}\right) = (\lambda\mu_2)(x) .$$

Case 2. $\lambda = 0$. If $x \neq 0$, then $(\lambda\mu_1)(x) = 0 = (\lambda\mu_2)(x)$. For $x = 0$, we have

$$(\lambda\mu_1)(x) = \sup_{y\in X}\mu_1(y) \leq \sup_{y\in X}\mu_2(y) = (\lambda\mu_2)(x)$$

□

Proposition 3. *Let X, Y be vector spaces over $\mathbb{K}$, $f : X \to Y$ be a linear mapping, $\lambda \in \mathbb{K}$ and $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$. Then*

1. $f(\mu_1 + \mu_2) = f(\mu_1) + f(\mu_2)$;
2. $f(\lambda\mu) = \lambda f(\mu)$.

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Proposition 4. *Let $\mu, \mu_1, \mu_2 \in \mathcal{F}(X)$ and $\alpha, \beta \in \mathbb{K}$. The following sentences are equivalent:*

1. $\alpha\mu_1 + \beta\mu_2 \leq \mu$;
2. *For all $x, y \in X$ we have $\mu(\alpha x + \beta y) \geq \mu_1(x) * \mu_2(y)$.*

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Definition 2.2. A fuzzy set μ in X is called *-fuzzy linear subspace of X if

1. $\mu + \mu \subseteq \mu$;
2. $\lambda\mu \subseteq \mu, (\forall)\lambda \in \mathbb{K}$.

Proposition 5. *Let $\mu \in \mathcal{F}(X)$. Then μ is a *-fuzzy linear subspace of X if and only in*

$$\mu(\alpha x + \beta y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)\alpha, \beta \in \mathbb{K} .$$

Proof. The proof is exactly the same as in (Katsaras & Liu, 1977). □

Definition 2.3. A fuzzy set μ in X is called *-convex if

$$\mu(tx + (1-t)y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)t \in [0, 1] .$$

Remark. If $\mu \in \mathcal{F}(X)$ is *-convex and crisp, then $\mu = \phi_A$ (ϕ_A is the characteristic function of the subset A of X) and

$$\mu(tx + (1-t)y) \geq \mu(x) * \mu(y), (\forall)x, y \in X, (\forall)t \in [0, 1] .$$

Thus

$$\mu(tx + (1-t)y) = 1, (\forall)x, y \in A, (\forall)t \in [0, 1] .$$

Hence

$$tx + (1-t)y \in A, (\forall)x, y \in A, (\forall)t \in [0, 1] .$$

So A is convex in the classical sence.

Proposition 6. *A fuzzy set μ in X is *-convex if and only if*

$$t\mu + (1-t)\mu \subseteq \mu, (\forall)t \in [0, 1] .$$

Proof. " $\Rightarrow$ " *Case 1.* $t = 0$.

$$(0\mu + 1\mu)(x) = \sup_{x_1+x_2=x} [(0\mu)(x_1) * (1\mu)(x_2)] = \sup_{y\in X} \mu(y) * \mu(x) \leq 1 * \mu(x) = \mu(x) .$$

Case 2. $t = 1$ is similar.
Case 3. $t \in (0, 1)$.

$$(t\mu + (1-t)\mu)(x) = \sup_{x_1+x_2=x} [(t\mu)(x_1) * ((1-t)\mu)(x_2)] =$$

$$= \sup_{x_1+x_2=x} \left[\mu\left(\frac{x_1}{t}\right) * \mu\left(\frac{x_2}{1-t}\right)\right] \leq$$

$$\leq \sup_{x_1+x_2=x} \mu\left[t \cdot \frac{x_1}{t} + (1-t) \cdot \frac{x_2}{1-t}\right] = \sup_{x_1+x_2=x} \mu(x_1 + x_2) = \mu(x) .$$

" $\Leftarrow$ " *Case 1.* $t \in (0, 1)$.

$$\mu(tx + (1-t)y) \geq (t\mu + (1-t)\mu)(tx + (1-t)y) \geq (t\mu)(tx) * ((1-t)\mu)((1-t)y) = \mu(x) * \mu(y) .$$

Case 2. $t = 0$.

$$\mu(0x + 1y) = \mu(y) = 1 * \mu(y) \geq \mu(x) * \mu(y) .$$

Case 3. $t = 1$ is similar. □

Remark. Some *-convexity properties of fuzzy sets, where * is a triangular norm on [0, 1], were investigated in papers (Yandong, 1984; Yuan & Lee, 2004; Nourouzi & Aghajani, 2008) etc.

3. Katsaras's type fuzzy norm

Let X be a vector space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $*$ be a continuous t-norm.

Definition 3.1. A fuzzy set ρ in X which is *-convex, balanced and absorbing will be called Katsaras's type fuzzy semi-norm. If in addition

$$\rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \;\Rightarrow\; x = 0 \,,$$

then ρ will be called Katsaras's type fuzzy norm.

Lemma 1. *If ρ is balanced and absorbing, then $\rho(0) = 1$.*

Proof. As ρ is balanced, we have that $\rho(0) = \rho(0 \cdot x) \geq \rho(x)$. Thus $\rho(0) = \bigvee_{x \in X} \rho(x)$. As ρ is absorbing, we have that $\bigvee_{t>0} \rho\left(\frac{x}{t}\right) = 1$. Hence $\bigvee_{x \in X} \rho(x) = 1$. Thus $\rho(0) = 1$. □

Lemma 2. *If ρ is balanced, then $\rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X, (\forall)\alpha \in \mathbb{K}$.*

Proof. If ρ is balanced, then $\rho(\lambda x) \geq \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| \leq 1$. Particularly, for $\lambda \in \mathbb{K}$: $|\lambda| = 1$, we have

$$\rho\left(\frac{1}{\lambda} x\right) \geq \rho(x), (\forall)x \in X \,.$$

Replacing x with λx, we obtain $\rho(x) \geq \rho(\lambda x), (\forall)x \in X$. Thus

$$\rho(\lambda x) = \rho(x), (\forall)x \in X, (\forall)\lambda \in \mathbb{K}, |\lambda| = 1 \,.$$

Take $\alpha \in \mathbb{K}, \alpha \neq 0$ (if $\alpha = 0$ it is obvious that $\rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X$). We put in previous equality $\lambda = \frac{\alpha}{|\alpha|}$. It results

$$\rho\left(\frac{\alpha}{|\alpha|} x\right) = \rho(x), (\forall)x \in X \Leftrightarrow \rho(\alpha x) = \rho(|\alpha| x), (\forall)x \in X \,.$$

□

Remark. The following theorems extend some results obtained in (Bag & Samanta, 2008).

Theorem 1. *If ρ is a Katsaras's type fuzzy norm, then*

$$N(x,t) := \begin{cases} \rho\left(\frac{x}{t}\right) & \text{if } \; t > 0 \\ 0 & \text{if } \; t = 0 \end{cases}$$

is a Bag-Samanta's type fuzzy norm.

Proof. $(N1)$ $N(x,0) = 0, (\forall)x \in X$ is obvious.
$(N2)$ $[N(x,t) = 1, (\forall)t > 0] \Rightarrow \rho\left(\frac{x}{t}\right) = 1, (\forall)t > 0 \Rightarrow x = 0$. Conversely, if $x = 0$, then $N(0,t) = \rho(0) = 1, (\forall)t > 0$.
$(N3)$ We suppose that $t > 0$ (if $t = 0$ $(N3)$ is obvious). Using previous lemma we have:

$$N(\lambda x, t) = \rho\left(\frac{\lambda x}{t}\right) = \rho\left(\frac{|\lambda| x}{t}\right) = \rho\left(\frac{x}{t/|\lambda|}\right) = N\left(x, \frac{t}{|\lambda|}\right) .$$

$(N4)$ If $t = 0$, then $N(x,t) = 0$ and $N(x,t) * N(y,s) = 0 * N(y,s) = 0$ and the inequality $N(x+y, t+s) \geq N(x,t) * N(y,s)$ is obvious. A similar situation occurs when $s = 0$. If $t > 0, s > 0$, then

$$N(x+y, t+s) = \rho\left(\frac{x+y}{t+s}\right) = \rho\left(\frac{t}{t+s} \cdot \frac{x}{t} + \frac{s}{t+s} \cdot \frac{y}{s}\right) \geq \rho\left(\frac{x}{t}\right) * \rho\left(\frac{y}{s}\right) = N(x,t) * N(y,s) .$$

$(N5)$ First, we note that $N(x,\cdot)$ is non-decreasing. Indeed, for $s > t$, we have

$$N(x,s) = N(x+0, t+s-t) \geq N(x,t) * N(0, s-t) = N(x,t) * 1 = N(x,t) .$$

We prove now that $N(x,\cdot)$ is left continuous in $t > 0$.
Case 1. $N(x,t) = 0$. Thus, for all $s \leq t$, as $N(x,s) \leq N(x,t)$, we have that $N(x,s) = 0$. Therefore $\lim\limits_{s\to t, s<t} N(x,s) = 0 = N(x,t)$.
Case 2. $N(x,t) > 0$. Let α be arbitrary such that $0 < \alpha < N(x,t)$. Let (t_n) be a sequence such that $t_n \to t, t_n < t$. We prove that there exists $n_0 \in \mathbb{N}$ such that $N(x,t_n) \geq \alpha, (\forall)n \geq n_0$. As $\alpha \in (0, N(x,t))$ is arbitrary, we will obtain that $\lim\limits_{n\to\infty} N(x,t_n) = N(x,t)$.

As $N(x,t) = t\rho(x) > 0$, we have that $\rho(x) > 0$. Let $s = \frac{\alpha}{\rho(x)}$. We note that $s < t$. Indeed,

$$s < t \Leftrightarrow \frac{\alpha}{\rho(x)} < t \Leftrightarrow \alpha < t\rho(x) = N(x,t) .$$

As $t_n \to t, t_n < t$ and $s < t$, there exists $n_0 \in \mathbb{N}$ such that $t_n > s, (\forall)n \geq n_0$. Then

$$N(x,t_n) = \rho\left(\frac{x}{t_n}\right) \geq \rho\left(\frac{x}{s}\right) = s\rho(x) = \alpha .$$

Since $\bigvee\limits_{t>0} t\rho(x) = 1$, we obtain that $\bigvee\limits_{t>0} N(x,t) = 1$. Thus $\lim\limits_{t\to\infty} N(x,t) = 1$. □

Theorem 2. *If N is a Bag-Samanta's type fuzzy norm, then $\rho : X \to [0,1]$ defined by*

$$\rho(x) = N(x,1), (\forall)x \in X$$

is a Katsaras's type fuzzy norm.

Proof. First, we note that, by $(N2)$, we have $\rho(0) = N(0,1) = 1$.
(1) ρ is *-convex.
Let $t \in (0,1)$. Then

$$\rho(tx + (1-t)y) = N(tx + (1-t)y, 1) = N(tx + (1-t)y, t + 1 - t) \geq$$

$$\geq N(tx, t) * N((1-t)y, 1-t) = N(x, 1) * N(y, 1) = \rho(x) * \rho(y) .$$

If $t = 0$, then $\rho(tx + (1-t)y) = \rho(y) = 1 * \rho(y) \geq \rho(x) * \rho(y)$. The case $t = 1$ is similar.
(2) ρ is balanced.
Let $x \in X, \lambda \in \mathbb{K}^*, |\lambda| \leq 1$. As $N(x, \cdot)$ is non-decreasing, we have that

$$\rho(\lambda x) = N(\lambda x, 1) = N\left(x, \frac{1}{|\lambda|}\right) \geq N(x, 1) = \rho(x) .$$

If $x \in X, \lambda = 0$, then $\rho(\lambda x) = \rho(0) = 1 \geq \rho(x)$.
(3) ρ is absorbing.
Using $(N5)$, we have that

$$\bigvee_{t>0} (t\rho)(x) = \bigvee_{t>0} \rho\left(\frac{x}{t}\right) = \bigvee_{t>0} N\left(\frac{x}{t}, 1\right) = \bigvee_{t>0} N(x, t) = 1 .$$

Finally,

$$\rho\left(\frac{x}{t}\right) = 1, (\forall) t > 0 \Rightarrow N\left(\frac{x}{t}, 1\right) = 1, (\forall) t > 0 \Rightarrow N(x, t) = 1, (\forall) t > 0 \Rightarrow x = 0 .$$

□

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Measure of Tessellation Quality of Voronoï Meshes

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Abstract

This article introduces a measure of the quality of Voronoï tessellations resulting from various mesh generators. Mathematical models of a number of mesh generators are given. A main result in this work is the identification of those mesh generators that produce the highest quality Voronoï tessellations. Examples illustrating the application of the quality measure are given in comparing Voronoï tessellations of digital images.

Keywords: Sites, Mesh Generation, Quality, Tessellations, Voronoï mesh.
2010 MSC: Primary 54E05, Secondary 20L05, 35B36.

1. Introduction

This article introduces a measure of the quality of Voronoï tessellations resulting from various mesh generators. A *Voronoï tessellation* is a collection of non-overlapping convex polygons called *Voronoï regions*. It is well-known that creating meshes is a fundamental and necessary step in several areas, including engineering, computing, geometric and scientific applications (Leibon & Letscher, 2000; Owen, 1998; Liu & Liu, 2004). Meshes assume simplex structures or volumes based on the geometry of the surfaces, dimension of the space and placement of sites of the meshes (see, *e.g.*, (Ebeida & Mitchell, 2012; Mitchell, 1993; Persson, 2004)). This work is a natural outgrowth of recent work on Voronoï tessellation (Persson, 2004; Persson & Strang, 2004; Bern & Plassmann, 1999; Du *et al.*, 1999; Brauwerman *et al.*, 1999; Peters, 2015*b*).

Seeds, generating points or sites of meshes for non-image domains may be chosen randomly, deterministically on grids (Persson, 2004), using distribution sampling *e.g.* (Ebeida & Mitchell, 2012), using the centroids of tessellation regions. In the search for a measure of mesh quality, it is

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useful to identify the best or most suitable mesh generators (also called sites) for mesh generation. A principal benefit of this work is the identification of those generators that produce the highest quality Voronoï tessellations.

A natural application of the proposed mesh quality measure is given in comparing and classifying digital images Voronoï tessellation covers. Each Voronoï region is a closed set of points in a convex polygon. A Voronoï tessellation *cover* of a digital image equals the union of the Voronoï regions.

2. Related Work

Numerous mesh generation algorithms have been developed for several purposes including image processing and segmentation (Arbeláez & Cohen, 2006), clustering (Ramella *et al.*, 1998), data compression, quantization and territorial behavior of animals (Persson, 2004; Persson & Strang, 2004; Du *et al.*, 1999). These methods tackle a wide variety of geometrical representations for meshing the surfaces. In addition, a number of mesh generation algorithms are iterative in nature, so that the algorithm adjusts the meshes iteratively to approach fulfilling predefined conditions, thereby terminating on meeting the criterion up to some limits (Persson, 2004; Persson & Strang, 2004). In some adaptive mesh algorithms (Persson, 2004; Persson & Strang, 2004; George, 2006), the mesh sites are variable. For example they may be displaced to attain force equilibrium. In one such scenario the algorithm starts by partitioning the space based on the initial distribution of sites but iterates through them according to preset conditions on an element size function until the equilibrium and stopping conditions are satisfied (Persson, 2004). This essentially is refining and solving for optimality of the meshes (Rajan, 1994; Peraire *et al.*, 1987; Rivara, 1984; Ruppert, 1995) according to the preset conditions.

Voronoï diagrams introduced by the Ukrainian mathematician G. Voronoï (Voronoï, 1903, 1907, 1908) (elaborated in the context of proximity spaces in (Peters, 2015*b*), (Peters, 2015*c*), (Peters, 2015*a*)) provide a means of covering a space with a polygonal mesh. In telecommunications, Voronoï diagrams have furnished a tool of analysis for binary linear block codes (Agrell, 1996) governing regions of block code, performance of Gaussian channel among others. In music, Voronoï diagrams have once again demonstrated their utility (McLean, 2007). For example, they been successfully applied in automatic grouping of polyphony (Hamanaka & Hirata, 2002). Other works bordering on applications of Voronoï meshes in reservoir modeling (Møller & Skare, 2001), attempts at cancer diagnosis (Demir & Yener, 2005) are also available.

Ever since utility of Voronoï diagrams was demonstrated in several applications including the post office problem where given a set of post office sites in furnished the answer in determining the nearest one to visit and other few works of application of 56 meshed surfaces an area, Voronoï tessellation has been useful in the study of the territorial behavior of animals, image compression, segmentation etc no works or very few have focused on their application in proximity and classification analysis of digital images. This work proposes to explore the utility of meshes in the study of proximal regions as a means of image classification.

Mesh applications in image analysis are not widely studied in the open literature. In that regard, we seek to contribute to the application of meshes in image classification by (a) identifying the best forms of generating points for meshes, (b) arriving at a measure of the quality of meshes, and

(c) characterizing meshes best suited for image classification. In addition, it is anticipated that this research will yield useful theorems that are a natural outcome of measuring mesh quality. Such theorems provide a formal foundation for the study of image mesh quality.

3. Preliminaries

Since the discovery that Voronoï tessellations are the secret working formula of bees, humans have sought to bring similar benefits and applications of Voronoï tessellations in their applications to image processing, image compression, clustering, territorial behaviour of animals etc (Persson, 2004; Persson & Strang, 2004; Bern & Plassmann, 1999; Du *et al.*, 1999; Brauwerman *et al.*, 1999). Several forms of polygonal meshes exist due to Voronoï and Delaunay tessellations and since Delaunay triangulation is a dual of Voronoï, our focus will be on the former.

Assume a finite set S of locations called sites s_i in a space $\mathbb{R}^n$. Computing the Voronoï diagram of S means partitioning the space into Voronoï regions $V(s_i)$ in such a way that $V(s_i)$ contains all points of S that are closer to s_i than to any other object s_j, $i \neq j$ in S.

Given the generator set

$$S = \{s_1, , s_k : i \in \mathbb{N}\},$$

where each member of S is called a mesh generating point, let $s_i \in S$. The Voronoï region $V(s_i)$ is defined by

$$V(s_i) = \{x \in \mathbb{R}^n : \|x - s_i\| \leq \|x - s_k\|, s_k \in S, i \neq k\},$$

where $\|., .\|$ is the Euclidean norm (distance between vectors). The set

$$V(S) = \bigcup_{s_i \in S} V(s_i)$$

is called the n-dimensional Voronoï diagram generated by the points in S. In $\mathbb{R}^2$, this effectively covers the plane with convex and non overlapping polygons, one for each generating point in S. A centroidal Voronoï tessellation is a special case of $V(s_i)$, where the sites are the mass centroids of regions computed by

$$c_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}$$

where $\rho(x)$ is the density function of $V(s_i)$. The mathematical utility of centroidal tessellations lie in their relationship to the energy function (Brauwerman *et al.*, 1999) defined as:

$$E(z, V) = \sum_{i=1}^{n} \int_{V_i} \rho(x)|x - z_i|^2 dx.$$

The energy function $E(z, V)$ for a Voronoï region depends on the density function $\rho(x)$ of the regions and the squared distances between the generating site z_i and nearby points x in the region. The total energy for a Voronoï diagram is the sum of integrals of the individual energies of the regions comprising the Voronoï diagram.

Voronoï regions are cells of growth from a certain view point. This view point is equivalent to considering the vertex of each region as a nucleus of a growing or expanding cell. Cells propagate simultaneously outward from their nuclei at uniform rates until they intersect with others. They then freeze giving the boundaries of the regions defined by the tessellation.

Ultimately, cells whose nuclei are on the convex hull of a vertex grow until they intersect the outgrowth of others. It is interesting to note that since all cells are growing at the same rate, their first points of contact will coincide with the midpoint of the two nuclei. This is exactly the locus of all equidistant points from the nuclei.In other words it the perpendicular bisector of the the nuclei from which all points are equidistant. The set of all points on these loci form the edges of the regions.

Voronoï cells that share an edge are said to be Voronoï neighbors. The aggregate of triangles formed by connecting the nuclei of all Voronoï neighbors tessellate the area within the convex hull of the set. Notice that the regions obtained are neighborhoods defined by the norm $\| \cdot \|$. This makes it possible to have a continuous image-like treatment of a dot pattern, thus permitting the application of general image processing techniques (Ahuja & Schachter, 1982). One such interpretation of the Voronoï tessellation is to view it as a cluster partition of a space (Ramella *et al.*, 1998). In this view, the space is segmented by the various Voronoï regions, with the size of the clusters given by the areas of the regions. Also, the Voronoï tessellation is useful for boundary extraction, with the tessellated space viewed as a mosaic.

Definition 3.1. Given a point set $S \subseteq \mathbb{R}^n$ and a distance function d_n, the set $\{V_i\}_{i=1}^k$ is called a Voronoï tessellation of S if $V_i \cap V_j \neq \varnothing$ for $i \neq j$.

Definition 3.2. The Voronoï region of a site is a polygon about that site. The set of all regions partition the plane of the sites S based on a distance function $\| \cdot \|$. This results in the plane being covered with polygons about those sites.

Definition 3.3. Given a set $S = \{s_1, ..., s_k\}$ of points, any plane (v_i, v_j) is a Voronoï edge of the Voronoï region V if and only if there exists a point x such that the circle centered at x and circumscribing v_i and v_j does not contain in its interior any other point of V.

Definition 3.4. A Voronoï tessellation is a set of polygons with their edges and vertices that partition a given space of sites.

Definition 3.5. A Voronoï edge is a half plane equidistant from two sites and which bounds some part of a Voronoï region. Every edge is incident upon exactly two vertices and every vertex upon at least three edges.

Definition 3.6. A Voronoï vertex is the center of a circle through three sites.

Definition 3.7. A set of points form a convex set if there is a line connecting each pair of points.

Definition 3.8. The convex hull of Voronoï regions about a set of sites is the smallest set which contains the Voronoï regions as well as the union of the regions.

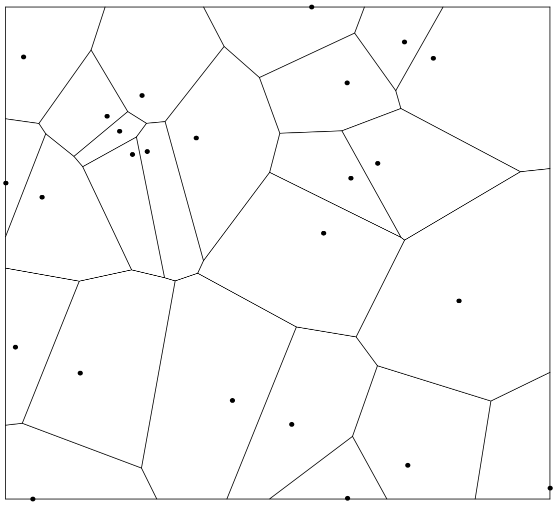


Figure 1. Voronoï diagram of a set of sites.

Definition 3.9. A dot or point pattern is a set of points of a signal representing spatial locations of signal features. For example sets of corners, keypoints etc are referred to as point or dot patterns.

Definition 3.10. The quality of a signal is defined as a characteristic of the signal which gives information about perceived signal degradation compared to an ideal.

Lemma 3.1. *The energy of a point located in a particular Voronoï region $V(s_i)$ is minimal with respect to all other regions $V(s_j)$ for $i \neq j$.*

Proof. Consider a point $y \in V(S_i)$. Its energy is evaluated as:

$$E(z, V) = \int_{V_i} \rho(y)|y - \hat{z}_i|^2 dy$$

$$E(z, V) = \int_{V_i} \rho(y)(0) dy = 0.$$

Since the distance between z and y is 0 by definition, $E = 0$. However E(z,V) of $y \notin V(S_j)$ is

$$E(z, V) = \int_{V_j} \rho(y)|y - \hat{z}_i|^2 dy \neq 0.$$

Since $\rho(y)$ nor $|y - \hat{z}_i|^2$ is non-zero. □

Theorem 3.1. *The energy function $E(z, V)$ is minimized at the centroid sites of the tessellations.*

Proof.

$$E(z, V) = \sum_{i=1}^{n} \int_{V_i} \rho(x)|x - \hat{z}_i|^2 dx.$$

The energy of a Voronoï region V_i is the integral of the product of the density function of that region $\rho(x)$ and the squared distances between the generating site $\hat{z}_i$ and points comprising the region. The total energy $E(z, V)$ is the sum of the energies of all Voronoï regions.

To obtain the minimum of $E(z, V)$, it requires that the derivative of the function with respect to the sites be equal to zero. The solution of the derivative are the sites $\hat{z}_i$.

$$\frac{dE}{d\hat{z}_i} = 2\sum_{i=1}^{n} \int_{V_i} \rho(x)(x - \hat{z}_i)dx = 0$$

$$\hat{z}_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}.$$

The solution $\hat{z}_i$ are the centroids. □

Theorem 3.2. *For a given set of sites $Z = \{z_i\}$, the energy is minimized when V is a centroidal Voronoï tessellation.*

Proof. Immediate from Lemma 3.1 and Theorem 3.1. □

4. Tessellation Generators

In the literature few works or none considers the choice of sites for meshing images using location of image features. We mostly go ahead and tessellate using chosen locations irrespective of locations of image features. Previously, sites have been chosen to correspond to center of masses of regions (Du *et al.*, 1999; Burns, 2009), random locations (Ebeida *et al.*, 2011; Aurenhammer, 1991), deterministic or regular locations (Persson, 2004; Persson & Strang, 2004; Aurenhammer, 1991). In short, majority of these locations do not take information of the sites into account. The method of centroids has evident advantages (Brauwerman *et al.*, 1999), but these are questionable if the regions from which we obtain the center of masses do not reflect image feature locations. In this work, various sites based on image features are first discovered and subsequently the sites give a tessellation of the space. With several sites found and tessellations performed, quality measures of the meshes are obtained with the view of helping ascertain the best sites for tessellations using an overall quality measure. This then forms a road map to meshed image analysis given a method of sites that yield high quality meshes. Important image features are known to reside at corner, edge, keypoints, centroids, extrema and modal image feature sites. These then would be used to discover mesh sites. They are treated next.

4.1. Image Corner Points

A very notable feature of digital images are image corner points. These define points where structures in the image intersect, as such they form a solid background for feature recognition and extraction.

One of the earliest criteria for corner point identification is a point that has low self-similarity (Moravec, 1980). Each pixel centered in a patch is compared to nearby pixels in an overlapping patch for corner point candidate examination. Since then, improvements in corner point identification by computing differential corner scores with respect to direction instead of patches in

(Moravec, 1980) resulted in combined corner and edge detection (Harris & Stephens, 1988). For more accuracy in subpixels in corner identification see (Förstner & Gülch, 1987). Recently, corner detection based on other methods: Multiple scales due originally to (Harris & Stephens, 1988), level curvature approach (Kitchen & Rosenfeld, 1980; Koenderink & Richards, 1988), difference of Laplacians, Gaussians, and Hessians (Lowe, 2004; Lindeberg, 1998, 2008), affine-adapted interest point operators (Lindeberg, 1993, 2008; Mikolajczyk & Schmid, 2004), curvature placement along edges (Wang & Brady, 1995), smallest univalue segment assimilating nucleus (Smith & Brady, 1997), direct testing of pixel self-similarity and feature accelerated segment (Rosten & Drummond, 2006), non-parametric and adaptive region processing methods (Guru & Dinesh, 2004), transform approaches (Kang *et al.*, 2005; Park *et al.*, 2004), adaptive approaches (Pan *et al.*, 2014), structure-based analysis (Kim *et al.*, 2012) have resulted . Corner points are identified here as points in which there is a significant change in intensity features in two or more directions:

$$C(u,v) = \sum w(x,y)[I(x+u,y+v) - I(x,y)]^2.$$

The window function $w(x,y)$ tends to be rectangular but could assume other suitable forms. So, a corner point is returned by the corner point function $C(x,y)$ if the squared intensity difference between the intensity at location (x,y) and location $(x+u,y+v)$ is large for any two directions.

4.2. Edge Sites

Edge maps are widely used in image processing for feature detection and object recognition. Besides, edge information is known to be crucial in feature detection and image analysis. These have been used due to the fact that the edges tend to localize an object of interest for target processing and feature detection. Edge points are points whose feature values differ sharply from those of neighboring points (Canny, 1986).

4.3. Image Keypoint Sites

Image keypoints are popular for extracting distinct image points (Lowe, 1999; Mitchell, 2010; Feng *et al.*, 2013; Woźniak & Marszałek, 2014). Scale-space extrema detection has been shown to yield image keypoints (Lowe, 1999). This process however usually yields numerous keypoint candidates therefore we have to resort to means of reducing their number to important ones. For example by eliminating low contrast points, the most important ones are retained. Detection of locations of keypoints invariant to scale change may be accomplished by obtaining stable features across scales of an image (Lowe, 1999). In that regard, the scale space $L(x,y,\sigma)$ is obtained by convolution of Gaussian functions $G(x,y,\sigma)$ with the image function $I(x,y)$ at several scales k.

$$L(x,y,\sigma) = G(x,y,\sigma) * I(x,y)$$

$$G(\vec{x},\sigma) = \frac{1}{2\pi\sqrt{\sigma}}\text{Exp}\{-\frac{1}{2}(\vec{x}-\mu)^T\sigma^{-1}(\vec{x}-\mu)\}.$$

We proceed below to obtain the difference of the convolved result in the previous step.

$$D(x,y,\sigma) = (G(x,y,k\sigma) - G(x,y,\sigma)) * I(x,y)$$

$$D(x, y, \sigma) = L(x, y, k\sigma) - L(x, y, \sigma).$$

In the final step, the extrema (minima and maxima) points are obtained by comparing points in the difference functions and selecting those that achieve the minimum or maximum values in their neighborhoods (Lowe, 1999). Keypoint locations originally found at $D(x, y)$ at scales of σ may be used as estimates in Taylor series expansion about the position vector $\vec{x} = (x, y)$ to obtain more accurate locations of the points (Brown & Lowe, 2002). Locations are extrapolated as follows.

$$D_a(\bar{x}) = D + \frac{\partial D^T}{\partial \vec{x}} + \frac{1}{2}\vec{x}\vec{x}^T \frac{\partial^2 D}{\partial \vec{x}^2 \vec{x}},$$

where $D_a(\bar{x})$ is the improved location of a keypoint.

4.4. Centroid Sites

Given a tessellated plane of points centroid points of regions are computed as follows.

$$c_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}.$$

The center of masses c_is are computed from Voronoï regions and then used to re-tessellate the regions. Given corner, edge and keypoint sites in images, their tessellations produce Voronoï regions corresponding to those generators. The centers of masses of these regions based on those image feature locations form a set of sites in the plane for centroidal tessellations.

4.5. Modal Pixel and Extrema Sites

The histogram distributions of images are readily available. They furnish information on the distribution of the pixels in the image. Pivotal points in an image can be sought by considering the modal pixel positions in the image. In this way, we get to find out the feature value that occurs most frequently in a digital image and use the locations of those as sites for tessellations. A variant to this approach is to find the modal feature value, displace it by a constant and then use the positions of the resulting value as sites for image tessellations. The most influential feature M is obtained from the histogram distribution $h(k)$ from the steps below.

$$h(k) = \sum_x \sum_y n_k$$

$$M = \text{Max}_k(\sum_x \sum_y n_k) = \text{Max}_k(hk).$$

For a given image function $I(x, y)$, the feature values are in the range $[0, k]$. From this set, there exists minimum and maximum feature values. Sites corresponding to the extrema can be used for tessellations. However, it should be noted that where any of the extrema is unique, only one site is returned for the tessellation. Extrema sites M_1, M_2, M_d are useful because they can give us geometrical information about objects that tend to have a particular distribution or feature value;

$$M_1 = \text{Min}(f(x, y) \forall x, y)$$

$$M_2 = \text{Max}(f(x,y)\forall x,y)$$
$$M_d = \text{Max}(\sum\sum n_{ij}) - a.$$

Extrema sites are discovered using M_1 and M_2, whilst displaced modal sites are discovered using M_d, given the constant a. Due to the numbers of modal and extrema sites and the nature of meshes they produce, they are not treated further in this work. Instead, displaced feature sites are discovered and used.

So far, we have identified sites of important features in digital images. A necessary step in determining the choice of sites lies in the quality of meshes produced by the particular choice of sites. In what follows, the quality of meshes produced by the sites is examined.

5. Voronoï Mesh Quality Analysis

Quality metrics for mesh analysis have been explored in the literature (Knupp, 2001; Shewchuk, 2002; Bhatia & Lawrence, 1990*a*). Most mesh generation approaches set a predefined quality factor for each cell as such they easily achieve meshes with high quality. This is not always useful or justifiable. For example, in images it's highly unlikely that mesh sites are deterministically or randomly distributed as assumed in these methods, thus this work discovers sites using image features with the view point of obtaining quality factors as high as possible.

Let X be a nonempty set of polygons in a Dirichlet tessellation, $x, y \in X$. A polygon $x \in X$ in a tessellation is called a **cell**. A question that arises naturally is that which sites are best or more favorable, leading to high quality cells? We will attempt to answer this in terms of the overall mesh quality factor q_{all} for the mesh cells produced by a particular set of sites. Qualities of individual mesh cells computed here are defined according to the geometry of the cell (Shewchuk, 2002; Bhatia & Lawrence, 1990*a*).

Let S be a set of tessellation cells, A the area of a tessellation containing a 3-sided polygon cell $s \in S$, l_1, l_2, l_3 the lengths of the sides of s with $Q(s)$ the quality of cell s. Then, for example, (Bhatia & Lawrence, 1990*b*), (Bank & Xu, 1996) as well as (Field, 2000) use the following smooth quality measure for a 3-sided cell.

$$Q_3(s) = 4\sqrt{3}\frac{A}{l_1^2 + l_2^2 +, l_3^2}.$$

For a four sided mesh cell, the quality factor $Q_4(s)$ is defined by

$$Q_4(s) = \frac{4A}{l_1^2 + l_2^2 + l_3^2 + l_4^2}.$$

The quality $Q(s)$ of $3D$ tetrahedron (polyhedron with 4 sides) cell in $\mathbb{R}^3$ is defined by

$$Q_{43D}(s) = \frac{6\sqrt{2}V}{l_{rms}^3}.$$

Here, V is the volume of the tetrahedron and l_{rms} is given by:

$$l_{rms} = \sqrt{\frac{1}{6}\sum_{i=1}^{6} l_i^2}.$$

Since the focus is on meshes of $2D$ surfaces in $\mathbb{R}^2$, we only briefly consider $3D$ meshes. An overall mesh quality indicator may be defined for any meshed surface by making use of the qualities of the individual cells. One such indicator is defined by q_{all} defined below.

$$q_{all} = \frac{1}{N}\sum_{i=1}^{N} q_i.$$

For a plane tessellated by a set of sites S, the indicator of the overall tessellation quality is influenced by the qualities of the individual cells q_i. This provides a useful tool for discriminating sites and their tessellation quality.

Theorem 5.1. *For any plane, there exists a set of sites for which the mesh quality is maximum.*

Proof. Consider an arbitrary n-sided mesh cell. Assume the cell is a quad cell without loss of generality. For maximum q, the partial derivatives of q with respect to the l_is should be equal.

$$\frac{\partial q}{\partial l_1} = \frac{\partial q}{\partial l_2} = \ldots = \frac{\partial q}{\partial l_n}$$
$$\frac{-8Al_1}{(l_1^2 + l_2^2 + \ldots + l_n^2)^2} = \frac{-8Al_2}{(l_1^2 + l_2^2 + \ldots + l_n^2)^2} \ldots = \frac{-8Al_n}{(l_1^2 + l_2^2 + \ldots + l_n^2)^2}.$$

This happens when $l_1 = l_2 = \ldots = l_n$. So generators chosen such that their half planes are equidistant from each other would satisfy this condition. □

To synthesize and crystallize the preceding deliberations on mesh quality analysis, the following algorithm is provided.

5.1. Algorithm for Mesh Quality Computation

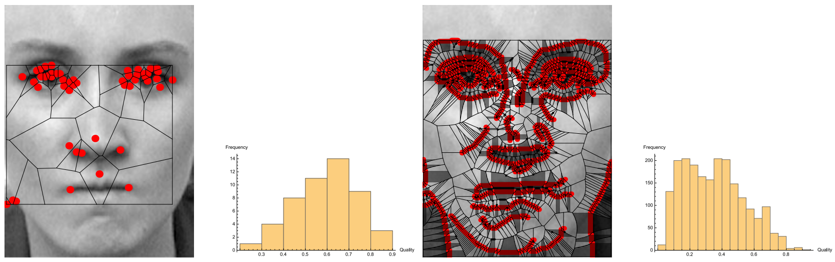
It is clear that a Voronoï diagram is a collection of several polygons of different dimensions in accordance with the criteria already laid out, thus the quality factor of each polygon is computed as follows:

Input: set of sites S, set of cells $V(S)$
Output: Mesh Quality(Q(s))
for each Voronoï region $V_i(s) \in V(S)$, site $s \in S$ **do**
  Access the number of sides and coordinates of the vertices of the polygon;
  Using the coordinates, compute the lengths l_i and Area A of the polygon;
  Use l_i and A in the appropriate expression to compute cell quality $Q(s)$, $s \in S$;
end for
$Q(S) = \{Q(s)\}$ ■

The results presented in the following sections show tessellated image spaces side-by-side with the distribution of the quality factors of their cells. These are shown for image data of several subjects. From the distribution of quality factors, we obtain the mean quality measure as an indicator of overall quality of meshes due to each set of generating sites.

6. Application: Digital Image Tessellation Quality

Mesh sites are obtained using the 2*D* face sets from (Craw, 2009). The images are monochrome with dimensions 181 by 241 showing subjects with different expressions and orientations. The sites so obtained from them are used to tessellate the regions and subsequently, quality factors of cells are computed. The quality factors are shown in histogram plots next to the tessellated images (Fig. 2-Fig. 21).



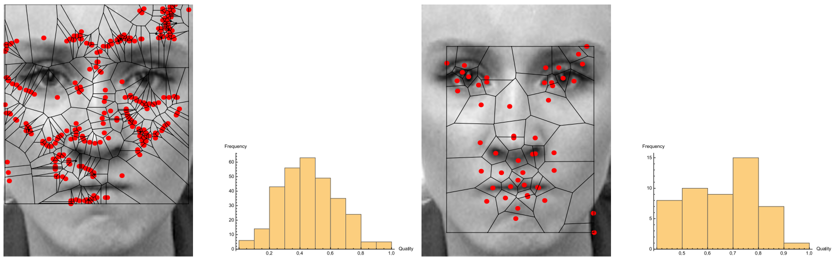
2.1: Corners 2.2: CorHist 2.3: Edges 2.4: EdHist

Figure 2. Corner & Edge Tessellations and Quality Histograms.

Remark 6.1. Corner-Based vs. Edge-Based Voronoï Tessellations.
A corner-based Voronoï tessellation of a face image is shown in Fig. 2.1. The corresponding corner-quality mesh histogram is given in Fig. 2.2. The horizontal axis represents mesh cell qualities and vertical axis represents the frequencies of the qualities. Because the number of image corners found are both sparse and grouped together in the facial high points representing pixel gradient changes in directions, the corresponding corner-based mesh consists of fairly large polygons surrounding the corners. Also, observe that the corner-based mesh histogram has a fairly normal distribution (skewed to the right).

An edge-based Voronoï tessellation of a face image is shown in Fig. 2.3. The corresponding edge-quality mesh histogram is given in Fig. 2.4. By contrast with image corners used as mesh generating points, the number of edge pixels found is large. In addition, the edge pixels are grouped closely together. Hence, the corresponding edge-based mesh contains many small Voronoï regions grouped closely together. The resulting edge-based mesh histogram has more than one maximum, which is an indicator that edge-based meshes have poor quality. □

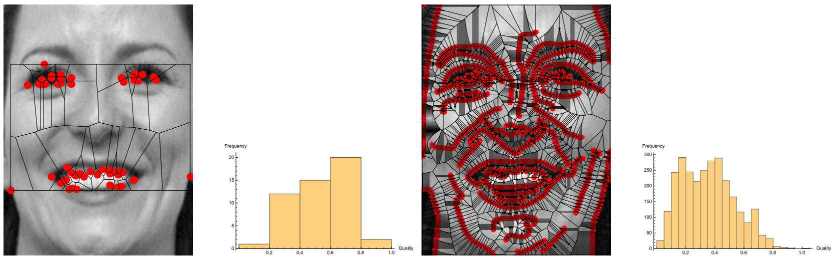
Remark 6.2. Dominant-Based vs. Keypoint-Based Voronoï Tessellations.
Fig. 3.1 shows dominant-based tessellations alongside their quality measures in Fig. 3.2. Similarly, keypoint-based tessellations and their qualities are shown in Fig. 3.3 and Fig. 3.4 respectively. Dominant-based cells tend to have quality values that fall within several ranges of the quality scale. Even though dominant generators tend to have higher numbers like edge generators, they generally give higher qualities compared to edge-based cells. Keypoint-based cells like dominant-based cells tend to have their qualities falling in several ranges of quality, only that the number of fragmented ranges is usually smaller compared to that of dominant-based cells. Even



3.1: Dominants 3.2: DomHist 3.3: Keypts 3.4: KeyHist

Figure 3. Dominant & Keypoint Tessellations and Quality Histograms.

though keypoint generators have smaller numbers compared to dominant generators, the location and distribution of the features favor creation of more perfect mesh cells. The qualities of dominant-based cells peak around mid scale whilst those of keypoints tend to be flat. □

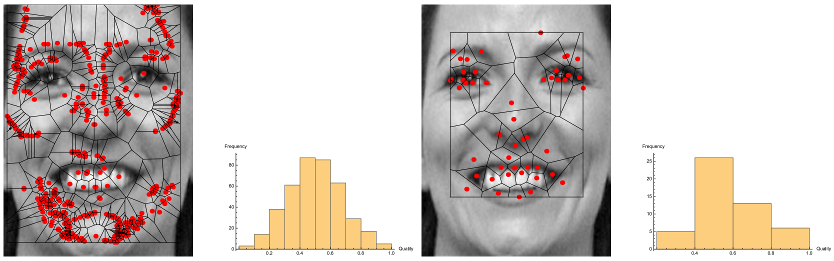


4.1: Corners 4.2: CorHist 4.3: Edges 4.4: EdHist

Figure 4. Corner & Edge Tessellations and Quality Histograms.

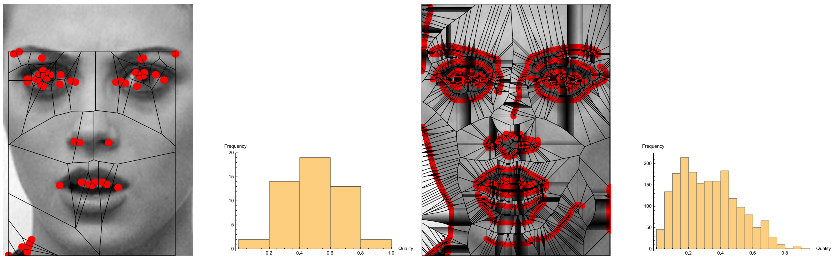
Remark 6.3. Corner-Based vs. Edge-Based Voronoï Tessellations.
The corner generators of Fig. 4.1 are concentrated in the mouth and eye regions of the subject. As a consequence, unequal distribution of the cell qualities in those regions result in Fig. 4.2. Edge-based cells on the other hand are distributed around the borders of the entire image giving peak cell quality in about mid range of the scale (see Fig. 4.3, Fig. 4.4). □

Remark 6.4. Dominant-Based vs. Keypoint-Based Voronoï Tessellations.
Dominant generators outweigh keypoints in number (see Fig. 5.1,Fig. 5.3). The generators are however concentrated around the mouth and eye regions but with extra points around the chin region in the case of dominant generators. Due to these distributions of the generators, both sets of generators have comparatively high qualities with peculiar quality factor distributions as shown in Fig. 5.2 and Fig. 5.4. □



5.1: Dominants 5.2: DomHist 5.3: Keypts 5.4: KeyptHist

Figure 5. Dominant & Keypoint Tessellations and Quality Histograms.



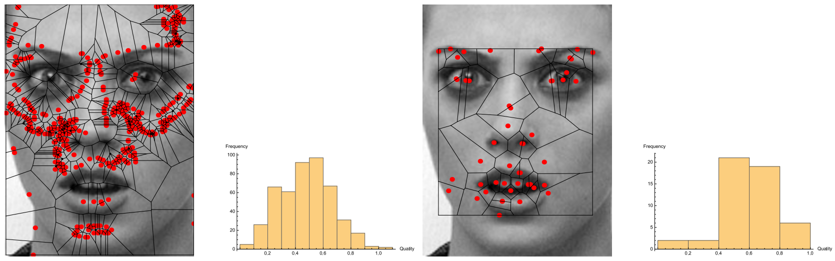
6.1: Corners 6.2: CorHist 6.3: Edges 6.4: EdHist

Figure 6. Corner & Edge Tessellations and Quality Histograms.

Remark 6.5. Corner-Based vs. Edge-Based Voronoï Tessellations.
Sets of generators of a subject are shown in Fig. 6.1 and Fig. 6.3. In addition to generators around the eye and mouth regions, generators around the nose and lower neck areas are returned for corner sites. Edges on the other hand are returned for regions of feature discontinuities. The quality plots of Fig. 6.2 and Fig. 6.4 represent the cells with more well laid out features on one hand and clustered generators on the other. □

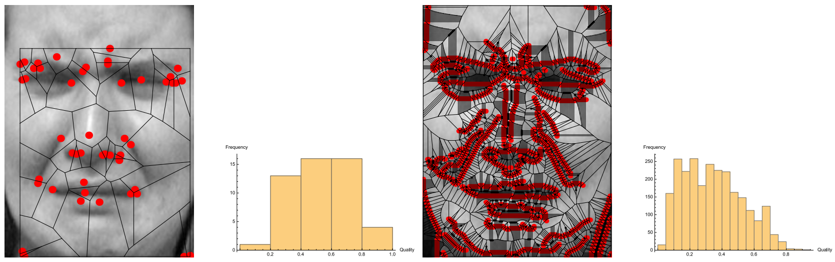
Remark 6.6. Dominant-Based vs. Keypoint-Based Voronoï Tessellations.
Dominant generators returned here exclude most of the keypoint generators (Fig. 7.1,Fig. 7.3). Also, keypoint generators are more localized as opposed to the more or less global distribution of dominant generators. These fundamentally different generators produced a somewhat flat distribution of cell qualities (taken in two halves) and an alternating distribution of qualities seen in Fig. 7.4 and Fig. 7.2 respectively. □

Remark 6.7. Corner-Based vs. Edge-Based Voronoï Tessellations.
Distinct generators are returned in the case of Fig. 8.1 as opposed to less distinct ones in Fig. 8.3. Although corner generators are smaller in number, they have covered a lot of distinct and important features in the image. Edge generators however are localized to boundary regions. Given the layout of generators, the corner generators favored creating meshes approaching perfect lengths than edge generators as seen in Fig. 8.2 and Fig. 8.4 respectively. □



7.1: Dominants 7.2: DomHist 7.3: Keypts 7.4: KeyptHist

Figure 7. Dominant & Keypoint Tessellations and Quality Histograms.



8.1: Corners 8.2: CorHist 8.3: Edges 8.4: EdHist

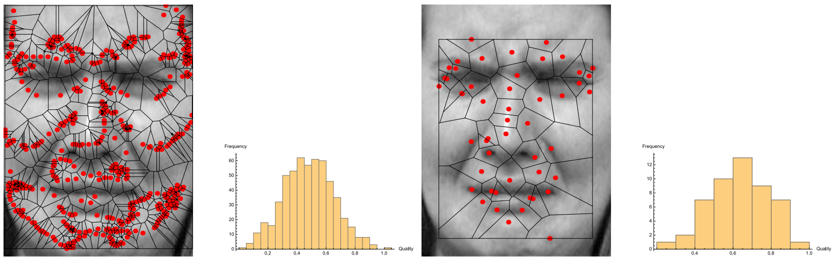
Figure 8. Corner & Edge Tessellations and Quality Histograms.

Remark 6.8. Dominant-Based vs. Keypoint-Based Voronoï Tessellations.
In Fig. 9.1 the layout of the generators does not favour polygons with equal lengths. This is evident in the fragmented nature of the quality factors in Fig. 9.2. Generators however in Fig. 9.3 performed better in their quality distributions in Fig. 9.4. □

Remark 6.9. Corner-Based vs. Edge-Based Voronoï Tessellations.
Examine for a brief moment the corner and edge generators in Fig. 10.1 and Fig. 10.3 alongside their quality factors in Fig. 10.2 and Fig. 10.4 respectively. Notice that some generators are clustered around the eye regions. The situation however is still better in terms of affording better overall quality as opposed to similar clustering of edge detectors in several areas. □

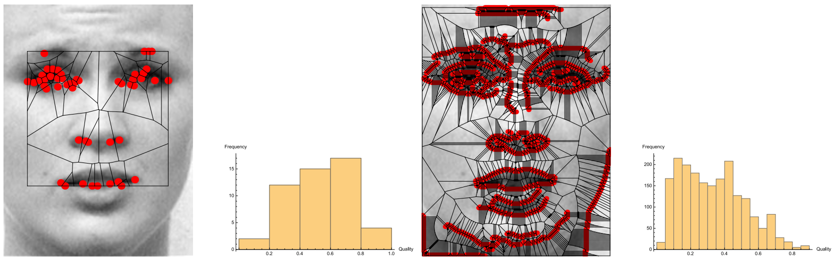
Remark 6.10. Dominant-Based vs. Keypoint-Based Voronoï Tessellations.
Dominants and their tessellations in Fig. 11.1 occupy a larger area compared to keypoints and their tessellations in Fig. 11.3. Although many cells result in Fig. 11.2, most of the quality factors are concentrated in the first half of the scale. Keypoints on the other hand are better laid out and although of a smaller number, they cover a comparable space and give a higher overall quality measure from Fig. 11.4. □

A complete set of results based on centroids of Voronoï regions specified by corner, edge, dominant and keypoint tessellations is shown in Fig. 12-Fig. 21. For those results shown, generators have been obtained corresponding to Voronoï regions of corner, edge, dominant and keypoint



9.1: Dominants 9.2: DomHist 9.3: Keypts 9.4: KeyptHist

Figure 9. Dominant & Keypoint Tessellations and Quality Histograms.



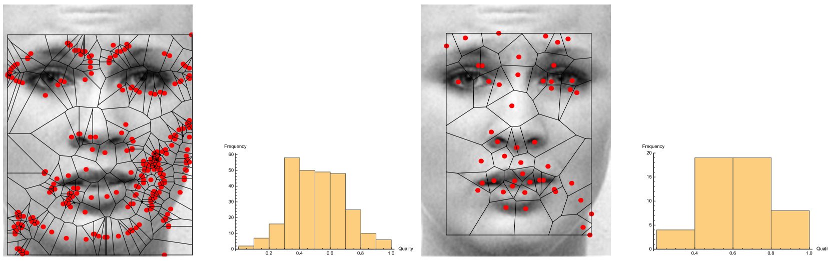
10.1: Corners 10.2: CorHist 10.3: Edges 10.4: EdHist

Figure 10. Corner & Edge Tessellations and Quality Histograms.

tessellations. These are the centroids of Voronoï regions in Fig. 2-Fig. 11. In the following text, remarks are included pertaining to the regions and their qualities.

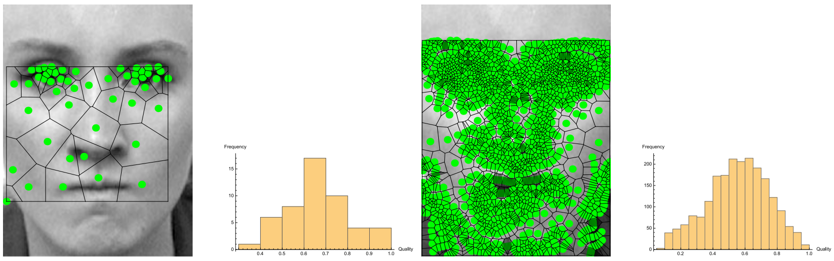
Remark 6.11. Corner centroid-Based vs. Edge centroid-Based Voronoï Tessellations.
In comparing Fig. 2.1 to Fig. 12.1 we notice that the numbers of generators is the same. However, an interesting situation arises. The polygons in the latter case have a reduced variability in their lengths. This led to better quality measures with quality distributions as in Fig. 12.2. In a similar vein the number of generators are the same for edge generators and edge-centroid generators. The overall quality has been improved from the neighborhoods of 0.3 to above 0.5 (Fig. 12.3,Fig. 12.4).
□

Remark 6.12. Dominant centroid-Based vs.Keypoint-Based Voronoï tessellations.
Notice that the locations of centroid generators are different from those of dominant generators. This distribution led to a mesh covering of about three quarters of the image as seen in Fig. 13.1. These generators favored better mesh qualities with the distribution seen in Fig. 13.2. Keypoint centroids of Fig. 13.3 and their tessellations however cover comparable areas with keypoint generators. The use of the centroid generators improved the mesh qualities as shown in Fig. 13.4.
□



11.1: Dominants 11.2: DomHist 11.3: Keypoints 11.4: KeyptHist

Figure 11. Dominant & Keypoint Tessellations and Quality Histograms.



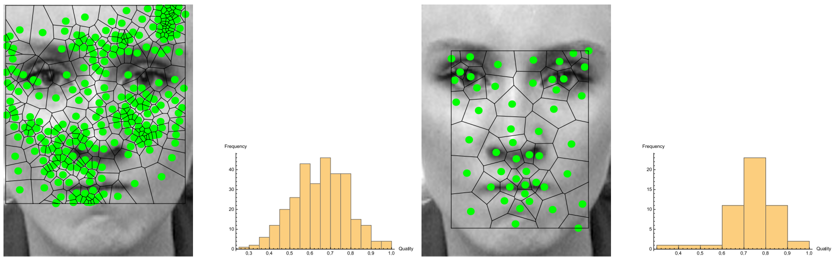
12.1: Corner-centroids 12.2: Cor-centHist 12.3: Edcen-troids 12.4: EdcentHist

Figure 12. Corner & Edge Centroid Tessellations and Quality Histograms.

Remark 6.13. Corner centroid-Based vs.Edge centroid-Based Voronoï Tessellations.
The centroid sites of Fig. 14.1 barely covered the eyes, nose and most of the mouth region. As expected, edge centroids cover and tessellate the entire image Fig. 14.3. Although the mesh qualities are improved, they are consistent with distribution of cell lengths in Fig. 14.2 and Fig. 14.3. □

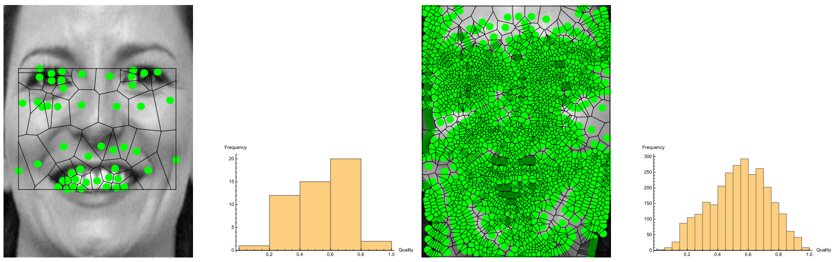
Remark 6.14. Dominant centroid-Based vs.Keypoint-Based Voronoï tessellations.
Observe in the tessellated spaces that the dominant centroid generators are mostly clustered in all areas except in the eyes and the mouth (see Fig. 15.1). Keypoint centroids however are distributed primarily around the facial features such as the mouth, nose and mouth as observed in Fig. 15.3. These generators are less clustered compared to their counterparts for dominant and keypoint generator tessellations. With the favorable condition for improved cell qualities obtained by using the centroids, the qualities are distributed across the entire quality scale as seen in Fig. 15.2 and Fig. 15.4. In most of the cases, the minimum cell quality for keypoint centroid-based generators is in the neighborhoods of 0.4-0.5. □

Remark 6.15. Corner centroid-Based vs.Edge centroid-Based Voronoï Tessellations.
Observe that several generators are located in the eye ball regions in Fig. 16.1. The tessellated regions cover up to the chin region. Less varying polygonal lengths led to distribution of mesh



13.1: Dominant-centroids 13.2: DomcentHist 13.3: Keyptcentroids 13.4: KeyptcentHist

Figure 13. Dominant & Keypoint Centroid Tessellations and Quality Histograms.



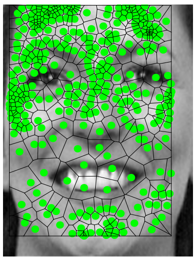
14.1: Cor-centroids 14.2: CorcentHist 14.3: Edgecentroids 14.4: EdcentHist

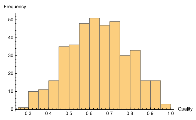
Figure 14. Corner & Edge Centroid Tessellations and Quality Histograms.

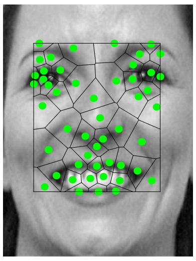
qualities in Fig. 16.2. Most of the image plane is however covered by edge centroid generators (Fig. 16.3). Although the quality factors are fragmented, they cover the entire scale with the distribution shown in Fig. 16.4 with minimum quality starting at about 0.3. □

Remark 6.16. Dominant centroid-Based vs.Keypoint-Based Voronoï tessellations.
Dominant generators cover most of the image plane. However, most of the generators tend to be concentrated just below the eyes and nose regions. Also notice that regions without clustering of the cells tend to be polygons whose lengths tend to be equal. This distribution of the generators affords cells and qualities in Fig. 17.1 and Fig. 17.2. Although there are keypoint centroid generators in the eye, nose and mouth regions, they are not clustered as in the previous case (see Fig. 17.3). They are better spaced out giving the qualities in Fig. 17.4. □

Remark 6.17. Corner centroid-Based vs.Edge centroid-Based Voronoï Tessellations.
Centroid corner generators tessellate the image region around the facial features. This placement of the generators yielded cells with flat distribution of qualities across the scale (see Fig. 18.1, Fig. 18.2). Edge centroid generators on the other hand tessellated the entire image but with the sites clustered together in most areas. Even though the whole image space is covered, the resulting cells do not promote better overall quality as compared with centroidal corner tessellations (see Fig. 18.3, Fig. 18.4). □


15.1: Dominant-centroids


15.2: Domcen-tHist


15.3: Keypoint-centroids

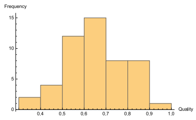
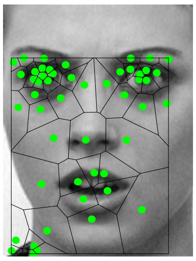
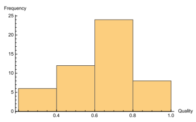
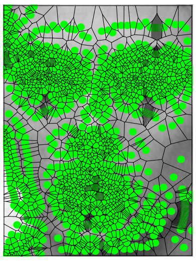

15.4: Keyptcen-tHist

Figure 15. Dominant & Keypoint Centroid Tessellations and Quality Histograms.


16.1: Corner-centroids


16.2: Cor-centHist


16.3: Edgecen-troids

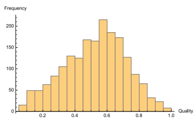
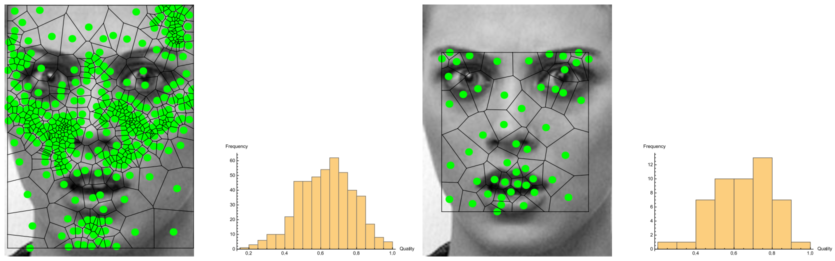

16.4: EdcentHist

Figure 16. Corner & Edge Centroid Tessellations and Quality Histograms.

Remark 6.18. Dominant centroid-Based vs.Keypoint-Based Voronoï tessellations.
Dominant generators tessellated most of the image with cells of improved qualities compared to previous situations (Fig. 19.1 and Fig. 19.2). The qualities of the cells cover the entire scale in both scenarios. However, you would notice that the cells of Fig. 19.3 are of better quality Fig. 19.4. □

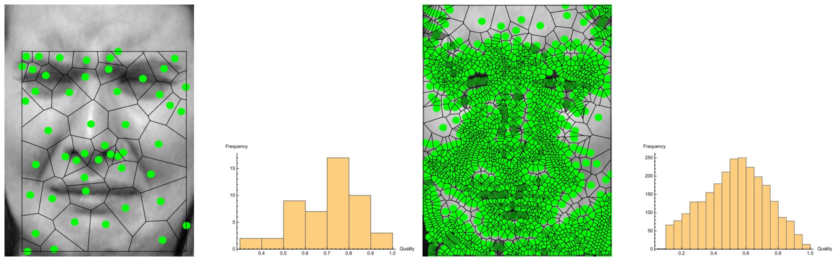
Remark 6.19. Corner centroid-Based vs.Edge centroid-Based Voronoï Tessellations.
Cell qualities cover the entire scale in Fig. 20.2 and Fig. 20.4. The difference however lies in the numbers of the generators and their positions as seen in Fig. 20.1 and Fig. 20.3. □

Remark 6.20. Dominant centroid-Based vs.Keypoint-Based Voronoï tessellations.
Although common generators are returned in Fig. 21.1 and Fig. 21.3, the concentration of points in the left cheek region and the lower neck region of the test subject favored better mesh qualities generation as seen in comparing Fig. 21.2 and Fig. 21.4. □



17.1: Dominant-centroids 17.2: Domcen-Hist 17.3: Keypoint-centroids 17.4: KeyptHist

Figure 17. Dominant & Keypoint Centroid Tessellations and Quality Histograms.

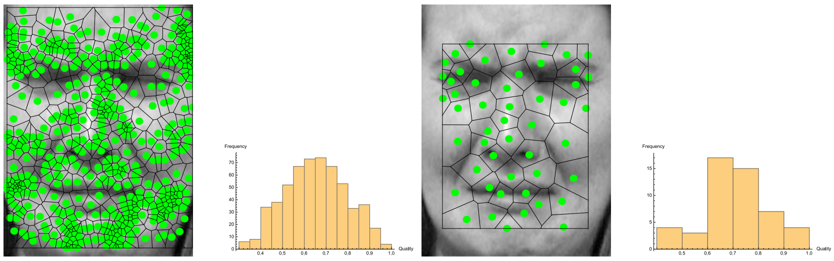


18.1: Corner-centroids 18.2: Cor-centHist 18.3: Edgecen-troids 18.4: EdcentHist

Figure 18. Corner & Edge Centroid Tessellations and Quality Histograms.

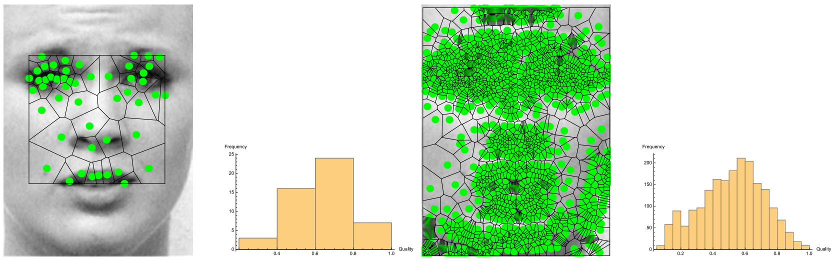
The overall quality measures of meshes for each test subject class based on corner, edge, dominant and keypoint sites is presented in Fig. 22. For several images of the same subject, q_{all} is computed for each set of generators. The plot therefore presented shows the relationships of sets of generators and the overall quality of meshes for tessellated images. In the plot, the trend indicates that for sets of generators and their tessellations, keypoints give the meshes with the highest qualities. This is due to the distribution of the sites in such a way that they tend to produce perfect polygons. Edge generators on the other hand consistently give low quality tessellations. This is the case because edge sites tend to be clustered together thus producing qualities on the lower side of the scale. The qualities of corner and dominant generators assume a place in between those of edge and keypoint tessellations. The qualities of the cells by sets of generators is in proportion to distributions that tend to give perfect polygons.
The method of centroids of regions defined by image centroids shows how mesh qualities may be improved (Fig. 23). In this figure, the qualities of the cells have been improved for all sets of generators. However, the order of mesh qualities has been preserved. This improvement results from the energy minimization property of centroids and the quasi perfect polygons centroids tend to produce.

Remark 6.21. What High Quality Meshes Reveal About Tessellated Images.
Sets of sites are used to generate a Voronoï diagram (also called a mesh) on a digital image. Each



19.1: Dominant-centroids 19.2: Domcen-tHist 19.3: Keypoint-centroids 19.4: Keyptcen-tHist

Figure 19. Dominant & Keypoint Centroid Tessellations and Quality Histograms.



20.1: Corner-centroids 20.2: Cor-centHist 20.3: Edgecen-troids 20.4: EdcentHist

Figure 20. Corner & Edge Centroid Tessellations and Quality Histograms.

region of a site is convex set represented geometrically by a polygon.

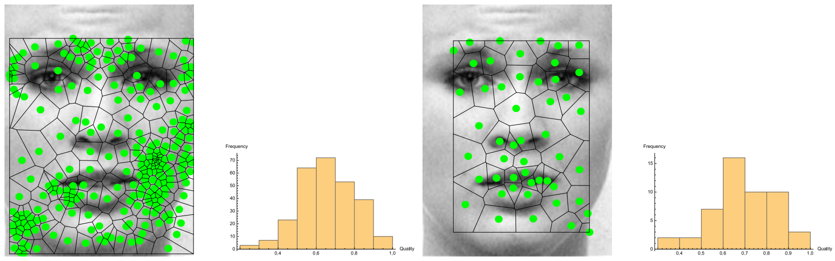
Associated with a set of sites are the qualities of the individual cells and their overall quality measure. So given sets of generators and overall tessellation qualities, the tessellation quality characterizes the underlying local structure of a collection of Voronoï regions. Quality of Voronoï polygons give us shape information about the region they cover. For the tessellated spaces, notice that the numbers of interior Voronoï polygons is greater than open border polygons. This shows that the mesh generation patterns are globular in nature. For example, the following quality expressions yielding $q = 1$ would indicate the presence of equilateral triangle and perfect quadrilateral respectively.

$$q = \frac{4\sqrt{3}A}{l_1^2 + l_2^2 + l_3^2}$$

$$q = \frac{4A}{l_1^2 + l_2^2 + l_3^2 + l_4^2}.$$

For small quality measurements as seen for edge point patterns sets, the measures are an indicator that the generators are on a curve and are closely spaced.

The qualities of cells and the overall quality of a tessellation characterizes the regularity and repeatability of a mesh generator set. If the space is covered with individual cells all of unit



21.1: Dominant-centroids

21.2: Domcen-tHist

21.3: Keypoint-centroids

21.4: Keyptcen-tHist

Figure 21. Dominant & Keypoint Centroid Tessellations and Quality Histograms.

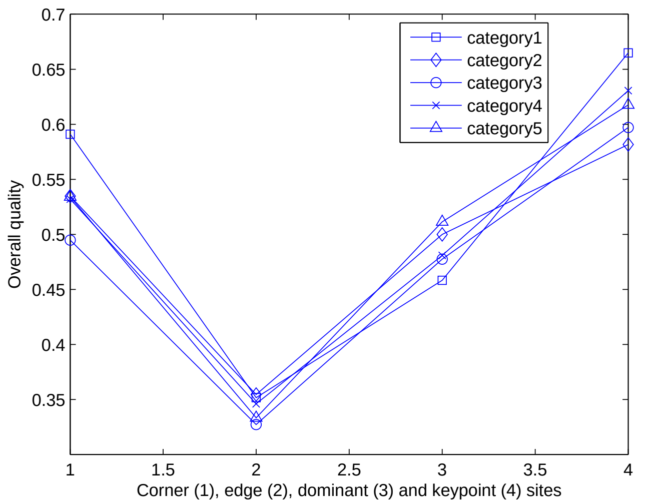


Figure 22. (a) Plot of overall quality factors against choice of sites for image categories.

quality, it indicates that the pattern points produce perfect polygons in the space. Associated with this is the simplicity of the design of the underlying pattern. Higher qualities indicate simple and predictable distributions whiles the converse holds for low qualities. This reveals the regularity of the points in the distribution of the pattern set. It also follows that the density of the points is uniform in the plane of the pattern space.

The quality of mesh cells and their associated image spaces give information on the separability of dot patterns. Generally, the higher the quality the greater the separation between points in the set. For example, the quality of edge generators is small compared to those of corners, keypoints and dominant generators and hence the separation of edge point pattern sets is poor compared to corner, keypoint and dominant pattern sets.

An extension of the separability of dot patterns and their associated factors is the notion of how adequately the point pattern represents the image space. For example if keypoints are used as mesh generators, then the generated mesh contains a distribution of polygons that surround objects in

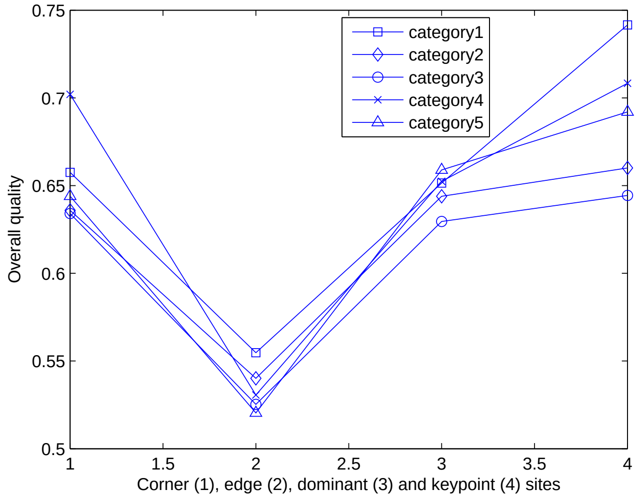
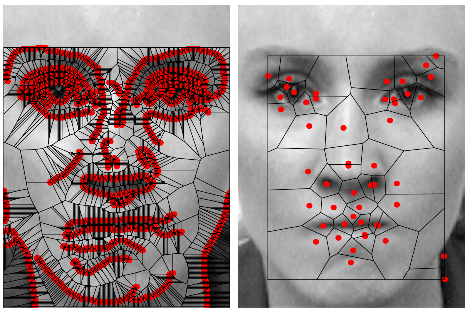


Figure 23. (a) Plot of overall quality factors against centroids of sites for image categories.

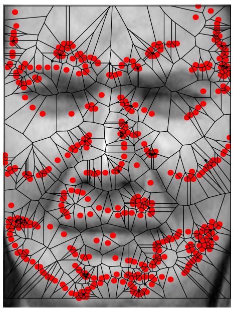
an image. In other words, the high quality of a keypoint-based mesh yields more information about image objects. To illustrate, edge tessellations give meshes with overall quality of 0.352 *versus* 0.665 *for keypoint tessellations in Fig. 24.1 and Fig. 24.2 respectively.*



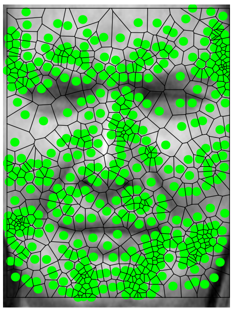
24.1: Sufficient1 24.2: Sufficient2

Figure 24. Meshes demonstrating sufficiency of coverings

Another important piece of information furnished by meshes relates to symmetry. The higher the quality, the greater the symmetry of image objects. High quality meshes tend to have connected sets of highly symmetrical polygons. To demonstrate symmetry, consider two generators $S_1(x_1, y_1)$ *and* $S_2(x_2, y_2)$ *on either side of a vertical line, y through a nose point. The generators* S_1 *and* S_2 *are symmetrical if and only if* $\|(x_1, y_1), y\| = \|(x_2, y_2), y\|$*. The mesh coverings in Fig. 25 are due to dominant generators and their centroids. If you draw a vertical line through the center of the nose region, you would notice that the generators in Fig. 25.2 demonstrate a better reflection of features than in Fig. 25.1. Symmetry thus furnishes us with a tool for feature location given features on one* *half of the space.*



25.1: Symmetry1



25.2: symmetry2

Figure 25. Symmetry of features

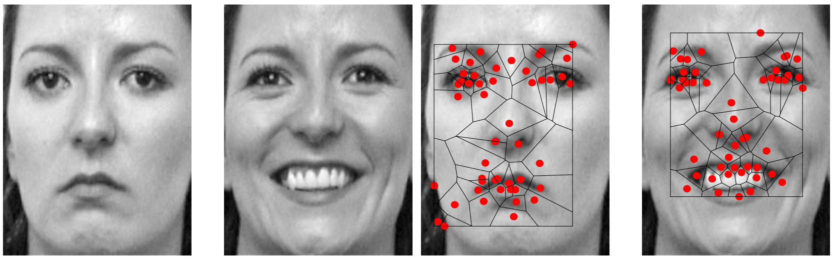
Last but not least, the quality of an image mesh covering may be used to estimate image quality. Two image quality assessment methods will be compared here: Image structural similarity index (SSIM) and image quality through Voronoï tessellations. SSIM compares normalized local pixel patterns (Wang et al., 2004). *For a signal pair x, y it is defined by* (Wang et al., 2004)

$$SSIM(x,y) = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}$$

In the definition above, the SSIM between x and y uses signal statistics; the mean values of the signals μ_x, μ_y, their variances σ_x^2, σ_y^2, cross correlation between signals σ_{xy} and constants C_1 and C_2.
Voronoï mesh image quality on the other hand is defined by using the geometry of the polygons enclosing image object points and regions in a tessellated space. Given the q measures of a tessellated image space, the image quality is defined using q_{all}.
Notice that $SSIM(x, y)$ uses the entire image spaces for image quality assessment and the images must be of the same size. Besides the size constraint, huge signal sizes can make it a computationally intensive approach. Voronoï analysis of image quality on the other hand uses a small set of the features used in SSIM. To demonstrate, four image signals and their mesh coverings are given in Fig. 26 and Fig. 27 respectively.

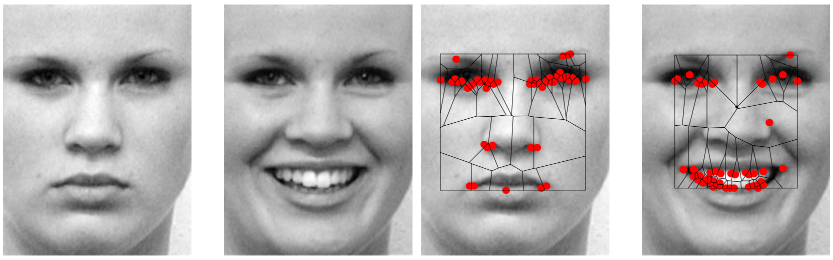
Table 1. SSIM and Quality Indexes

Image	SSIM	q_{all}
1	0.5569	0.562207
2	-	0.581664
3	0.5969	0.538463
4	-	0.538030



26.1: Image 1 26.2: Image 2 26.3: Meshed Covering 1 26.4: Meshed Covering 2

Figure 26. An image pair and their meshed domains for SSIM and quality comparisons



27.1: Image 3 27.2: Image 4 27.3: Meshed Covering 3 27.4: Meshed Covering 4

Figure 27. An image pair and their meshed domains for SSIM and quality comparisons

Image quality indicators of the signals in Fig. 26 and Fig. 27 is reported in Table. 1. Note that although the entire feature space is utilized in the calculation of the SSIM, they are comparable with those obtained with Voronoï tessellations. Notwithstanding, the Voronoï image quality method provides a quality indicator for each image whereas two signals are needed to output their SSIM. The ideal index possible is 1. *An image Voronoï index of 1 means that the point pattern spatial geometry or arrangement is perfectly regular while an SSIM of 1 would mean a perfect match between the two images.*

□

Note that there are differences between the SSIM and q_{all} indexes, although small. The differences stem from the fact that the mesh approach uses polygonal regions and their shape information to capture image quality, whilst the SSIM index depends on all pixel values in the image. Also, the SSIM index depends on constants, as it it affects the indicators. The mesh approach however depends on image features only.

Note that mesh qualities show that low overall quality generators are not sufficient descriptors of image features. On the other hand high mesh qualities indicate important and influential image features useful for image and mesh analysis. Also, note that the numbers of generator sites are different for corner, edge, dominant and keypoint sites. Even though edges tend to have the highest numbers, their qualities are very poor in comparison to the other generators. This shows that high quality meshes such as those generated by keypoints identify better the most influential, more well laid out and important features in image characterization and representation.

7. Conclusion

Voronoï generating points useful in image tessellations and visual image quality analysis have been identified. Previous works focused on generating points based on random distributions, sites without consideration of feature locations with scant attention given to resulting mesh quality. Various mathematical results pertaining to meshes, quality have been identified and proved. Centroidal tessellations have been used as a means to improve tessellation quality. This appears to be a new way of tessellation quality improvement as the literature hardly considers feature-based centroidal tessellations and resulting qualities. The measurement of image mesh quality offers a means of choosing suitable mesh generating points or sites based on their sufficiency in characterizing features of digital images. An important limitation of the model here is that slight perturbation of generators would mean changing locations leading to possibly different polygonal lengths and areas, hence the need to readjust or compute quality measures. Choosing a subset of the entire signal space for generators does not seem to be a significant limitation since it usually covers a significant portion of the space. However, we can always increase the numbers of generators to cover larger spaces although at higher computational costs.

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Initial Maclaurin Coefficients Bounds for New Subclasses of Bi-univalent Functions

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Abstract

In this work we introduce the subclasses $\mathcal{L}_\Sigma(\theta,\alpha)$ and $\mathcal{L}_\Sigma(\theta,\gamma)$ of bi-univalent functions. Furthermore, we obtain coefficient bounds involving the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes.The results presented in this paper would generalize those in related works of several earlier authors.

Keywords: Analytic and univalent functions, Bi-univalent functions, Starlike and convex functions, Coefficients bounds.
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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ with the conditions $f(0) = 0$ and $f'(0) = 1$ and having form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \qquad (z \in \mathcal{U}). \tag{1.1}$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

For each θ, $-\pi < \theta \leq \pi$, Silverman and Silvia (Silverman & Silvia, 1999) introduced the class

$$\mathcal{L}(\theta) = \left\{ f \in \mathcal{A}\colon \mathrm{Re}\left(f'(z) + \frac{1+e^{i\theta}}{2} zf''(z)\right) > 0, \qquad z \in \mathcal{U}\right\}$$

and they proved that $\mathcal{L}(\theta) \subset \mathcal{L}(\pi)$, where $\mathcal{L}(\pi)$ is the well known class $\mathcal{R}$ that consists of univalent functions in whose derivatives have positive real part in $\mathcal{U}$ (Alexander, 1915). The class $\mathcal{L}(0)$

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was studied by Singh and Singh (Singh & Singh, 1989), Lewandowski *et al.* (Lewandowski *et al.*, 1976), Chichra (Chichra, 1977), and Silverman (Silverman, 1994).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a brief history and interesting examples in the class Σ, see (Srivastava *et al.*, 2010).

Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively (see (Brannan & Taha, 1988)). Thus, following Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see (Brannan & Taha, 1988; Taha, 1981)).

Recently, Srivastava *et al.* (Srivastava *et al.*, 2010), Frasin (Frasin, 2014), Frasin and Aouf (Frasin & Aouf, 2011), Goyal and Goswami (Goyal & P.Goswami, 2012), Li and Wang (Li & Wang, 2012), Siregar and Raman (Siregar & Raman, 2012) and Caglar *et al.*(Caglar *et al.*, 2012) introduced new subclasses of bi-univalent functions and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these classes.

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

In order to establish our main results, we shall require the following lemma:

Lemma 1. *(Pommerenke, 1975) If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in $\mathcal{U}$ for which*

$$Re\,(p(z)) > 0,\;\; p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}).$$

2. Coefficient bounds for the function class $\mathcal{L}_\Sigma(\theta, \alpha)$

We now introduce the subclass $\mathcal{L}_\Sigma(\theta, \alpha)$ of the functions in the class $\mathcal{A}$ as follows.

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{L}_\Sigma(\theta, \alpha)$ *where* $0 < \alpha \leq 1$ and $\theta \in (-\pi, \pi]$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(f'(z) + \frac{1+e^{i\theta}}{2} z f''(z)\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathcal{U}) \tag{2.1}$$

and

$$\left|\arg\left(g'(w) + \frac{1+e^{i\theta}}{2} w g''(w)\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathcal{U}), \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2.3}$$

We first state and prove the following result.

Theorem 1. *Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(\theta, \alpha)$ where $0 < \alpha \leq 1$ and $\theta \in (-\pi, \pi]$. Then*

$$|a_2| \leq \frac{2\alpha}{[(3\alpha + 9 + (1-\alpha)\cos 2\theta + 6\cos\theta)^2 + ((1-\alpha)\sin 2\theta + 6\sin\theta)^2]^{1/4}} \tag{2.4}$$

and

$$|a_3| \leq \frac{2\alpha^2}{5 + 3\cos\theta} + \frac{2\alpha}{3\sqrt{5 + 4\cos\theta}}. \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$f'(z) + \left(\frac{1+e^{i\theta}}{2}\right) z f''(z) = [p(z)]^\alpha \tag{2.6}$$

and

$$g'(w) + \left(\frac{1+e^{i\theta}}{2}\right) w g''(w) = [q(w)]^\alpha \tag{2.7}$$

where $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots . \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(3 + e^{i\theta})a_2 = \alpha p_1, \tag{2.10}$$

$$3(2 + e^{i\theta})a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, \tag{2.11}$$

$$-(3 + e^{i\theta})a_2 = \alpha q_1 \tag{2.12}$$

and

$$3(2 + e^{i\theta})(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(3 + e^{i\theta})^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\begin{aligned} 6(2 + e^{i\theta})a_2^2 &= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) \\ &= \alpha(p_2 + q_2) + \frac{(\alpha - 1)(3 + e^{i\theta})^2}{\alpha} a_2^2. \end{aligned}$$

Thus

$$a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{6\alpha(2 + e^{i\theta}) - (\alpha - 1)(3 + e^{i\theta})^2}$$

that is

$$\left|a_2^2\right| = \frac{\alpha^2 \left|p_2 + q_2\right|}{\left|6\alpha(2 + e^{i\theta}) - (\alpha - 1)(3 + e^{i\theta})^2\right|}$$

Applying Lemma 1 for the coefficients p_2 and q_2, we have

$$|a_2| \le \frac{2\alpha}{[(3\alpha + 9 + (1 - \alpha)\cos 2\theta + 6\cos\theta)^2 + ((1 - \alpha)\sin 2\theta + 6\sin\theta)^2]^{1/4}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$6(2 + e^{i\theta})a_3 - 6(2 + e^{i\theta})a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2\right). \tag{2.16}$$

Upon substituting the value of a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = \frac{\alpha^2 p_1^2}{(3 + e^{i\theta})^2} + \frac{\alpha(p_2 - q_2)}{6(2 + e^{i\theta})}.$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{2\alpha^2}{5+3\cos\theta} + \frac{2\alpha}{3\sqrt{5+4\cos\theta}},$$

which completes the proof of Theorem 1. □

Choosing $\theta = \pi$ in Theorem 1, we obtain the following particular case due to Srivastava *et al.*(Srivastava *et al.*, 2010):

Corollary 2.1. *(Srivastava* et al.*, 2010) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(\pi,\alpha)$; $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+1}} \tag{2.17}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}. \tag{2.18}$$

Putting $\theta = 0$ in Theorem 1, we obtain the following particular case due to Frasin (Frasin, 2014):

Corollary 2.2. *(Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\alpha)$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+8}} \tag{2.19}$$

and

$$|a_3| \leq \frac{9\alpha^2+8\alpha}{36}. \tag{2.20}$$

3. Coefficient bounds for the function class $\mathcal{L}_\Sigma(\theta,\gamma)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{L}_\Sigma(\theta,\gamma)$ *where* $0 \leq \gamma < 1$, $\theta \in (-\pi,\pi]$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \operatorname{Re}\left(f'(z) + \frac{1+e^{i\theta}}{2} z f''(z)\right) > \gamma \qquad (z \in \mathcal{U}) \tag{3.1}$$

and

$$\operatorname{Re}\left(g'(w) + \frac{1+e^{i\theta}}{2} w g''(w)\right) > \gamma \qquad (w \in \mathcal{U}), \tag{3.2}$$

where the function g is given by (2.3).

Theorem 2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(\theta,\gamma)$, where $0 \leq \gamma < 1, \theta \in (-\pi,\pi]$. Then*

$$|a_2| \leq \sqrt{\frac{4(1-\gamma)}{6\sqrt{5+4\cos\theta}}} \tag{3.3}$$

and

$$|a_3| \leq \frac{2(1-\gamma)^2}{5+3\cos\theta} + \frac{2(1-\gamma)}{3\sqrt{5+4\cos\theta}}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist p and $q \in \mathcal{P}$ such that

$$f'(z) + \left(\frac{1+e^{i\theta}}{2}\right) zf''(z) = \gamma + (1-\gamma)p(z) \tag{3.5}$$

and

$$g'(w) + \left(\frac{1+e^{i\theta}}{2}\right) wg''(w) = \gamma + (1-\gamma)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(3+e^{i\theta})a_2 = (1-\gamma)p_1, \tag{3.7}$$

$$3(2+e^{i\theta})a_3 = (1-\gamma)p_2, \tag{3.8}$$

$$-(3+e^{i\theta})a_2 = (1-\gamma)q_1 \tag{3.9}$$

and

$$3(2+e^{i\theta})(2a_2^2 - a_3) = (1-\gamma)q_2 \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(3+e^{i\theta})^2 a_2^2 = (1-\gamma)^2(p_1^2 + q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$6(2+e^{i\theta})a_2^2 = (1-\gamma)(p_2+q_2).$$

Thus, we have

$$\begin{aligned} \left|a_2^2\right| &\leq \frac{(1-\gamma)}{6\,|2+e^{i\theta}|}(|p_2| + |q_2|) \\ &\leq \frac{4(1-\gamma)}{6\sqrt{5+4\cos\theta}} \end{aligned}$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$6(2+e^{i\theta})a_3 - 6(2+e^{i\theta})a_2^2 = (1-\gamma)(p_2-q_2)$$

or, equivalently,

$$a_3 = a_2^2 + \frac{(1-\gamma)(p_2-q_2)}{6(2+e^{i\theta})}.$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{(1-\gamma)^2(p_1^2+q_1^2)}{2(3+e^{i\theta})^2} + \frac{(1-\gamma)(p_2-q_2)}{6(2+e^{i\theta})}.$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{2(1-\gamma)^2}{5+3\cos\theta} + \frac{2(1-\gamma)}{3\sqrt{5+4\cos\theta}}$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Choosing $\theta = \pi$ in Theorem 2, we obtain the following particular case due to Srivastava *et al.*(Srivastava *et al.*, 2010):

Corollary 3.1. *(Srivastava et al., 2010) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\gamma)$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\gamma)}{3}} \tag{3.13}$$

and

$$|a_3| \le \frac{(1-\gamma)(5-3\gamma)}{3}. \tag{3.14}$$

Putting $\theta = 0$ in Theorem 2, we obtain the following particular case due to Frasin (Frasin, 2014):

Corollary 3.2. *(Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0,\gamma)$, $0 \le \gamma < 1$. Then*

$$|a_2| \le \frac{1}{3}\sqrt{2(1-\gamma)} \tag{3.15}$$

and

$$|a_3| \le \frac{(1-\gamma)(9(1-\gamma)+8)}{36}. \tag{3.16}$$

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Zweier I-Convergent Double Sequence Spaces Defined by a Sequence of Modulii

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Abstract

In the present article we have introduced the double sequence spaces ${}_2\mathcal{Z}^I(F)$, ${}_2\mathcal{Z}^I_0(F)$ and ${}_2\mathcal{Z}^I_\infty(F)$ for a sequence of modulii $F = (f_{ij})$. We have also studied their topological as well as algebraic properties.

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2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \ or \ \mathbb{C}\}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively normed by $\|x\|_\infty = \sup_k |x_k|$.

The concept of statistical convergence was first introduced by Fast (Fast, 1951) in 1951 and also independently by Buck (Buck, 1953) and Schoenberg (Schoenberg, 1959) for real and complex sequences. Further this concept was studied by Connor (Connor, 1998, 1989; Connor & Kline, 1996), Connor, Fridy and Kline (Connor *et al.*, 1994) and many others. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. A sequence $x = (x_k)$ is said to be Statistically convergent to L if for a given $\epsilon > 0$

$$\lim_k \frac{1}{k} |\{i : |x_i - L| \geq \epsilon, i \leq k\}| = 0.$$

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Later on it was studied by Fridy (Fridy, 1985, 1993) from the sequence space point of view and he linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, Wilczyński (Kostyrko *et al.*, 2000). Later on it was studied by Šalát, Tripathy, Ziman (Šalát *et al.*, 2004) and Demirci (Demirci, 2001).

Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. A set I $\subseteq 2^X$(2^X denoting the power set of X) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if $\phi \notin \mathcal{F}$, for $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$.

An Ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I, i.e $\mathcal{F}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$ is called a sequence space. A sequence space λ with linear topology is called a K-space provided each of maps $p_i \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an FK-space provided λ is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ and we denote it by writing $A : \lambda \longrightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$.

Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar and Mursaleen (Altay *et al.*, 2006), Başar and Altay (Başar & Altay, 2003), Malkowsky (Malkowsky, 1997), Ng and Lee (Ng & Lee, 1978) and Wang (Wang, 1978). Şengönül (Şengönül, 2007) defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, (i = k), \\ 1 - p, (i - 1 = k); (i, k \in \mathbb{N}), \\ 0, \text{otherwise}. \end{cases}$$

Following Basar and Altay (Başar & Altay, 2003), Şengönül Şengönül (2007) introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}$$

Definition 1.1. (Khan & Khan, 2013) A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}), where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \epsilon, \qquad for\ all\ i, j \geq N$$

Definition 1.2. A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij} x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.3. E is said to be monotone if it contains the canonical preimages of all its stepspaces.

Definition 1.4. E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 1.5. E is said to be a sequence algebra if $(x_{ij} y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I$. In this case we write I-lim $x_{ij} = L$. The space ${}_2c^I$ of all I-convergent double sequences to L is given by

$${}_2c^I = \{(x_k) \in \omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I, \text{ for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_{ij}) \in \omega$ is said to be I-null if $L = 0$. In this case we write I-lim $x_k = 0$.

Definition 1.8. A sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m, n dependent on ϵ such that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I$.

Definition 1.9. A sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\} \in I$.

Definition 1.10. A modulus function f is said to satisfy $\triangle_2$ condition if for all values of u there exists a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.

Definition 1.11. Take for I the class I_f of all finite subsets of $\mathbb{N}$.Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.

Definition 1.12. For I= I_δ and $A \subset \mathbb{N} \times \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal, I_δ-convergence is said to be logarithmic statistical convergence.

Definition 1.13. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 1.14. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_{ij}) \in {}_2\ell_\infty : \text{there exists } I - \lim x \in \mathbb{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of ${}_2\ell_\infty$ with respect to the supremum norm, $F(I) = {}_2\ell_\infty \cap {}_2c^I$. Define a function $\hbar : F(I) \to \mathbb{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \mathbb{R}$ is a Lipschitz function. The following Lemmas will be used for establishing some results of this article.

Lemma 1. *Let E be a sequence space. If E is solid then E is monotone.*

Lemma 2. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I. (Tripathy & Hazarika, 2011).*

Lemma 3. *If $I \subset 2^N$ and M⊆N. If M ∉I, then M∩N ∉I. (Tripathy & Hazarika, 2011).*

The idea of modulus was structured in 1953 by Nakano (Nakano, 1953).
A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is nondecreasing, and
(4) f is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967)used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle (Ruckle, 1973) proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (Ruckle, 1973) proved that, for any modulus f:

$$X(f) \subset \ell_1 \text{ and } X(f)^\alpha = \ell_\infty$$

where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling (Garling, 1966), Köthe (Köthe, 1970) and many more. After then Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

Recently Khan et al (Khan *et al.*, 2013), introduced the following classes of sequences

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$
$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$
$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f).$$

In this article we introduce the following sequence spaces.

$${}_2\mathcal{Z}^I(F) = \{(x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L|) \geq \varepsilon, \text{ for some L} \in \mathbb{C}\ \} \in I\},$$
$${}_2\mathcal{Z}_0^I(F) = \{(x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij}|) \geq \varepsilon\} \in I\},$$
$${}_2\mathcal{Z}_\infty^I(F) = \{(x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij}|) \geq M, \text{ for each fixed M>0}\} \in I\}.$$

We also denote by

$${}_2m_{\mathcal{Z}}^I(F) = {}_2\mathcal{Z}_\infty(F) \cap {}_2\mathcal{Z}^I(F)$$

and

$${}_2m_{\mathcal{Z}_0}^I(F) = {}_2\mathcal{Z}_\infty(F) \cap {}_2\mathcal{Z}_0^I(F).$$

2. Main Results

Theorem 1. *For a sequence of modulii $F = (f_{ij})$, the classes of sequences ${}_2\mathcal{Z}^I(F)$, ${}_2\mathcal{Z}_0^I(F)$, ${}_2m_{\mathcal{Z}}^I(F)$ and ${}_2m_{\mathcal{Z}_0}^I(F)$ are linear spaces.*

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I(F)$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I(F)$ and let α, β be scalars. Then

$$I - \lim f_{ij}(|x_{ij} - L_1|) = 0, \text{ for some} L_1 \in \mathbb{C};$$
$$I - \lim f_{ij}(|y_{ij} - L_2|) = 0, \text{ for some} L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{2.1}$$

$$A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|y_{ij} - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{2.2}$$

Since f_{ij} is a modulus function, we have

$$\begin{aligned} f_{ij}(|(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2) &\leq f_{ij}(|\alpha||x_{ij} - L_1|) + f_{ij}(|\beta||y_{ij} - L_2|) \\ &\leq f_{ij}(|x_{ij} - L_1|) + f_{ij}(|y_{ij} - L_2|) \end{aligned}$$

Now, by (2.1) and (2.2), $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I(F)$. Hence ${}_2\mathcal{Z}^I(F)$ is a linear space. □

We state the following result without proof in view of Theorem 2.1.

Theorem 2. *The spaces* ${}_2m^I_{\mathcal{Z}}(F)$ *and* ${}_2m^I_{\mathcal{Z}_0}(F)$ *are normed linear spaces, normed by*

$$\|x_{ij}\|_* = \sup_{i,j} f_{ij}(|x_{ij}|). \tag{2.3}$$

Theorem 3. *A sequence* $x = (x_{ij}) \in {}_2m^I_{\mathcal{Z}}(F)$ *I-converges if and only if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}\times\mathbb{N}$ *such that*

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I). \tag{2.4}$$

Proof. Suppose that $L = I - \lim x$. Then $B_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : |x_{ij} - L| < \frac{\epsilon}{2}\} \in \mathcal{F}(I)$. For all $\epsilon > 0$, fix an $N_\epsilon \in B_\epsilon$ such that we have $|x_{N_\epsilon} - x_{ij}| \leq |x_{N_\epsilon} - L| + |L - x_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $(i,j) \in B_\epsilon$. Hence $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$.

Conversely, suppose that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$. That is $\{(i,j) \in \mathbb{N}\times\mathbb{N} : (|x_{ij} - x_{N_\epsilon}|) < \epsilon\} \in \mathcal{F}(I)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in \mathcal{F}(I)$ for all $\epsilon > 0$.

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in \mathcal{F}(I)$ as well as $C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$.

Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}(F)$. This implies that $J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in J\} \in \mathcal{F}(I)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J.

In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_{ij} \supseteq ...$ with the property that $diamI_{ij} \leq \frac{1}{2}diamI_{(i-1)(j-1)}$ for (i,j=2,3,4, ...) and $\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in {}_2m^I_{\mathcal{Z}}(F)\} \in I_{ij}$ for (i,j=1,2,3,4, ...). Then there exists a $\xi \in \cap I_{ij}$ where $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $\xi = I - \lim x$. So that $f_{ij}(\xi) = I - \lim f_{ij}(x)$, that is $L = I - \lim f_{ij}(x)$. □

Theorem 4. *Let* (f_{ij}) *and* (g_{ij}) *be modulus functions for some fixed k that satisfy the* $\triangle_2$*-condition. If X is any of the spaces* ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0$, ${}_2m^I_{\mathcal{Z}}$ *and* ${}_2m^I_{\mathcal{Z}_0}$ *etc., then the following assertions hold.*
(a)$X(g_{ij}) \subseteq X(f_{ij}.g_{ij})$,
(b)$X(f_{ij}) \cap X(g_{ij}) \subseteq X(f_{ij} + g_{ij})$

Proof. (a) Let $(x_{mn}) \in {}_2\mathcal{Z}^I_0(g_{ij})$. Then

$$I - \lim_{m,n} g_{ij}(|x_{mn}|) = 0. \tag{2.5}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_{ij}(t) < \epsilon$ for $0 < t < \delta$. Write $y_{mn} = g_{ij}(|x_{mn}|)$ and consider

$$\lim_{m,n} f_{ij}(y_{mn}) = \lim_{m,n} f_{ij}(y_{mn})_{y_{mn}<\delta} + \lim_{m,n} f_{ij}(y_{mn})_{y_{mn}\geq\delta} \tag{2.6}$$

Now for $y_{mn} < \delta$, we already have $\lim_{m,n} f_{ij}(y_{mn}) < \epsilon$. For $y_{mn} \geq \delta$, we have $y_{mn} < \frac{y_{mn}}{\delta} < 1 + \frac{y_{mn}}{\delta}$.
Since f_{ij} is non-decreasing, it follows that $f_{ij}(y_{mn}) < f_{ij}(1 + \frac{y_{mn}}{\delta}) < \frac{1}{2}f_{ij}(2) + \frac{1}{2}f_{ij}(\frac{2y_{mn}}{\delta})$

Since f_{ij} satisfies the $\triangle_2$-condition, therefore for $y_{mn} \geq \delta > 0$ we can choose some $K > 0$ such that $f_{ij}(y_{mn}) < \frac{1}{2}K\frac{y_{mn}}{\delta}f_{ij}(2) + \frac{1}{2}K\frac{y_{mn}}{\delta}f_{ij}(2) = K\frac{y_{mn}}{\delta}f_{ij}(2)$

Hence $\lim_{m,n} f_{ij}(y_{mn}) \leq \max(1,K)\delta^{-1}f_{ij}(2)\lim_{m,n}(y_{mn}) = \epsilon'(say)$. Substituting in equation (2.6), we get

$$\lim_{m,n} f_{ij}(y_{mn}) = \epsilon + \epsilon'. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $I - \lim_{m,n} f_{ij}(g_{ij}(|x_{mn}|)) = 0$.

Hence $(x_{mn}) \in {}_2\mathcal{Z}_0^I(f_{ij}.g_{ij})$. Thus ${}_2\mathcal{Z}_0^I(g_{ij}) \subseteq {}_2\mathcal{Z}_0^I(f_{ij}.g_{ij})$. The other cases can be proved similarly.

(b) Let $(x_{mn}) \in {}_2\mathcal{Z}_0^I(f_{ij}) \cap {}_2\mathcal{Z}_0^I(g_{ij})$. Then $I - \lim_{m,n} f_{ij}(|x_{mn}|) = 0$ and $I - \lim_{m,n} g_{ij}(|x_{mn}|) = 0$

The rest of the proof follows from the following equality $\lim_{m,n}(f_{ij} + g_{ij})(|x_{mn}|) = \lim_{m,n} f_{ij}(|x_{mn}|) + \lim_{m,n} g_{ij}(|x_{mn}|)$. □

Corollary 2.1. *$X \subseteq X(f_{ij})$ for some fixed (i, j) and $X = {}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}_0^I$, ${}_2m_{\mathcal{Z}}^I$ and ${}_2m_{\mathcal{Z}_0}^I$.*

Theorem 5. *The spaces ${}_2\mathcal{Z}_0^I(F)$ and ${}_2m_{\mathcal{Z}_0}^I(F)$ are solid and monotone .*

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I(F)$.

Let $(x_{ij}) \in {}_2\mathcal{Z}_0^I(F)$. Then

$$I - \lim_{(i,j)} f_{ij}(|x_{ij}|) = 0. \tag{2.8}$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from (2.8) and inequality $f_{ij}(|\alpha_{ij}x_{ij}|) \leq |\alpha_{ij}|f_{ij}(|x_{ij}|) \leq f_{ij}(|x_{ij}|)$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. The space ${}_2\mathcal{Z}_0^I(F)$ is monotone follows from Lemma 1. For ${}_2m_{\mathcal{Z}_0}^I(F)$ the result can be proved similarly. □

Theorem 6. *The spaces ${}_2\mathcal{Z}^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are sequence algebras.*

Proof. We prove the result for ${}_2\mathcal{Z}_0^I(F)$. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I(F)$. Then

$$I - \lim f_{ij}(|x_{ij}|) = 0$$

and

$$I - \lim f_{ij}(|y_{ij}|) = 0$$

Therefore, we have

$$I - \lim f_{ij}(|(x_{ij}.y_{ij})|) = 0.$$

Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I(F)$ and hence ${}_2\mathcal{Z}_0^I(F)$ is a sequence algebra. In a similar way we can prove the result for the space ${}_2\mathcal{Z}^I(F)$. □

Theorem 7. *The spaces ${}_2\mathcal{Z}^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are not convergence free in general.*

Proof. Here we give a counter example. Let $I = I_f$ and $f_{ij}(x) = x^3$ for some fixed (i,j) and for all $x = (x_{mn}) \in [0, \infty)$. Consider the sequence (x_{mn}) and (y_{mn}) defined by

$$x_{mn} = \frac{1}{m+n} \quad \textit{and} \;\; y_{mn} = m + n \;\text{ for all } (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Then $(x_{mn}) \in {}_2\mathcal{Z}^I(F) \cap {}_2\mathcal{Z}_0^I(F)$, but $(y_{mn}) \notin {}_2\mathcal{Z}^I(F) \cap {}_2\mathcal{Z}_0^I(F)$. Hence the spaces ${}_2\mathcal{Z}_0^I(F)$ and ${}_2\mathcal{Z}_0^I(F)$ are not convergence free. □

Theorem 8. ${}_2\mathcal{Z}_0^I(F) \subset {}_2\mathcal{Z}^I(F) \subset {}_2\mathcal{Z}_\infty^I(F)$.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I(F)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_{ij}(|x_{ij} - L|) = 0$. We have

$$f_{ij}(|x_{ij}|) \leq \frac{1}{2} f_{ij}(|x_{ij} - L|) + \frac{1}{2} f_{ij}(|L|).$$

Taking the supremum over (i, j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}^I_\infty(F)$. The inclusion ${}_2\mathcal{Z}^I_0(F) \subset {}_2\mathcal{Z}^I(F)$ is obvious. □

Theorem 9. *The function* $\hbar : {}_2m^I_{\mathcal{Z}}(F) \to \mathbb{R}$ *is the Lipschitz function, where* ${}_2m^I_{\mathcal{Z}}(F) = {}_2\mathcal{Z}^I_\infty(F) \cap {}_2\mathcal{Z}^I(F)$*, and hence uniformly continuous.*

Proof. Let $x = (x_{ij}), y = (_{ij}) \in {}_2m^I_{\mathcal{Z}}(F)$, $x \neq y$.
Then the sets

$$A_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| \geq \|x - y\|_*\} \in I,$$
$$A_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \|x - y\|_*\} \in \mathcal{F}(I),$$
$$B_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \|x - y\|_*\} \in \mathcal{F}(I).$$

Hence also $B = B_x \cap B_y \in m^I{}_{{}_2\mathcal{Z}}(F)$, so that $B \neq \phi$.
Now taking $(i, j) \in$ we have, B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y_{ij} - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus $\hbar$ is a Lipschitz function.
For the space ${}_2m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly. □

Theorem 10. *If* $x, y \in {}_2m^I_{\mathcal{Z}}(F)$*, then* $(x.y) \in {}_2m^I_{\mathcal{Z}}(F)$ *and* $\hbar(xy) = \hbar(x)\hbar(y)$*.*

Proof. For $\epsilon > 0$

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < \epsilon\} \in \mathcal{F}(I),$$
$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < \epsilon\} \in \mathcal{F}(I).$$

Now,

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| = |x_{ij}y_{ij} - x_{ij}\hbar(y) + x_{ij}\hbar(y) - \hbar(x)\hbar(y)| \leq |x_{ij}||y_{ij} - \hbar(y)| + |\hbar(y)||x_{ij} - \hbar(x)| \quad (2.9)$$

As ${}_2m^I_{\mathcal{Z}}(F) \subseteq {}_2\mathcal{Z}^I_\infty(F)$, there exists an $M \in \mathbb{R}$ such that $|x_{ij}| < M$ and $|\hbar(y)| < M$.
Using equation (2.9), we get

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i, j) \in B_x \cap B_y \in {}_2m^I(F)$.
Hence $(x.y) \in {}_2m^I_{\mathcal{Z}}(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For the space ${}_2m^I_{\mathcal{Z}_0}(F)$ the result can be proved similarly. □

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Some Families of q-Series Identities and Associated Continued Fractions

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Abstract

In this paper, by using some known q-identities, the authors derive several results involving q-series and associated continued fractions. Some other closely-related q-identities are also considered.

Keywords: q-Series, q-Identities, q-Series identities, Rogers-Ramanujan identities, Continued fractions.
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1. Introduction, Definitions and Notations

For $q, \lambda, \mu \in \mathbb{C}$ $(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, (Slater, 1966); see also the recent works (Cao & Srivastava, 2013), (Choi & Srivastava, 2014), (Srivastava, 2011), (Srivastava & Choi, 2012) and (Srivastava & Karlsson, 1985) dealing with the q-analysis)

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \qquad (|q| < 1;\ \lambda, \mu \in \mathbb{C}), \tag{1.1}$$

so that

$$(\lambda; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

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and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty}\left(1-\lambda\, q^j\right) \qquad (|q|<1;\ \lambda\in\mathbb{C}), \tag{1.3}$$

where, as usual, $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{N}$ denotes the set of positive integers (*with* $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$). For convenience, we write

$$(a_1,\cdots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n \tag{1.4}$$

and

$$(a_1,\cdots,a_k;q)_\infty = (a_1;q)_\infty\cdots(a_k;q)_\infty. \tag{1.5}$$

In the literature on q-series, there usually are two types of identities as follows:

Type 1. Series = Product

and

Type 2. Series = Series.

The most famous identities of Type 1 are the following Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.7}$$

The identities (1.6) and (1.7) have a remarkably fascinating history. They were first proved in 1894 by Rogers (Rogers, 1894), but his paper was completely overlooked. They were rediscovered (*without any published proof*) by Ramanujan sometime before 1913. These identities were discovered again in 1917 and proved independently by Schur (Schur, 1973).

There are numerous q-identities that are similar to the Rogers-Ramanujan identities (1.6) and (1.7). These include (for example) the q-identities due to Jackson (Jackson, 1928), Rogers (see (Rogers, 1894) and (Rogers, 1917)), Bailey (see (Bailey, 1947) and (Bailey, 1949)), and Slater (Slater, 1952) (see also (McLaughlin & Sills, 2009)). In particular, Slater's paper (Slater, 1952) contains a list of 130 q-identities of the Rogers-Ramanujan type. On the other hand, in terms of continued fractions, Ramanujan stated for $|q|<1$ that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}} = 1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots. \tag{1.8}$$

There are numerous q-identities of Type 2 in the 'Lost' Notebook of Ramanujan (see (Ramanujan, 1988)) and also in other places in the literature on q-series. Our aim in this paper is to consider various q-identities of Type 2 in order to establish a number of results involving continued fractions of the form involved in (1.8).

2. A Set of Main Results

In this section, we propose to derive continued-fraction expressions for the quotients of the series involved in some known q-identities.

First of all, we consider the following identity (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.10)) *and* (McLaughlin *et al.*, 2008, p. 41, Eq. (6.1.7))):

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}. \tag{2.1}$$

and its companion identity given by (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.11)) *and* (McLaughlin *et al.*, 2008, p. 41, Eq. (6.1.8)))

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma/q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}. \tag{2.2}$$

I. We now investigate the quotient of the right-hand sides of (2.1) and (2.2) as follows:

$$\begin{aligned}
\frac{(\gamma/q;q^2)_\infty}{(\gamma q;q^2)_\infty} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}} &= \cfrac{1-\frac{\gamma}{q}}{\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}} \\
&= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n} - \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}} \\
&= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{(-\gamma)^n q^{n(n-2)}\,(1-q^n)}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}}
\end{aligned}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{\displaystyle\sum_{n=1}^{\infty}\frac{(-\gamma)^n q^{n(n-2)}}{(q;q)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}}$$

$$= \cfrac{1-\frac{\gamma}{q}}{1+\cfrac{(-\gamma/q)}{\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n(n-1)}}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-\gamma)^n q^{n^2}}{(q;q)_n}}}}. \tag{2.3}$$

Proceeding in the same way, we find that

$$\frac{(\gamma/q;q^2)_\infty}{(\gamma q;q^2)_\infty}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n-2)}(-\gamma)^n}{(q;q)_n}} = \frac{1-\frac{\gamma}{q}}{1-}\;\frac{(\gamma/q)}{1-}\;\frac{\gamma}{1-}\;\frac{\gamma q}{1-}\;\frac{\gamma q^2}{1-}\;\frac{\gamma q^3}{1-}\;\cdots. \tag{2.4}$$

From (2.1), (2.2) and (2.4), we have

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n}} = \frac{1-\frac{\gamma}{q}}{1-}\;\frac{(\gamma/q)}{1-}\;\frac{\gamma}{1-}\;\frac{\gamma q}{1-}\;\frac{\gamma q^2}{1-}\;\frac{\gamma q^3}{1-}\;\cdots. \tag{2.5}$$

The following special cases and consequences of (2.5) are worthy of note. Firstly, upon setting $\gamma = -q$ in (2.5), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n}} = \frac{2}{1+}\;\frac{1}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\cdots, \tag{2.6}$$

which, in light of the known result (Andrews & Berndt, 2005, p. 87, Entry (3.2.3)), yields

$$\frac{2(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n} = 1+\frac{1}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\cdots. \tag{2.7}$$

If we use another known result (Andrews & Berndt, 2005, p. 153, Corollary (6.2.6)) in (2.7), we obtain

$$2\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q^2)_n(q^2;q^2)_n}=\frac{(q^2;q^4)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}+\frac{(q^2;q^4)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.8}$$

We next set $\gamma=-q^3$ in (2.5) and obtain

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(-q^4;q^2)_n(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(q^4;q^4)_n}}=\frac{1+q^2}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\cdots, \tag{2.9}$$

which, in conjunction with a known result (Andrews & Berndt, 2005, p. 87, Entry (3.2.3)), yields the following consequence of (2.5):

$$\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+2)}}{(-q^2;q^2)_{n+1}(q^2;q^2)_n}=\frac{1}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^4}{1+}\;\cdots. \tag{2.10}$$

II. Let us consider the following q-identity of Type 2 (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.7)) *and* (McLaughlin *et al.*, 2008, p. 40, Eq. (6.1.4))):

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}=(-\gamma;q)_\infty\sum_{n=0}^{\infty}\frac{(-a\gamma)^n\, q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}, \tag{2.11}$$

which, upon replacing γ by γq, yields

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}=(-\gamma q;q)_\infty\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n^2}}{(-\gamma q;q)_n(q;q)_n}. \tag{2.12}$$

By taking the quotient of the left-hand sides of (2.11) and (2.12), we find that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}}=\cfrac{1}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}-\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}}$$

$$=\cfrac{1}{1+\cfrac{\displaystyle\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_{n-1}}\gamma^n q^{n(n-1)/2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}}$$

$$\cdot \frac{1}{1+\frac{\gamma(1-a)}{\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}}}. \tag{2.13}$$

It is easily observed that

$$\begin{aligned}
\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}} &= 1+\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}-\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}\\
&= 1+\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_{n-1}(-a)(1-q^n)}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}\\
&= 1-\frac{a\gamma q}{\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+3)/2}}},
\end{aligned} \tag{2.14}$$

which, when combined with (2.14), yields

$$\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}} = \frac{1}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{\frac{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}\gamma^n q^{n(n+3)/2}}}. \tag{2.15}$$

Finally, by iterating the above process, we get the following result:

$$\frac{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n+1)/2}}{\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\gamma^n q^{n(n-1)/2}} = \frac{1}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{1+}\,\frac{\gamma q(1-aq)}{1-}\,\frac{a\gamma q^3}{1+}\,\cdots. \tag{2.16}$$

Applying the q-identities (2.11), (2.12) and (2.16), we find that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n^2}}{(-\gamma q;q)_n(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}} = \frac{(1+\gamma)}{1+}\,\frac{\gamma(1-a)}{1-}\,\frac{a\gamma q}{1+}\,\frac{\gamma q(1-aq)}{1-}\,\frac{a\gamma q^3}{1+}\cdots, \tag{2.17}$$

which, upon setting $\gamma = -q$, yields

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)}}{(q^2;q)_n(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{[(q;q)_n]^2}} = \frac{(1-q)}{1-}\,\frac{q(1-a)}{1+}\,\frac{aq^2}{1-}\,\frac{q^2(1-aq)}{1+}\,\frac{aq^4}{1-}\cdots. \tag{2.18}$$

In its *further* special case when $a = 1$, (2.18) yields

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_{n+1}(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}. \tag{2.19}$$

For $\gamma = 1$ and $a = -q$, we find from (2.17) that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q)_n(q;q)_n}} = \frac{2}{1+}\,\frac{(1+q)}{1+}\,\frac{q^2}{1+}\,\frac{q(1+q^2)}{1+}\,\frac{q^4}{1+}\cdots. \tag{2.20}$$

If, instead, we put $\gamma = 1$ and $a = q$ in (2.17), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-1;q)_n(q;q)_n}} = \frac{2}{1+}\,\frac{(1-q)}{1-}\,\frac{q^2}{1+}\,\frac{q(1-q^2)}{1-}\,\frac{q^4}{1+}\cdots. \tag{2.21}$$

We now recall a known result (Andrews & Berndt, 2005, p. 152, Entry (6.2.32)) *with* $a = -1$ as follows:

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n} = \frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2;q^2)_\infty, \tag{2.22}$$

which, in combination with (2.21), yields

$$\frac{2}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-1;q)_n(q;q)_n} = 1 + \frac{1-q}{1-}\,\frac{q^2}{1+}\,\frac{q(1-q^2)}{1-}\,\frac{q^4}{1+}\cdots. \tag{2.23}$$

III. Let us consider the following known q-identity (see (Bowman & McLaughlin, 2006, p. 4, Theorem 1, Eq. (2.9)) *and* (McLaughlin *et al.*, 2008, p. 40, Eq. (6.1.6))):

$$\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\ \gamma^n}{(\gamma;q^2)_n(q;q)_n} = \frac{1}{(\gamma;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n-1)}\gamma^n}{(q^2;q^2)_n}, \tag{2.24}$$

which, upon replacing γ by γq^2, yields

$$\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}\ \gamma^n}{(\gamma q^2;q^2)_n(q;q)_n} = \frac{1}{(\gamma q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}. \tag{2.25}$$

For the quotient of of the right-hand sides of (2.24) and (2.25), we have

$$\begin{aligned} \frac{(1-\gamma)\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n}} &= \frac{(1-\gamma)}{1+\dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_n}(1-q^{2n})}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+1)}}{(q^2;q^2)_n}}} \\ &= \frac{(1-\gamma)}{1+\dfrac{\displaystyle\sum_{n=1}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n-1)}}{(q^2;q^2)_n}}} = \frac{(1-\gamma)}{1+\dfrac{\gamma q}{\dfrac{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+1)}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{\gamma^n q^{n(2n+3)}}{(q^2;q^2)_n}}}}. \end{aligned} \tag{2.26}$$

Proceeding in the above way, we obtain

$$\frac{(1-\gamma)\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\gamma^n}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}\gamma^n}{(q^2;q^2)_n}} = \frac{(1-\gamma)}{1+}\ \frac{\gamma q}{1+}\ \frac{\gamma q^3}{1+}\ \frac{\gamma q^5}{1+}\ \frac{\gamma q^7}{1+}\ \cdots. \tag{2.27}$$

Finally, by applying (2.24), (2.25) and (2.27), we get

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n(3n+1)/2}\gamma^n}{(\gamma q^2;q^2)_n(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\gamma^n}{(\gamma;q^2)_n(q^2;q^2)_n}} = \frac{1-\gamma}{1+}\ \frac{\gamma q}{1+}\ \frac{\gamma q^3}{1+}\ \frac{\gamma q^5}{1+}\ \frac{\gamma q^7}{1+}\ \cdots. \tag{2.28}$$

In its special case when $\gamma = -1$, we find from (2.28) that

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(-1;q^2)_n (q^2;q^2)_n}} = \frac{2}{1-}\,\frac{q}{1-}\,\frac{q^3}{1-}\,\frac{q^5}{1-}\,\frac{q^7}{1-}\cdots. \tag{2.29}$$

Many other similar results involving q-series and associated continued fractions can also be derived analogously.

3. Concluding Remarks and Observations

While q-identities of Type 1 include such important and widely-investigated results as the celebrated Rogers-Ramanujan identities, we have successfully derived several families of q-identities of Type 2 involving q-series and associated continued fractions. We have also considered some other closely-related q-identities of Types 1 and 2.

Such q-series identities of Type 2 as (for example) (2.1), (2.2), (2.11) and (2.24), upon which our present investigation depends remarkably heavily, are derivable as special or limit cases of relatively more familiar known q-identities (see, for details, (Bowman & McLaughlin, 2006, pp. 4–7) and (McLaughlin *et al.*, 2008, p. 42)).

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Primality Testing and Factorization by using Fourier Spectrum of the Riemann Zeta Function

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Abstract

In number theory, integer factorization is the decomposition of a composite number into a product of smaller integers, for which there is not known efficient algorithm. In this article, the author tries to make primality testing and factorization of integers by using Fourier transform of a correlation function generated from the Riemann zeta function. From the theoretical analysis, we can see that prime factorization for the integer composed of two different primes can be conducted within a polynomial time and it can be seen that this special case belongs to the P class.

Keywords: Primality testing, prime factorization, Fourier transform, Riemann zeta function.
2010 MSC: 11A51, 11M06, 11Y05, 11Y11, 42A38.

1. Introduction

In number theory, integer factorization is the decomposition of a composite number into a product of smaller integers. When the numbers are very large, no efficient, non-quantum integer factorization algorithm is known. However, it has not been proven that no efficient algorithm exists (Klee & Wagon, 1991). The presumed difficulty of this problem is at the heart of widely used algorithms in cryptography such as RSA. Many areas of mathematics and computer science have been brought to bear on the problem, including elliptic curves, algebraic number theory, and quantum computing.

Recently, Shor's algorithm has been proposed by Peter Shor, which is a quantum algorithm for integer factorization. On a quantum computer, it has been proved that Shor's algorithm runs in polynomial time (Shor, 1997). But the polynomial time factoring algorithm of integers has not been found for ordinary computing systems.

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In optics, we know that white light consists of all visible frequencies mixed together and the prism breaks them apart so we can see the separate frequencies. It is like the Riemann zeta function be consisted of primes shown as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{(1 - p^{-s})}$$

where p runs over all of primes.

From which, we can consider the white light as a zeta function and separate component frequencies are primes. As we use a prism to decompose visible light into components of different frequencies, we can use Fourier transforms as a prism to decompose the zeta function into primes. In this paper, the method of primality testing and prime factorization by using Fourier transforms of the Riemann zeta function is presented.

2. Frequency spectrum of a correlation function generated from the Riemann Zeta function

Riemann zeta function is an analytic function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. From which, we define the Fourier transform of $z_\sigma(t, \tau)$ shown as

$$Z_\sigma(t, \omega) = \lim_{T \to \infty} \int_{-T}^{+T} z_\sigma(t, \tau) e^{-i\omega\tau} d\tau, \tag{2.1}$$

where $z_\sigma(t, \tau)$ is a time-dependent autocorrelation function (Yen, 1987) given by

$$z_\sigma(t, \tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2)). \tag{2.2}$$

In this formula, $\zeta^*(s)$ is a conjugate of $\zeta(s)$.

In the previous paper of author's (Musha, 2014), $Z_\sigma(t, \omega)$ can be shown as

$$Z_\sigma(t, \omega) = \sum_{n=1}^{\infty} \frac{a(n, t)}{n^\sigma} 2\pi\delta(\omega - \tfrac{1}{2} \log n), \tag{2.3}$$

where $a(n, t)$ is a real valued function given by $a(n, t) = \sum_{n=kl} \cos[\log(k/l)t]$ and $\delta(\omega)$ is a Dirac's delta function.

As $a(n, t)$ is a multiplicative on n which satisfies $a(mn, t) = a(m, t)a(n, t)$ for the case when satisfying $(m, n) = 1$, we have the following equation for the integer n given by $n = p^a q^b r^c \cdots$ (Musha, 2012):

$$Z_\sigma\left(t, \tfrac{1}{2} \log n\right) = \frac{2\pi\delta(0)}{n^\sigma} \frac{\sin[(a + 1)t \log p]}{\sin(t \log p)} \cdot \frac{\sin[(b + 1)t \log q]}{\sin(t \log q)} \frac{\sin[(c + 1)t \log r]}{\sin(t \log r)} \cdots . \tag{2.4}$$

From the Fourier transform of $Z_\sigma\left(t, \tfrac{1}{2} \log n\right)$ given by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma\left(t, \tfrac{1}{2} \log n\right) e^{-i\omega t} dt$, we can obtain the following Lemma.

Lemma 1. *If $n = p_1 p_2 p_3 \cdots p_k$, where $p_1, p_2, p_3, \cdots, p_k$ are different primes, $F_n(\omega)$ for $\omega > 0$ is consisted of 2^{k-1} discrete spectrum shown as:*

$$F_n(\omega) = 2\pi \sum_{i=1}^{2^k} \delta(\omega - \lambda_{i\,1} \log p_1 - \lambda_{i\,2} \log p_2 - \cdots - \lambda_{i\,k} \log p_k), \tag{2.5}$$

where $\lambda_{i\,k}$ equals to -1 or $+1$.

Proof.
As $a(n,t) = \sum_{i=1}^{2^k} [\cos(\lambda_{i\,1} \log p_1) + \cos(\lambda_{i\,2} \log p_2) + \cdots\cdots + \cos(\lambda_{i\,k} \log p_k)]$, where $\log p_1, \log p_2, \log p_3, \cdots, \log p_k$ are linearly independent over $\mathbb{Z}$ (Kac, 1959), thus $F_n(\omega)$ is consisted of 2^{k-1} different spectrum shown as

$$\begin{aligned} F_n(\omega) =& 2\pi \sum_{i=1}^{2^k} \delta(\omega - \lambda_{i\,1} \log p_1 - \lambda_{i\,2} \log p_2 - \cdots - \lambda_{i\,k} \log p_k) \\ &+ 2\pi \sum_{i=1}^{2^k} \delta(\omega + \lambda_{i\,1} \log p_1 + \lambda_{i\,2} \log p_2 + \cdots + \lambda_{i\,k} \log p_k). \end{aligned}$$

□

Then we obtain following Theorems.

Theorem 1. *If and only $F_n(\omega)$ is consisted of a single spectra for $\omega \geq 0$, then n is a prime.*

Proof.
It is clear from Lemma 1. □

Theorem 2. *If and only $F_n(\omega)$ is consisted of two spectrum for $\omega \geq 0$, then n has either form of $n = p \cdot q \; (p \neq q)$, $n = p^2$ or $n = p^3$.*

Proof.
From Theorem 1, there is only a case for the integer $n = p_1 p_2 \cdots p_k$, when$F_n(\omega)$ is consisted of two spectrum, that is $n = p \cdot q \; (p \neq q)$.

For $r \geq 1$ of the function $a(p^r, t)$:

$$\begin{aligned} r = 1,\; a(p,t) &= 2\cos(t \log p) \\ r = 2,\; a(p^2,t) &= 1 + 2\cos(2t \log p) \\ r = 3,\; a(p^3,t) &= 2\cos(t \log p) + 2\cos(3t \log p) \\ r = 4,\; a(p^4,t) &= 1 + 2\cos(2t \log p) + 2\cos(4t \log p) \\ r = 5,\; a(p^5,t) &= 2\cos(t \log p) + 2\cos(3t \log p) + 2\cos(5t \log p) \\ r = 6,\; a(p^6,t) &= 1 + 2\cos(2t \log p) + 2\cos(4t \log p) + 2\cos(6t \log p) \end{aligned}$$

$$r = 7,\ a(p^7, t) = 2\cos(t\log p) + 2\cos(3t\log p) + 2\cos(5t\log p) + 2\cos(7t\log p)$$

$$\vdots$$

Including the spectra at $\omega = 0$, there are cases for $r = 2$ and $r = 3$ when $a(n, t)$ has two spectrum. □

3. Method for primality testing and factorization by using Fourier spectrum

From Theorems 1 and 2, we can make a primality testing and a factorization of the integer n consisted of two primes from the Fourier spectrum $F_n(\omega)$ $(\omega \geq 0)$ by following calculations:

$$\text{①}\quad z_\sigma(t,\tau) = \zeta(\sigma - i(t + \tau/2)) \cdot \zeta^*(\sigma - i(t - \tau/2)), \tag{3.1}$$

$$\text{②}\quad Z_\sigma(t,\omega) = \int_{-\infty}^{+\infty} z_\sigma(t,\tau) e^{-i\omega\tau} d\tau, \tag{3.2}$$

$$\text{③}\quad F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma(t, \tfrac{1}{2}\log n) e^{-i\omega t} dt. \tag{3.3}$$

From which we can obtain the Fourier spectrum by $F_n(\omega) = \int_{-\infty}^{+\infty} Z_\sigma(t, \frac{1}{2}\log n) e^{-i\omega t} dt$. Then we can make a primality testing and integer factorization for an integer n, the process of which is shown in Figure 1.

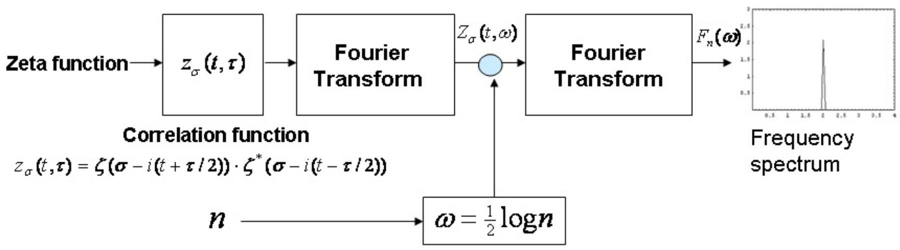

Figure 1. Process to conduct a primality testing for the integer n.

From this process, we can recognize the prime as a single spectra from the frequency analysis result. If there are two spectrum observed from the calculation result, n has either form of $n = p \cdot q$ $(p \neq q)$, $n = p^2$ or $n = p^3$ from Theorem 2.

3.1. Numerical Calculation Method to obtain $\boldsymbol{F_n(\omega)}$

To conduct calculations to obtain the values of $F_n(\omega)$ by using discrete Fourier transform, we can select the value for frequency resolution as $\Delta f = 1/4\pi n$ from $\Delta\omega = \left|\frac{1}{2}\log n - \frac{1}{2}\log(n \pm 1)\right| \approx 1/2n$, for large numbers.

Then we select the maximum frequency of DFT analysis to be $f_{\max} = 4[\log n/4\pi]$ (where $[\ ]$ is a Gauss's symbol), which makes $\omega = \log n$ to be at the center of frequency range.

The element number N for DFT calculation satisfies $f_{\max} = N \cdot \Delta f/2$, then we have $N = [\,8n \log n\,] + 1$.

As Eq.(3.1) can be written in a discrete form as

$$z_\sigma(m,l) = \zeta(\sigma - i(m\Delta t + l\Delta\tau/2))\cdot\zeta^*(\sigma - i(m\Delta t - l\Delta\tau/2)). \tag{3.4}$$

From the relation of $\Delta f \cdot \Delta t = 1/N$, we obtain

$$z_\sigma(m,l) = \zeta\left(\sigma - i\left(\frac{4\pi n}{N}m + \frac{2\pi n}{N}l\right)\right)\zeta^*\left(\sigma - i\left(\frac{4\pi n}{N}m - \frac{2\pi n}{N}l\right)\right). \tag{3.5}$$

As the total time $T_0 = N \cdot \Delta t = 4\pi n$, then Eq.(3.2) in a discrete form can be given by

$$Z_\sigma(m,k) = \frac{4\pi n}{N}\sum_{l=0}^{N-1} z_\sigma(m,l)\exp[-i2\pi(k\Delta f)\cdot(l\Delta\tau)]. \tag{3.6}$$

At the frequency of $\omega = \frac{1}{2}\log n$, we have

$$k\Delta f \cdot l\,\Delta\tau = \frac{\log n}{4\pi} \times \frac{\pi}{2\log n}\,l = \frac{l}{8}, \tag{3.7}$$

then we have

$$y(m) = \frac{4\pi n}{N}\sum_{l=0}^{N-1} z_\sigma(m,l)\exp\left(-i\frac{\pi}{4}l\right), \tag{3.8}$$

which corresponds to $Z_\sigma\left(t, \frac{1}{2}\log n\right)$.

From which, we have the discrete form of Eq.(3.3) given by

$$Y(k) = \frac{4\pi n}{N}\sum_{m=0}^{N-1} y(m)\exp\left(-i2\pi\frac{km}{N}\right), \tag{3.9}$$

which shows the spectrum of $F_n(\omega)$.

Thus we need the following three steps for calculations to obtain $F_n(\omega)$.

① Input the integer n and we let $N = [\,8n\log n\,] + 1$,

② $z_\sigma(m,l) = \zeta\left(\sigma - i\left(\frac{4\pi n}{N}m + \frac{2\pi n}{N}l\right)\right)\cdot\zeta^*\left(\sigma - i\left(\frac{4\pi n}{N}m - \frac{2\pi n}{N}l\right)\right)$,

③ For $m = 0 \sim N-1$, calculate $y(m) = \frac{4\pi n}{N}\sum_{l=0}^{N-1} z_\sigma(m,l)\exp\left(-i\frac{\pi}{4}l\right)$,

④ For $k = 0 \sim N-1$, calculate $Y(k) = \frac{4\pi n}{N}\sum_{m=0}^{N-1} y(m)\exp\left(-i2\pi\frac{km}{N}\right)$.

3.2. Some examples of primality testing

To confirm the validity of discrete computational algorism given in this paper, we try to compute some examples shown as follows:

To generate the Riemann zeta function, we use Mathematica by Wolfram Research.

At the calculation, we set $\sigma = 1.1$ to compute $F_n(\omega)$ to minimize the noise generated by DFT calculations.

(Calculation program by using Mathematica).

```
σ=1.1; (Real Part of Zeta function)
n0=17; (Input an Integer)
n1=Ceiling[8*n0*Log[n0]];(Element number for calculation)
x[m_,l_]:=Zeta[σ-I*(4*Pi*n0*m/n1+2*Pi*n0*l/n1)]*
Conjugate [Zeta[σ-I*(4*Pi*n0*m/n1-2*Pi*n0*l/n1)]];(Autocorrelation function of
zeta function)
data=Table[N[4*Pi*n0/n1*Sum[x[m,l]*Exp[-I*Pi*l/4],{l,0,n1 -1}]],{m,0,n1-1}];
ListPlot[Abs[InverseFourier[data]],PlotJoined→True,PlotR
ange→{{0,n1/2}, {0,200}}, Frame→True]; (DFT calculation and plot results)
```

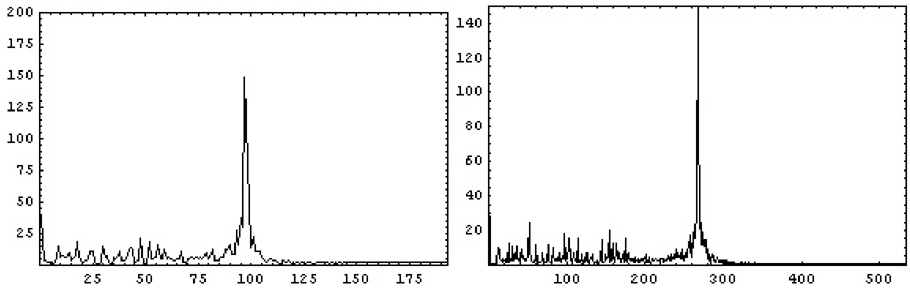

Figure 2. Computational result for n=17(left) and n=37(right).

From calculation, we can see that there exists only one spectrum at the center and it can be shown that both of numbers, 17 and 37 are primes.

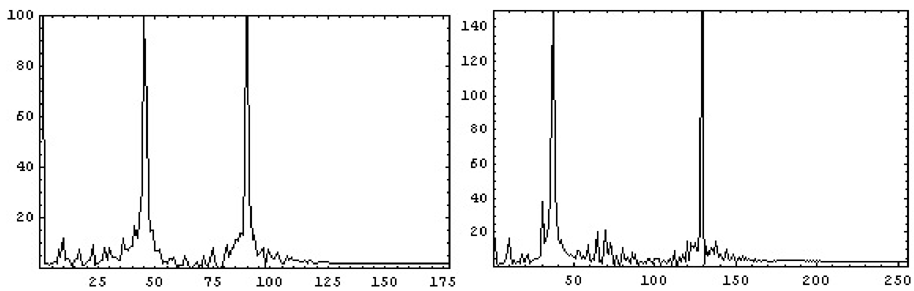

Figure 3. Computational result for n=16(left) and n=21(right).

These results corresponds to $a(p^4, t) = 1 + 2\cos(2t\log p) + 2\cos(4t\log p)$ (three spectrum including $\omega = 0$) and $a(p\cdot q, t) = \cos[(\log q - \log p)t] + \cos[(\log q - \log p)t]$ (two spectrum), and we can see that they are composite numbers.

4. Running time for prime factorization by using DFT algorithms

As the value of Riemann zeta function can be generated by the formula (Gourdon & Sebah, 2003)

$$\zeta(s) = \frac{1}{d_0(1-2^{1-s})}\sum_{k=1}^{m}\frac{(-1)^{k-1}d_k}{k^s} + \gamma_m(s), \tag{4.1}$$

where $d_k = m\sum_{j=k}^{m}\frac{(m+j-1)!\,4^j}{(m-j)!\,(2j)!}$, we can compute $\zeta(\sigma+it)$ with d decimal digits of accuracy, which requires number of term m roughly equal to $m \approx 1.3d + 0.9|t|$ (Gourdon & Sebah, 2003).

To calculate Eq.(3.4), we need to compute up to $t = N \cdot \Delta t = 4\pi n$, hence we need $m \approx 1.3d + 3.6\pi n$ to obtain the value of $\zeta(\sigma+it)$ with d decimal digits of accuracy, which has expected running time $O(n^2)$.

As the running time to require DFT calculation is $O(N^2)$, thus we need the running time to complete calculations of steps from ① to ④, to be estimated as $O(n^2(\log n)^2)$. Hence it can be seen that primality testing of integer can be conducted in a polynomical time by using this algorithm.

Moreover, we can factor the integer which is composed of two different primes by steps from ① to ④, because the calculated result of $F_n(\omega)$ has only two spectrum according to Theorem 2.

As two spectrum obtained can be given by $\omega_1 = \log q - \log p$ and $\omega_2 = \log q + \log p$ (Musha, 2014), we obtain $p = \sqrt{n \cdot \exp(-\omega_1)}$ and $q = \sqrt{n \cdot \exp(\omega_1)}$ $(q > p)$ from them if we let ω_1 is a small spectrum obtained from the calculation of $F_n(\omega)$. From these obtained values for p and q, we have finally to examine whether they satisfy $n = p \cdot q$ or not.

Thus it can be seen that prime factorization for the integer composed of two different primes can be conducted in a polynomial time. There is no efficient, non-quantum integer factorization algorithm is not known now (Yang, 2002), and it has been widely believed that no algorithm is existed that can factor all integers in polynomial time. Thus the presumed difficulty of this problem is at the heart of widely used algorithms in cryptography such as RSA. Contrary to this, we can see that prime factorization for the integer composed of two different primes can be conducted within a polynomial time and it can be shown that this special case belongs to the P class from the theoretical analysis.

However, the validity of this factoring algorithm has been confirmed for only small integers by the restriction of a computer capacity and hence it is necessary to confirm the validity of this algorithm for large integers by using more powerful computer systems.

5. Conclusion

From the spectrum obtained by the Fourier transform of a correlation function generated from the Riemann zeta function, we can see the primality of a integer n if and only the $F_n(\omega)$ has a single spectra for $\omega \geq 0$. Furthermore, it can be shown that the prime factorization can be conducted within a polynomial time for the special case that the integer is composed of two different primes and hence we can conclude that that prime factorization for the integer composed of two different primes is in the P class.

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