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An Application of Fuzzy Sets to Veterinary Medicine

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Abstract

In this paper, firstly, the waves P and T in ECG of kittens and adult cats were converted to fuzzy sets. After, using to entropy definition for fuzzy sets, we have assigned an entropy to waves P and T for kittens and adult cats. Also, using to some new formulates, the graphical representation of waves P and T for normal or diseased heart of cats were given.

Keywords: Entropy, ecg, fuzzy set, similarity degree, kitten, cats.
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1. Introduction

The theoretical and practical applications of fuzzy sets have increased considerably since Zadeh's paper, (see (Abdollahian *et al.*, 2010) ; (Bilgin, 2003); (Dhar, 2013); (Diamond & Kloeden, 1994); (Goetschel & Voxman, 1986); (Li *et al.*, 1995); (Iwamoto & et al, 2007); (Kosko, 1986); (Matloka, 1986); Tong *et al.* (2007); (Zadeh, 1965) and (Zararsız & Şengönül, 2013)). In medicine, cardiologists are try to predetermine some heart diseases from electrocardiographs and this processes is also valid for veterinary medicine. Some fine details may not be seen in graphical representation of the waves electrocardiographs of human or animals. It is a fact that, long time can be spent for interpreting electrocardiographs (shortly; ECG) and sometimes small but important details can be unnoticed or ECG's can be misleading for junior vet or cardiologists. In this paper, by using entropy concept, we have obtained numerical values for ECGs of kittens and adult cats. These numerical values are the best way to observe fine details in the waves such as P, PQR complex and T. The numerical values are also very clear and can be easily interpreted for any person according to graphical representation of ECG's. It will be seen that these computations are

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completely different than computations of (Czogala & Leski, 2000). Let us give some background information on fuzzy sets and entropy of the fuzzy sets.

Let $\mathscr{X}$ be nonempty set. According to Zadeh, a fuzzy subset of $\mathscr{X}$ is a nonempty subset $\{(x, u(x)) : x \in \mathscr{X}\}$ of $\mathscr{X} \times [0, 1]$ for some function $u : \mathscr{X} \to [0, 1]$, (Diamond & Kloeden, 1994). Consider a function $u : \mathbb{R} \to [0, 1]$ as a subset of a nonempty base space $\mathbb{R}$. The function u is called membership function of the fuzzy set u.

Furthermore, we know that shape similarity of the membership functions does not reflect the conception of itself, but it will be used for examining the context of the membership functions. Whether a particular shape is suitable or not can be determined only in the context of a particular application. However, that many applications are not overly sensitive to variations in the shape. In such cases, it is convenient to use a simple shape, such as the triangular shape of membership function. Let us define fuzzy set u on the set $\mathbb{R}$ with membership function as follows:

$$u(x) = \begin{cases} \frac{h_u}{u_1 - u_0}(x - u_0), & x \in [u_0, u_1) \\ \frac{-h_u}{u_2 - u_1}(x - u_1) + h_u, & x \in [u_1, u_2] \\ 0, & others \end{cases}, \tag{1.1}$$

where the notations h_u denotes height of the fuzzy sets u. For brief, we write triple $(u_0, u_1 : h_u, u_2)$ for fuzzy set u. Notation $\mathscr{F}$ be the set of the all fuzzy sets in the form $u = (u_0, u_1 : h_u, u_2)$ on the $\mathbb{R}$.

Define the function S as follows:

$$S : \mathscr{F} \times \mathscr{F} \to \mathbb{R}, \quad S(u, v) = \frac{\min\{h_u, h_v\}}{\max\{h_u, h_v\}}[1 - \frac{1}{3}\sum_{k=0}^{2}|u_k - v_k|]. \tag{1.2}$$

The function S is called similarity degree between the fuzzy sets u and v. If $S(u, v) = 1$ then we say that u is full similar to v or vice versa, we say that v is completely similar to u. If $0 < S(u, v) < 1$ then we say that the fuzzy set u is S- similar to the fuzzy set v (or the fuzzy set v is S- similar to the fuzzy set u), if $S(u, v) \leq 0$ we say that, u is not similar to v. Similar definitions can be found in (Sridevi & Nadarajan, 2009) and (Yıldız & Şengönül, 2014).

If we capture numerous ECG for any human or animal, it can be considered as a finite sequence of ECG's. Therefore we will give some definitions and properties about sequences of fuzzy sets.

The set

$$w(\mathscr{F}) = \{(u_k) \mid u : \mathbb{N} \to \mathscr{F}, u(k) = (u^k) = ((u_0^k, u_1^k : h_{u^k}, u_2^k))\} \tag{1.3}$$

is called sequence of fuzzy sets. Any element of the set $w(\mathscr{F})$ is called sequences of fuzzy sets, where $u_0^k, u_1^k, u_2^k \in \mathbb{R}$, $u_0^k \leq u_1^k \leq u_2^k$ and the mean of notation $u_1^k : h_{u^k}$ is the k^{th} therm of the sequence (u^k) takes highest membership degree at u_1^k and this membership degree is equal to h_{u^k} . If for all $k \in \mathbb{N}$, $h_{u^k} = 1$ then the set $w(\mathscr{F})$ turns into sequence set of fuzzy numbers and if $u_0^k = u_1^k = u_2^k$ and $h_{u_1^k} = 1$ the set $w(\mathscr{F})$ turns in to ordinary sequence space of the real numbers, respectively.

An another important class of the sequence set of the fuzzy sets is defined by

$$\varphi(\mathscr{F}) = \{(u_k) \in w(\mathscr{F}) \mid \exists k_0 \in \mathbb{N}, \forall k \geq k_0 : u^k = 0\}. \tag{1.4}$$

Clearly, the sequences of fuzzy sets can obtain by fuzzification of the term by term of sequence of real numbers with a suitable method.

Definition 1.1. Let us define the function $\mathscr{S}$ as follows:

$$\mathscr{S} : w(\mathscr{F}) \times w(\mathscr{F}) \to \mathbb{R}, \quad \mathscr{S}(u_n, v_n) = \frac{\inf\{h_{u_n}, h_{v_n}\}}{\sup\{h_{u_n}, h_{v_n}\}}[1 - \frac{1}{3} \lim_n \sum_{k=0}^{2} |u_k^n - v_k^n|] = \lambda. \tag{1.5}$$

The function $\mathscr{S}$ is called similarity degree between sequences of fuzzy sets (u_n) and (v_n). If $\mathscr{S}(u_n, v_n) = 1$ then we say that (u_n) is completely similar to the sequence (v_n), if $0 < \mathscr{S}(u_n, v_n) = \lambda < 1$ then we say that the sequence (u_n) is λ- similar to the sequence (v_n), if $\lambda \leq 0$ we say that, (u_n) is not similar to (v_n).

In the fuzzy set theory, the fuzziness of a fuzzy set is a important matter and there are many method to measure the fuzziness of a fuzzy set. At first, the fuzziness was thought to be the distance between fuzzy set and its nearest nonfuzzy set. Later, the entropy was used instead of of fuzziness (de Luca & Termini, 1972) and has received attention, recently (Wang & Chui, 2000). Well, then what is the entropy?

Definition 1.2. (Zimmermann, 1991) Let $u \in \mathscr{F}$ and $u(x)$ be the membership function of the fuzzy set u and consider the function $H : \mathscr{F} \to \mathbb{R}^+$. If the function H satisfies conditions below,

1. $H(u) = 0$ iff u is crisp set,
2. $H(u)$ has a unique maximum, if $u(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$
3. For $u, v \in \mathscr{F}$, if $v(x) \leq u(x)$ for $u(x) \leq \frac{1}{2}$ and $u(x) \leq v(x)$ for $u(x) \geq \frac{1}{2}$ then $H(u) \geq H(v)$,
4. $H(u^c) = H(u)$, where u^c is the complement of the fuzzy set u

then the $H(u)$ is called entropy of the fuzzy set u.

Let suppose that $u = u(x)$ be membership function of the fuzzy set u and the function $h : [0, 1] \to [0, 1]$ satisfies the following properties:

1. Monotonically increasing at $[0, \frac{1}{2}]$ and decreasing $[\frac{1}{2}, 1]$,
2. $h(x) = 0$ if $x = 0$ and $h(x) = 1$ if $x = \frac{1}{2}$.

The function h is called entropy function and the equality $H(u(x)) = h(u(x))$ holds for $x \in \mathbb{R}$. Some well known entropy functions are given as follows:

$h_1(x) = 4x(1 - x)$, $h_2(x) = -x \ln x - (1 - x) \ln(1 - x)$, $h_3(x) = \min\{2x, 2 - 2x\}$ and

$$h_4(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ 2(1 - x), & x \in [\frac{1}{2}, 1] \end{cases}.$$

Note that the function h_1 is the logistic function, h_2 is called Shannon function and h_3 is the tent function.

Let $\mathscr{X}$ be a continuous universal set. The total entropy of the fuzzy set u on the $\mathscr{X}$ is defined

$$e(u) = \int_{x \in \mathscr{X}} h(u(x))p(x)dx \tag{1.6}$$

where $p(x)$ is the probability density function of the available data in $\mathscr{X}$ (Pedrycz, 1994), (Pedrycz & Gomide, 2007). If we take $p(x) = 1$ in the (1.6) then the $e(u)$ is called entropy of the fuzzy set

u. It is known that the value of $e(u)$ is depend on support of the fuzzy set u. Let u be fuzzy set on the set $\mathbb{R}$ with membership function (1.1), then we see that the total entropy of fuzzy set u is equal to

$$e(u) = c(2h_u - \frac{4}{3}h_u^2)\ell(u) \tag{1.7}$$

for $p(x) = c$ and $h = h_1$, where $\ell(u) = \max\{x - y : x, y \in \overline{\{x \in \mathbb{R} : u(x) > 0\}}\}$. We know that each fuzzy set or a fuzzy number correspond to the fuzzy thoughts in the idea of user. So, any sequence of the fuzzy sets can be seen as sequence of thoughts or sequence fuzzy information. This sequence of fuzzy information may contain an useful information or not contain an useful information. But we can use terms of this sequence to obtain meaningful information from this sequence.

Definition 1.3. Let h be an entropy function, (u^k) be a sequence of fuzzy sets (or fuzzy thought) and $p_k(x)$ be probability density function of the available data in $\mathbb{R}$ for every $k \in \mathbb{N}$. Then sequence

$$e(u^k) = \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \tag{1.8}$$

is called total entropy sequence of the fuzzy sets (u^k). If the probability density function $p_k(x) = 1$ is fix, for all $k \in \mathbb{N}$, then the (1.8) is called entropy sequence of the fuzzy sets $u = (u^k)$.

If we take $u = (u^k) \in w(\mathscr{F})$, $p_k(x) = c_k \in (0, 1]$ and $h(u) = h_1(u)$ then from (1.8) we have

$$e(u^k) = (c_k(2h_{u^k} - \frac{4}{3}h_{u^k}^2)\ell(u^k)), \tag{1.9}$$

here and other places in the text, the notation $2h_{u^k}^2$ denotes second power of the h_{u^k}. If we choose the probability density functions $p_k(x) = c \in (0, 1]$ for all $k \in \mathbb{N}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$ in the (1.9) then we see that $e(u^k) = \frac{2}{3}c\ell(u^k)$.

Let us suppose that $u = (u^k)$ be sequences of the fuzzy numbers (that is $h_{u^k} = 1$), $h(u) = h_1(u)$ and $p_k(x) = c_k = 1 \in (0, 1]$ for all $k \in \mathbb{N}$. Then the entropy $e(u^k)$ of the sequence of fuzzy numbers (u^k) is equal to

$$e(u^k) = \frac{2}{3}\ell(u^k). \tag{1.10}$$

Clearly, if $\ell(u^k) = 0$ for every $k \in \mathbb{N}$ then the sequence (u^k) returns to sequence of real numbers. In this case the entropy of the total entropy sequence is zero for sequences of real numbers. For example, let $u = (u^k)$ be $((1, 1 : 1, 1))$, then from (1.10) we obtain zeros sequence. Furthermore, the entropy sequence (e_k) can not be convergent but be bounded.

Definition 1.4. Let $\mathscr{A} = (a_{nk})$ be a lower triangular infinite matrix of real or complex numbers and

$$\sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \to E, \quad n \to \infty. \tag{1.11}$$

The real number E is called total $\mathscr{A}$-entropy of the sequence (u^k) of fuzzy sets, if it exists.

Definition 1.5. Let suppose that the $u = (u^k)$ be a sequence of fuzzy sets, $p_k(x) = c_k$, $(c_k \in (0, 1])$ for all $k \in \mathbb{N}$ and

$$\lim_n \sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = \lim_n \sum_k a_{nk}c_k(2h_{u^k} - \frac{4}{3}h^2_{u^k})\ell(u^k) = E_1. \tag{1.12}$$

The real number E_1 is called total $\mathscr{A}$- entropy according to entropy function h and $p_k(x) = c_k$ is probability density functions of the sequence $u = (u^k)$ of fuzzy sets, and it is shown by $T_e^{\mathscr{A}}(u^k)$.

Let $n, k \in \mathbb{N}$, $\alpha > -1$, $p_k(x) = c_k$ and $\binom{n-k+\alpha-1}{n-k}$, $\binom{n+\alpha}{n}$ are binomial confidence. Let us define infinite matrices $A = (a_{nk})$ and $C^\alpha = (c^\alpha_{nk})$ as follows:

$$a_{nk} = \begin{cases} 1, & 0 \le k \le n \\ 0, & otherwise \end{cases} \quad \text{and} \quad c^\alpha_{nk} = \begin{cases} \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}}, & 0 \le k \le n \\ 0, & otherwise \end{cases}.$$

If we write the matrices A and C^α instead of $\mathscr{A}$ in the expression (1.12) then we have

$$\lim_n \sum_{k=0}^{n} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = T_e^A(u^k) \tag{1.13}$$

and

$$\lim_n \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} \int_{x\in\mathbb{R}} h(u^i(x))p_k(x)dx = T_e^{C^\alpha}(u^k), \tag{1.14}$$

respectively.

The expressions (1.13) and (1.14) are called A- total entropy and total Cesàro entropy of order α of the sequence $u = (u^k)$ of fuzzy sets, according to probability density functions $p_k(x)$, respectively. If we take $\alpha = 1$ and $p_k(x) = c_k$ from (1.14) we see that

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) \tag{1.15}$$

which is called Cesàro normalized entropy of order 1 (shortly, Cesàro entropy) of the sequence $u = (u^k)$ of fuzzy sets.

It is easily prove that, if

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

then

$$T_e^{C^1}(u^k) = \lim_n \frac{s}{n+r} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

where $r, s \in \mathbb{R}$. For example, the Cesàro entropy of sequence $(u^k) = ((\frac{k}{k+1} - t_1, \frac{k}{k+1} : 1, \frac{k}{k+1} + t_2))$ is

$$T_e^C(u^k) = \lim_n \frac{2(t_2 + t_1)}{3(n+1)} \sum_{k=0}^{n} c_k, \tag{1.16}$$

where we assume that $t_1 < t_2$ and $t_1, t_2 \in \mathbb{R}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$. If the series $\sum_k c_k$ is convergent then the value $T_{e^{C^1}}(u^k)$ exists every time. As a comment of the (1.13) and (1.16), we point out that we can obtain an useful information from infinite fuzzy information by a suitable method. But, the total entropy and Cesàro entropy of the sequence v defined by $v = ((v_0^k, v_1^k, v_2^k)) = ((-k, 1 : 1, k+2))$ is infinite. This means that, the sequence v does not contain any useful information for us.

Since, every real number is also a fuzzy number then we can give following corollary:

Corollary 1.1. *Let the sequence $r = (r^k)$ be a convergent or divergent sequence of real numbers. Then the all entropies of the $r = (r^k)$ are zero.*

Corollary 1.1 can be interpreted as, in the any information sequence, if the elements of information sequence are crisp information then we obtain a crisp information from this sequence.

Proposition 1.1. If the fuzziness of the any sequence of fuzzy set is constantly increasing then the entropy is constantly grow and maybe is infinite. On the contrary if the fuzzyness of the any sequence of fuzzy set is constantly decreasing then the entropy is decreases and becomes 0.

It is calculated in (Chin, 2006) that the entropy of any fuzzy number is $\frac{2c(u_2-u_0)}{3}$. Therefore, in generally, if we take $h = h_1$ and $p_i(x) = c$, for every $i \in \mathbb{N}$, then entropy of the sequence of fuzzy numbers is given with (1.10).

In next section, we will investigate entropy of the electrocardiogram for cats and give some comments. We know that, an electrocardiogram is an important test for any relevant heart diseases of human or animals, the shortest way of identifying heart problems and you can detects cardiac (heart) abnormalities, as an example heart attacks, an enlarged heard or abnormal heart rhythms may cause heart failure, abnormal position of heart can be given, by measuring the electrical activity generated by the heart as it contacts, (for more, see (de Luna, 1987)).

2. The Applications to ECG's of the Idea Entropy and Some Comments

It is a fact that, the long time can be spent for interpreting electrocardiographs results by cardiologists or vet and sometimes small but important details can be unnoticed because of complexity of the *ECG*. The same situation is also valid for computerized electrocardiography. According to us, numerical values for *ECG* outputs can be more reliable for cardiologists and vet for interpreting *ECG* results. Furthermore, if the outputs are numerical then the consultation may be easy than consultation of the *ECG* papers. In this section we have proposed a new consultation method for cardiac problems of cats which will be based upon numerical value of *ECG*s, (see (Brady & Rosen, 2005); (Khan, 2003) for ECG).

Quite simply every heart beats can be considered as term of a sequence. Using to the waves P, QRS complex and T, we can construct the waves sequence $((P_k, (QRS)_k), T_k))$, where k is beat number or number of measurements and is finite. The graphical shapes of the waves P, QRS complex and T can imagine a membership functions a fuzzy set. With this idea, we can appoint an entropy value using to these membership functions which will be described below.

The entropy of the sequence $((P_k, (QRS)_k), T_k))$ can compute for finite or infinite many k and this computation gives to us a numerical value, not graphical. From numerical value, we can determine some cardiac problems. Namely, the sequence $((P_k, (QRS)_k), T_k))$ can divide three part for calculate entropy as follows:

1. The entropy of the sequence (P_k) waves,
2. The entropy of the sequence $((QRS)_k))$ complexes,
3. The entropy of the sequence (T_k) waves.

In this case, we can assume that the total entropy of the heart is equal to

$$\mathscr{E} = e(P_k) + e((QRS)_k) + e(T_k). \tag{2.1}$$

Now we will summarize some information about electrocardiographs without deepening the subject.

The electrocardiograph records the electrical activity of the heart muscle and displays this data as a trace on a screen or on paper and, later, this data is interpreted by a medical practitioner. ECG's from healthy hearts have a characteristic shape. Any irregularity in the heart rhythm or damage to the heart muscle can change the electrical activity of heart which leads to change in the shape of ECG's according to patients. Using this changes, we can investigate entropy of the heart rhythm or damage entropy of the heart muscle. It is known that, the QRS complex reflect the rapid depolarization of the right and left ventricles. The ventricles have a large muscle mass compared to the atria so the QRS complex usually has a much larger amplitude than the P- wave.

Furthermore, the heart movements are kept in check by various charges and pulses that change slightly on exertion, blood chemistry and strain. According to us, residence of skin and conductivity of blood are important for ECG, too. The conductivity and residence of the skin are vary according to some minerals in the blood plasma such as calcium, chloride, potassium or glucose concentration in a diabetic patients blood. So we have to consider the conductivity of blood in the calculations of transmitting electric current and therefore in the entropy calculations for a heart. For blood conductivity properties, you can read to (Hirsch & et al, 1950).

2.1. The Entropy of The Waves Sequence (P_k) and Some Comments

Primary wave of a heart in ECG, is called P wave and shortly denoted with P, have an entropy value and it can be compute as follows:

$$e(P) = \int_{x\in\mathbb{R}} h_1(P(x))r(x)dx, \tag{2.2}$$

where the function $P(x)$ is membership function of the fuzzy $\mathscr{P}$ set that we will correspond to wave P and the function $r(x)$ is conductivity function (generally the function r is fix) of the body .

Experimental measurements showed that to us for kittens, the wave P has maximal height about $0.12mV$, duration is shorter than 0.3 seconds but these values for adult cats are $0.2mV$ second and 0.04 (Lourenço & Ferreira, 2003).

Using the maximal height and duration of wave P as 0.12 second and 0.3 mV, respectively, the membership function $P_1(x)$ of the fuzzy $\mathscr{P}_1$ set which is correspond to wave P for kittens can write as follows:

$$P_1(x) = \begin{cases} 0.8x, & x \in [0, 0.15] \\ 0.24 - 0.8x, & x \in (0.15, 0.30] \\ 0, & \text{otherwise} \end{cases} . \tag{2.3}$$

Furthermore, the membership function $P_2(x)$ of the fuzzy $\mathscr{P}_2$ set which is correspond to wave P for adult cats is

$$P_2(x) = \begin{cases} 10x, & x \in [0, 0.02] \\ 0.4 - 10x, & x \in (0.02, 0.04] \\ 0, & \text{otherwise} \end{cases} . \tag{2.4}$$

It is clear that the support of the fuzzy set $\mathscr{P}_1$ is duration of the wave P and height is maximum height of wave P.

Let us take $supp\mathscr{P}_1 \approx]0, 0.30[$, $supp\mathscr{P}_2 \approx]0, 0.04[$ and closure of the $supp\mathscr{P}_1$ and $supp\mathscr{P}_2$ be $\overline{supp\mathscr{P}_1} = [0, 0.30]$ and $\overline{supp\mathscr{P}_2} = [0, 0.04]$ where the notations $supp\mathscr{P}_1$ and $supp\mathscr{P}_2$ denotes support of the $\mathscr{P}_1$ and $\mathscr{P}_2$.

In this case, we see that $h_1(P_1(x)) = \begin{cases} 3.2x - 2.56x^2, & x \in [0, 0.15] \\ 0.7296 - 1.664x - 2.56x^2, & x \in (0.15, 0.30] \\ 0, & \text{otherwise} \end{cases}$. Similarly to $h_1(P_1(x))$, we have $h_1(P_2(x)) = \begin{cases} 40x - 400x^2, & x \in [0, 0.02] \\ 0.96 - 8x - 400x^2, & x \in (0.02, 0.04] \\ 0, & \text{otherwise} \end{cases}$.

Let us denote P_1 and P_2 of wave P for kittens and adult cats, respectively. If we choose $r(x) = c$ in (2.2) then we see that the the entropy of wave P_1 is equal to

$$e(P_1) = 662.4 \times 10^{-4} c \tag{2.5}$$

for normal wave P for kittens. The P_2 wave entropy for adult cats is

$$e(P_2) = 138.667 \times 10^{-4} c. \tag{2.6}$$

If we compare (2.5) and (2.6) then we see that the P wave entropies of the kittens and adult cats are different.

Definition 2.1. The total Cesàro entropy of the sequence (P_k) is

$$T_e^{C^1}(P_k) = \frac{1}{k+1} \sum_{i=0}^{k} c_i a_2^i (2h_{a_1^i} - \frac{4}{3} h_{a_1^i}^2) S(P_k, P), \tag{2.7}$$

where c_i is resistance of the dry skin in the ith sample, k is number of sample of P wave and $S(P_k, P)$ is similarity degree between of the waves P_k and P.

Table 1. Non-clinical P waves data for adult cats

Gender: Male	Age:xx	Weight:xx	Height:xx							
Days	1	2	3	4	5	6	7	8	9	10
$m(h_{a_k^1})$	0.2	0.2	0.19	0.21	0.23	0.23	0.19	0.2	0.18	0.15
$m(a_2^k)$	0.04	0.04	0.03	0.05	0.05	0.05	0.045	0.044	0.043	0.043
$e(P_k)$	0,0138672	0,0138672	0,009956361	0,018060735	0,019474215	0,019474215	0,017526794	0,014602663	0,013622864	0,011610323
$S(P_k, P)$	1	1	0,94525	0,947619048	0,865217391	0,865217391	0,867391304	0,9481	0,89865	0,748875

Let the resistance of the dry skin be fix that is if c_i equal to c at the each every i. place then the (2.7) is turn to

$$T_e^{C^1}(P_k) = \frac{c}{k+1}\sum_{i=0}^{k} a_2^i(2h_{a_1^i} - \frac{4}{3}h_{a_1^i}^2)S(P_k, P). \tag{2.8}$$

Example 2.1. Let us suppose, the wave P values as height and width as given in Table 1 for any adult cat for 10 measurements with fix conductivity of blood and residence of the skin. Note that these data are not clinical measures. In this mean, the sequence (P_k) is in the set $\varphi(\mathscr{F})$. The notations $m(h_{a_k^1})$ and $m(a_2^k)$ in Table 1 denotes measured height and durations of the wave P in day. Then from (2.8), we see that the Cesàro total entropy of the wave P of adult cats according to Table 1 is

$$T_e^{C^1}(P_k) = 137.94345 \times 10^{-4}c \tag{2.9}$$

for 10 beats. If we compare (2.5) and (2.9), the P wave properties of the adult cat heart which given above example is very low than normal value. Using to (1.7), we can give a graphic for 10 sample of wave P which given in the Table 1 (see, Figure 2) .

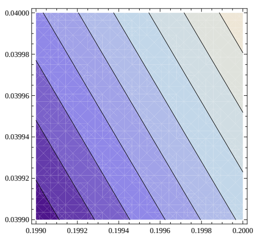

Figure 1
Graphical representation of $e(P_k)$ of the normal P wave for adult cats.

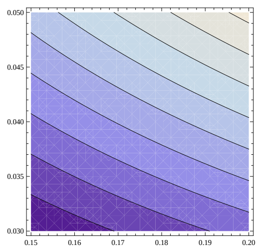

Figure 2
Graphical representation of $e(P_k)$ for Table 1 values for adult cats.

The Figure 1 is entropy graphic for the normal wave P of adult cats. If we compare the Figures 1 and 2 then we see that the height and duration of the P wave when changed with any effect, the all entropy zones are curl to upward at adult cats as in humans. It can be consider that the magnitude of the curl is P wave degenerations.

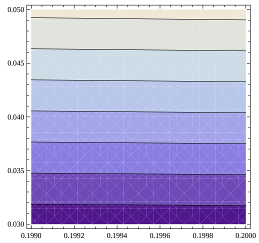

Figure 3
The values $h_{a_k^2}$ nearly fix but values a_2^k variable for adult cats.

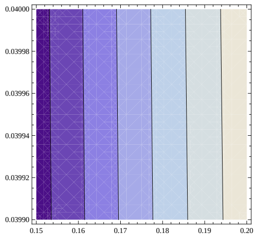

Figure 4
The values a_2^k nearly fix but the values $h_{a_k^2}$ variable for adult cats.

If the $h_{a_k^1}$ is fix but the value a_2^k be variable and conversely the $h_{a_k^1}$ is variable but the value a_2^k be fix then graphical representation of the entropy zones are shown as in Figure 3 and Figure 4, respectively.

As similar to (2.7), the A- entropy of the sequence wave P is

$$T_e^A(P_k) = 1379.43446 \times 10^{-4} c \tag{2.10}$$

from (1.13). But normal A-entropy value for 10 beats of adult cats should be $6624 \times 10^{-4} c$ and the P wave value in (2.10) very low than $6624 \times 10^{-4} c$. where c is resistance of the dry skin in the ith time.

Comment 1.

We know that the value of the $S(P_k, P)$ must be $0 \leq S(P_k, P) \leq 1$ for every $k \in \mathbb{N}$. After a certain place, if P_k waves is not exists, or the similarity values $S(P_k, P)$ nearly to the zero then the entropy of atrial depolarization of the heart, the $T_e^A(P_k)$ is near to zero. In this case we can say that this is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement for this heart in the future as in human.

Comment 2.

Respectively, if the values $e(P_1)$ and $e(P_2)$ less than $662.4 \times 10^{-4} c$ and $138.667 \times 10^{-4} c$ for kitten and adult cats then, we can say that, there is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement as in human for this heart in the future.

3. Comparison with the ECG

1. Long time can be spent for interpreting electrocardiographs results by cardiologists or vets and sometimes small but important details can be unnoticed because of the complexity of ECG.
2. Numerical values are more reliable than graphical representations.
3. If the outputs are numerical then the consultation may be easy than consultation of the ECG papers.

4. Weakness of This Model

The weakness of this model is that the data may be incomplete and not accurate enough because of the system that we use when we collect the data. Kittens adaptation to ECG machines is an important factor in the measurement phase since heart rates can change under stress and different circumstances. The numerical values may not reflect the reality if the information is not in the near proximity of real world assessment, shortly wrong inputs can produces misleading results.

5. Conclusions and Suggestions

The conclusions can be summarized as follows:

1. The entropy of the wave P for normal heart of the kitten should be $1379.43446 \times 10^{-4}c$ and should be $6624 \times 10^{-4}c$ for adult cats.
2. The graphical representation of the normal wave P of kittens should similar to Figure 1.
3. If the duration is fix but height is being altered by any reason then lines in graphical representation of the wave P becomes steeper.
4. The lines in the graphical representation of the wave T should be almost parallel to horizontal axis.

As a suggestion, clearly, one can define entropy value and graphical representations of QRS complex and wave T to similar entropy value wave P. So any numerical value can obtain for (2.1). If entropy value of the QRS complex and wave P are calculate then we can give a numerical entropy value for (2.1).

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On Certain Properties for Hadamard Product of Uniformly Univalent Meromorphic Functions with Positive Coefficients

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Abstract

In this paper we study some results concerning the Hadamard product of certain classes related to uniformly starlike and convex univalent meromorphic functions with positive coefficients.

Keywords: Univalent, meromorphic, starlike, convex, uniformly, Hadamard product.
2010 MSC: 30C45.

1. Introduction

Throughout this paper, let the functions of the form

$$\varphi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0; c_n \geq 0), \tag{1.1}$$

and

$$\psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \qquad (d_1 > 0; d_n \geq 0) \tag{1.2}$$

which are analytic and univalent in the unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\};$$

also, let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0; a_n \geq 0), \tag{1.3}$$

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$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \qquad (a_{0,i} > 0; a_{n,i} \geq 0), \tag{1.4}$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (b_0 > 0; b_n \geq 0), \tag{1.5}$$

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{0,j} > 0; b_{n,j} \geq 0). \tag{1.6}$$

which are analytic and univalent in the punctured unit disc

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < \ |z| < 1\}.$$

A function $f(z) \in \Sigma$ is meromorhpically starlike of order α if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \, (z \in U^*; 0 \leq \alpha < 1). \tag{1.7}$$

A function f of the form (1.3) is said to be in the class $U\Sigma S_0^*(\alpha,\beta)$ of meromorphic uniformly β -starlike functions of order α if it satisfies the condition:

$$-Re\left\{\frac{zf'(z)}{f(z)} + \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} + 1\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.8}$$

Also, a function f of the form (1.3) is said to be in the class $U\Sigma C_0(\alpha,\beta)$ of meromorphic uniformly β -convex functions of order α if it satisfies the condition:

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)} + \alpha\right\} > \beta\left|2 + \frac{zf''(z)}{f'(z)}\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$f \in U\Sigma C_0(\alpha,\beta) \iff -zf' \in U\Sigma S_0^*(\alpha,\beta). \tag{1.10}$$

The classes $U\Sigma S_0^*(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$ have been studied by (Aouf *et al.*, 2014), (Atshan & Kulkarni, 2007), and others. We note that
(i) $U\Sigma S_0^*(\alpha,0) = S_n^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = C_n(\alpha)$ (see (Aouf & Silverman, 2008), *with* $n = 1$);
(ii) $U\Sigma S_0^*(\alpha,0) = \Sigma_p S_n^*(\alpha,\gamma)$ and $U\Sigma C_0(\alpha,0) = \Sigma_p C_n(\alpha,\gamma)$ (also see (R. M. El-Ashwah & Hassan, 2013), with $n = p = \gamma = 1$);
(iii) $U\Sigma S_0^*(\alpha,0) = \Sigma S_0^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = \Sigma K_0(\alpha,\beta)$ (see (Mogra, 1991)).

Lemma 1.1. *Let the function f defined by (1.3). Then $f \in U\Sigma S_0^*(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} [n(1+\beta) + (\alpha+\beta)] a_n \leq (1-\alpha) a_0. \tag{1.11}$$

Lemma 1.2 (3)**.** *Let the function f defined by (1.3). Then $f \in U\Sigma C_0(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} n[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0. \tag{1.12}$$

Definition 1.1. Let the function f defined by (1.3). Then $f \in U\Sigma S_m(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} n^m[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0, \tag{1.13}$$

where $(0 \le \beta < \infty)$, $(0 \le \alpha < 1)$ and m any positive integer number.

We note that $U\Sigma S_1(\alpha,\beta) = U\Sigma C_0(\alpha,\beta)$ and $U\Sigma S_0(\alpha,\beta)$ is equivalent to $U\Sigma S_0^*(\alpha,\beta)$. Further, $U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_r(\alpha,\beta)$ if $m > r \ge 0$, the containment beign proper. Whence, for any positive integer m, we have the inclusion relation

$$U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_{m-1}(\alpha,\beta) \subset \ldots \subset U\Sigma S_2(\alpha,\beta) \subset U\Sigma C_0(\alpha,\beta)\, U\Sigma S_0^*(\alpha,\beta).$$

Also, we note that for nonnegative real number m the class $U\Sigma S_m(\alpha,\beta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} \frac{(1-\alpha)a_0}{n^m[n(1+\beta)+(\alpha+\beta)]}\lambda_n z^n,$$

where $a_0 > 0$, and $\sum_{n=1}^{\infty} \lambda_n \le 1$, satisfy the inequality (1.13). For the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \left(a_{n,j} \ge 0;\, j = 1, 2\right). \tag{1.14}$$

We denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2} z^n. \tag{1.15}$$

Similarly, we can define the Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions $\varphi(z)$ and $\psi(z)$ given by (1.1) and (1.2), (see (Kumar, 1987)).

$$(\varphi * \psi)(z) = c_1 d_1 z - \sum_{n=2}^{\infty} c_n d_n z^n \tag{1.16}$$

In this paper, we can discuss certain results concerning the Hadamard product of functions in the classes $U\Sigma S_0^*(\alpha,\beta)$, $U\Sigma S_m(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$.

2. Main results

Theorem 2.1. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1,2,...,m$, and suppose that the functions $g_j(z)$ defined by (1.6) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $j = 1,2,...,q$. Then the Hadamard product $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z)$ belongs to the class $U\Sigma S_{2m+q-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\}\leq(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,i}\right]. \tag{2.1}$$

Since $f_i(z)\in U\Sigma C_0(\alpha,\beta)$, we get

$$\sum_{n=1}^{\infty}n[n(1+\beta)+(\alpha+\beta)]a_{n,i}\leq(1-\alpha)a_{0,i}\quad(i=1,2,...,m). \tag{2.2}$$

Therefore,

$$a_{n,i}\leq\frac{(1-\alpha)}{n[n(1+\beta)+(\alpha+\beta)]}a_{0,i} \tag{2.3}$$

which implies that

$$a_{n,i}\leq n^{-2}a_{0,i}\quad(i=1,2,...,m). \tag{2.4}$$

Similarly, for $g_j(z)\in U\Sigma S_0^*(\alpha,\beta)$, we obtain

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]b_{n,j}\leq(1-\alpha)b_{0,j}, \tag{2.5}$$

for $j = 1,2,...,q$. Hence we have

$$b_{n,j}\leq n^{-1}b_{0,j}\quad(j=1,2,...,q). \tag{2.6}$$

Using (2.4) for $i = 1,2,...,m$, (2.6) for $j = 1,2,...,q-1$, and (2.5) for $j = q$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\}\\ \leq\;&\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2m}n^{-(q-1)}\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]b_{n,q}\right\}\\ =\;&\left[\prod_{i=1}^{m-1}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}b_{n,q}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,j}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z)\in U\Sigma S_{2m+q-1}(\alpha,\beta)$. The proof of Theorem 1 is completed. □

Theorem 2.2. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1, 2, ..., m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{2m-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m} a_{n,i}\right]\right\} \leq (1-\alpha)\left[\prod_{i=1}^{m} a_{0,i}\right]. \tag{2.7}$$

Since $f_i(z) \in U\Sigma C_0(\alpha,\beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1, 2, ..., m$.

Using (2.2) for $i = 1, 2, ..., m-1$, and (2.1) for $i = 1, 2, ..., m$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m} a_{n,i}\right]\right\} \\
\leq\ &\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2(m-1)}\prod_{i=1}^{m-1} a_{0,i}\right]a_{n,m}\right\} \\
=\ &\left[\prod_{i=1}^{m-1} a_{0,i}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right] \leq (1-\alpha)\prod_{i=1}^{m} a_{0,i}.
\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z) \in U\Sigma S_{2m-1}(\alpha,\beta)$. The proof of Theorem 2 is completed. □

Theorem 2.3. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $i = 1, 2, ..., m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{m-1}(\alpha,\beta)$.*

Proof. Since $f_i(z) \in U\Sigma S_0^*(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]a_{n,i} \leq (1-\alpha)a_{0,i}, \tag{2.8}$$

for every $i = 1, 2, ..., m$. Therefore, we obtain $a_{n,i} \leq \frac{(1-\alpha)}{n(1+\beta)+(\alpha+\beta)}a_{0,i}$ which implies that

$$a_{n,i} \leq n^{-1}a_{0,i} \quad (i = 1, 2, ..., m). \tag{2.9}$$

Using (2.9) for $i = 1, 2, ..., m-1$, and (2.8) for $i = 1, 2, ..., m$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m} a_{n,i}\right]\right\} \\
\leq\ &\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-(m-1)}\prod_{i=1}^{m-1} a_{0,i}\right]a_{n,m}\right\} \\
=\ &\left[\prod_{i=1}^{m-1} a_{0,i}\right]\sum_{n=1}^{\infty}\left[\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right] \leq (1-\alpha)\prod_{i=1}^{m} a_{0,i}.
\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z) \in U\Sigma S_{m-1}(\alpha,\beta)$, which completes the proof of Theorem 3. □

Remark. Taking $\beta = 0$ in our main results, we obtain the results obtained by Mogra (Mogra, 1991).

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On Uniform h-Stability of Evolution Operators in Banach Spaces

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Abstract

The paper treats the general concept of uniform h-stability, as a generalization of uniform exponential stability for evolution operators in Banach spaces.
The main aim is to give necessary and sufficient conditions of Datko-type and Barbashin-type for this property and also criterias for uniform h-stability using Lyapunov functions. As particular cases, we obtain the results for uniform exponential stability.

Keywords: uniform stability, growth rates, evolution operators.
2010 MSC: 34D05, 34D20, 34D23.

1. Preliminaries

One of the most important asymptotic properties studied for evolution operators is the uniform exponential stability. This concept was treated in a large number of papers and of the most important we recall (Coppel, 1965), (Lupa *et al.*, 2010), (Megan *et al.*, 2001), (van Neerven, 1995) and (Stoica & Megan, 2010).

In the last years, are considered more general concepts of stability, as h-stability (see (Megan, 1995)) or (h, k)-stability (see (Fenner & Pinto, 1997), (Megan & Cuc, 1997), (Minda & Megan, 2011)), where h and k are growth rates (i.e. nondecreasing functions with different properties).

In this paper is considered the concept of uniform h-stability, with $h : \mathbb{R}_+ \to [1, +\infty)$ a growth rate (more precisely a nondecreasing function with $\lim\limits_{t\to+\infty} h(t) = +\infty$), for evolution operators in Banach spaces.
Are obtained necessary and sufficient conditions for this notion and as consequences, we emphasize the results for the case of uniform exponential stability.

In what follows, X represents a real or complex Banach space, X^* its topological dual and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. We will denote the norms on X, on X^*

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and on $\mathcal{B}(X)$ by $\|\cdot\|$.
Also, Δ is the set of all the pairs $(t, s) \in \mathbb{R}_+^2$ with $t \geq s$ and I represents the identity operator on X.

Definition 1.1. A mapping $\Phi : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(eo_1) $\Phi(t, t) = I$, for every $t \geq 0$;
(eo_2) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$, for all (t, s) and $(s, t_0) \in \Delta$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ an evolution operator and $h : \mathbb{R}_+ \to [1, +\infty)$ a growth rate.

Definition 1.2. We say that Φ has a *uniform h-growth* if there exists $N \geq 1$ such that for all $(t, s, x) \in \Delta \times X$:

$$h(s)\|\Phi(t, s)x\| \leq Nh(t)\|x\|.$$

If $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we say that Φ has a *uniform exponential growth.*

Definition 1.3. The evolution operator Φ is called *uniformly h-stable* if there exists $S \geq 1$ such that for all $(t, s, x) \in \Delta \times X$:

$$h(t)\|\Phi(t, s)x\| \leq S h(s)\|x\|.$$

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we recover the concept of *uniform exponential stability* and α is called *stability constant*.

Remark. If Φ is uniform h-stable, then it has a uniform h-growth. In general, the converse implication is not valid.

Example 1.1. Considering the evolution operator $\Phi : \Delta \to \mathcal{B}(X)$, defined by

$$\Phi(t, s) = \frac{h(t)}{h(s)}, \quad \text{for all} \quad (t, s) \in \Delta,$$

it is easy to observe that Φ has a uniform h-growth, but Φ is not uniformly h-stable.

Remark. The evolution operator Φ has a uniform h-growth if and only if there exists $N \geq 1$ with

$$h(s)\|\Phi(t, t_0)x_0\| \leq Nh(t)\|\Phi(s, t_0)x_0\|,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$.

Remark. Φ is uniformly h-stable if and only if there is $S \geq 1$ such that

$$h(t)\|\Phi(t, t_0)x_0\| \leq S h(s)\|\Phi(s, t_0)x_0\|,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$.

Definition 1.4. We say that $\Phi : \Delta \to \mathcal{B}(X)$ is

(i) *strongly measurable* if for all $(s, x) \in \mathbb{R}_+ \times X$ the mapping

$$t \mapsto \|\Phi(t, s)x\| \text{ is measurable on } [s, +\infty);$$

(ii) **-strongly measurable* if for all $(t, x^*) \in \mathbb{R}_+ \times X^*$ the mapping

$$s \mapsto \|\Phi(t, s)^* x^*\| \text{ is measurable on } [0, t].$$

2. Necessary conditions for uniform *h*-stability

In this section we will denote by $\mathcal{H}$ the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there is a constant $M \geq 1$ such that

$$\int_s^{+\infty} \frac{dt}{h(t)} \leq \frac{M}{h(s)}, \quad \text{for all } s \geq 0.$$

Also, $\mathcal{H}_1$ represents the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0.$$

Remark. Denoting by $\mathcal{E}$ the set of functions $h : \mathbb{R}_+ \to [1, +\infty)$, $h(t) = e^{\alpha t}$, with $\alpha > 0$, it results that $\mathcal{E} \subset \mathcal{H} \cap \mathcal{H}_1$.

Remark. The growth rate $h : \mathbb{R}_+ \to [1, +\infty)$ is in $\mathcal{H}_1$ if and only if there exists a growth rate $h_2 : \mathbb{R}_+ \to [1, +\infty)$, defined by $h_2(t) = \dfrac{h(t)}{h_1(t)}$, for all $t \geq 0$ such that $h_2 \in \mathcal{H}$.

A first result concerning the connections between the uniform exponential stability and uniform h-stability of an evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 2.1. *Following statements are equivalent:*

(i) *Φ is uniformly exponentially stable;*
(ii) *there exists $h \in \mathcal{H}_1$ such that Φ is uniformly h-stable;*
(iii) *there exists $h \in \mathcal{H}$ such that Φ is uniformly h-stable.*

Proof. (1) ⇒ (2). It results for $h(t) = e^{\alpha t}$, with $\alpha > 0$.
(2) ⇒ (3). From the hypothesis, there is a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0$$

and using the second *Remark* from this section it follows that $h_2 \in \mathcal{H}$.

Thus, for all $(t, s, x) \in \Delta \times X$ we have

$$h_2(t)||\Phi(t, s)x|| = \frac{h(t)}{h_1(t)}||\Phi(t, s)x|| \leq$$

$$\leq S \frac{h(s)}{h_1(t)}||x|| \leq S h_2(s)||x||,$$

which shows that Φ is h_2-stable.
(3) ⇒ (1). It is immediate from the first *Remark* of this section. □

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a first necessary condition of Datko-type, due to R. Datko ((Datko, 1972)) is

Theorem 2.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly h-stable with $h \in \mathcal{H}_1$ then there are a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $D \geq 1$ such that*

$$\int_s^{+\infty} h_1(t)\|\Phi(t,t_0)x_0\|dt \leq Dh_1(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$.

Proof. It is immediate for $D = M_1 S$, where M_1 and h_1 are given by definition of $\mathcal{H}_1$ and S is given by Definition 1.3. □

Corollary 2.1. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly exponentially stable, then there are the constants $\beta > 0$ and $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\beta t}\|\Phi(t,t_0)x_0\|dt \leq De^{\beta s}\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$.

Proof. It is a particular case of Theorem 2.2. □

Definition 2.1. A mapping $L : \Delta \times X \to \mathbb{R}_+$ is said to be a *h-Lyapunov function for* Φ if

$$L(t,t_0,x_0) + \int_s^t h(\tau)\|\Phi(\tau,t_0)x_0\|d\tau \leq L(s,t_0,x_0),$$

for all $(t,s),\ (s,t_0) \in \Delta$, $x_0 \in X$.

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then the function L is called *exponential Lyapunov function*.
The importance of the Lyapunov functions in the study of the stability property is described for instance in (Barreira & Valls, 2008), (Barreira & Valls, 2013).

Another significant result for the uniform h-stability of an evolution operator is given by

Theorem 2.3. *If the evolution operator Φ is uniformly h-stable with $h \in \mathcal{H}_1$, then there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$, a h_1-Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s,s,x_0) \leq Dh_1(s)\|x_0\|,$$

for all $(s,x_0) \in \mathbb{R}_+ \times X$.

Proof. Let $L : \Delta \times X \to \mathbb{R}_+,\ L(t,s,x_0) = \int\limits_t^{+\infty} h_1(\tau)\|\Phi(\tau,s)x_0\|d\tau.$

Thus, L is a h_1-Lyapunov function for Φ and using Theorem 2.2 we obtain

$$L(s,s,x_0) = \int\limits_s^{+\infty} h_1(\tau)\|\Phi(\tau,s)x_0\|d\tau \le Dh_1(s)\|x_0\|,$$

for all $(s,x_0) \in \mathbb{R}_+ \times X$. □

In particular, we obtain

Corollary 2.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly exponentially stable, then there are the constants $\beta > 0$, $D \ge 1$ and an exponential Lyapunov function L for Φ with*

$$L(s,s,x_0) \le De^{\beta s}\|x_0\|,$$

for all $(s,x_0) \in \mathbb{R}_+ \times X$.

We consider now the set $\tilde{\mathcal{H}}$ of the growth rates $h : \mathbb{R}_+ \to [1,+\infty)$ with the property that there is a growth rate $h_1 : \mathbb{R}_+ \to [1,+\infty)$ and a constant $\tilde{M} \ge 1$ with

$$\int\limits_0^t \frac{h(\tau)}{h_1(\tau)}d\tau \le \tilde{M}\frac{h(t)}{h_1(t)}, \quad \text{for all } t \ge 0.$$

Remark. It is easy to see that the functions $h \in \mathcal{E}$ (considered in Remark 2) are in $\tilde{\mathcal{H}}$.

Let $\Phi : \Delta \to \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator. A first result for this type of evolution operators is proved by E. A. Barbashin ((Barbashin, 1967)) in the case of uniform exponential stability.
Concerning the uniform h-stability, we prove

Theorem 2.4. *If Φ is uniformly h-stable with $h \in \tilde{\mathcal{H}}$, then there is a growth rate $h_1 : \mathbb{R}_+ \to [1,+\infty)$ and $B \ge 1$ with*

$$\int\limits_0^t \frac{\|\Phi(t,\tau)^*x^*\|}{h_1(\tau)}d\tau \le \frac{B}{h_1(t)}\|x^*\|,$$

for all $(t,x^*) \in \mathbb{R}_+ \times X^*$.

Proof. It results using Definition 1.3 and the definition of $\tilde{\mathcal{H}}$, for $B = S\tilde{M}$. □

As a consequence of the above result, we obtain

Corollary 2.3. *If Φ is uniformly exponentially stable, then there are the constants $\gamma > 0$ and $B \ge 1$ such that*

$$\int\limits_0^t e^{-\gamma\tau}\|\Phi(t,\tau)^*x^*\|d\tau \le Be^{-\gamma t}\|x^*\|,$$

for all $(t,x^*) \in \mathbb{R}_+ \times X^*$.

3. Sufficient conditions for uniform *h*-stability

In what follows, we will denote by $\mathcal{H}_2$ the set of the functions $h : \mathbb{R}_+ \to [1, +\infty)$ with the property

$$\sup_{s \geq 0} \frac{h(s+1)}{h(s)} = M_2 < +\infty.$$

Remark. We observe that all the functions $h \in \mathcal{E}$ (defined in Remark 2) are in $\mathcal{H}_2$, i.e. $\mathcal{E} \subset \mathcal{H}_2$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a sufficient criteria of Datko-type is

Theorem 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} h(t)||\Phi(t, t_0)x_0||dt \leq Dh(s)||\Phi(s, t_0)x_0||,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$, then Φ is uniformly h-stable.

Proof. Let $S = M_2^2 ND$.
Case 1. We consider $(t, s),\ (s, t_0) \in \Delta$ with $t \geq s + 1$, $x_0 \in X$. Thus,

$$h(t)||\Phi(t, t_0)x_0|| \leq \int_{t-1}^{t} h(t)||\Phi(t, \tau)|| \cdot ||\Phi(\tau, t_0)x_0||d\tau \leq$$

$$\leq N \int_{t-1}^{t} h(t)\frac{h(t)}{h(\tau)}||\Phi(\tau, t_0)x_0||d\tau \leq$$

$$\leq NM_2^2 \int_s^{+\infty} h(\tau)||\Phi(\tau, t_0)x_0||d\tau \leq S\,h(s)||\Phi(s, t_0)x_0||.$$

It results that

$$h(t)||\Phi(t, t_0)x_0|| \leq S\,h(s)||\Phi(s, t_0)x_0||,$$

for all $(t, s),\ (s, t_0) \in \Delta$ with $t \geq s + 1$, $x_0 \in X$.

Case 2. Let $(t, s),\ (s, t_0) \in \Delta$ with $t \in [s, s + 1]$, $x_0 \in X$. We have

$$h(t)||\Phi(t, t_0)x_0|| \leq h(t)||\Phi(t, s)|| \cdot ||\Phi(s, t_0)x_0|| \leq$$

$$\leq N\frac{h^2(t)}{h^2(s)}h(s)||\Phi(s, t_0)x_0|| \leq S\,h(s)||\Phi(s, t_0)x_0||.$$

In conclusion,

$$h(t)||\Phi(t, t_0)x_0|| \leq S\,h(s)||\Phi(s, t_0)x_0||,$$

for all $(t, s),\ (s, t_0) \in \Delta$, $x_0 \in X$, which shows that Φ is uniformly h-stable. □

Corollary 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\alpha t}||\Phi(t, t_0)x_0||dt \leq De^{\alpha s}||\Phi(s, t_0)x_0||,$$

for all $(t, s),\ (s, t_0) \in \Delta,\ x_0 \in X$, then Φ is uniformly exponentially stable.

Proof. It results from Theorem 3.1. □

Theorem 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there exist a h-Lyapunov function for Φ and $D \geq 1$ with*

$$L(s, s, x_0) \leq Dh(s)||x_0||,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly h-stable.

Proof. From Definition 2.1, for $s = t_0$ we obtain

$$\int_s^{t} h(\tau)||\Phi(\tau, s)x_0||d\tau \leq L(s, s, x_0) \leq Dh(s)||x_0||,$$

for all $(t, s, x_0) \in \Delta \times X$ and for $t \to +\infty$, it follows that Φ is uniformly h-stable. □

In particular, a sufficient condition for the uniform exponential stability is given by

Corollary 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there exist an exponential Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s, s, x_0) \leq De^{\alpha t}||x_0||,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly exponentially stable.

A sufficient condition of Barbashin-type for the uniform h-stability of a $*$-strongly measurable evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 3.3. *We consider Φ an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $B \geq 1$ with*

$$\int_0^{t} \frac{||\Phi(t, \tau)^* x^*||}{h(\tau)} d\tau \leq \frac{B}{h(t)}||x^*||,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly h-stable.

Proof. We consider $S = NM_2^2 B$.

Let $(t, s) \in \Delta$, $t \geq s + 1$ and $(x, x^*) \in X \times X^*$. Then,

$$h(t)| < x^*, \Phi(t, s)x > | = \int_s^{s+1} h(t)| < \Phi(t, \tau)^* x^*, \Phi(\tau, s)x > |d\tau \leq$$

$$\leq h(t) \int_s^{s+1} \|\Phi(t,\tau)^* x^*\| \cdot \|\Phi(\tau,s)x\| d\tau \leq$$

$$\leq N h(t) \int_s^{s+1} \frac{\|\Phi(t,\tau)^* x^*\|}{h(\tau)} \frac{h^2(\tau)}{h^2(s)} h(s) d\tau \|x\| \leq$$

$$\leq S h(s)\|x\| \cdot \|x^*\|.$$

Considering the supremum relative to $\|x^*\| \leq 1$ it results that

$$h(t)\|\Phi(t,s)x\| \leq S h(s)\|x\|, \text{ for all } t \geq s+1, \; x \in X.$$

Let now $t \in [s, s+1], \; x \in X$. We obtain

$$h(t)\|\Phi(t,s)x\| \leq N \frac{h^2(t)}{h(s)} \|x\| \leq S h(s)\|x\|,$$

for all $t \in [s, s+1], \; x \in X$.
In conclusion, Φ is uniformly h-stable. □

As a particular case, we obtain

Corollary 3.3. *Let Φ be an evolution operator with uniform exponential growth. If there is $B \geq 1$ with*

$$\int_0^t e^{-\alpha\tau} \|\Phi(t,\tau)^* x^*\| d\tau \leq B e^{-\alpha t} \|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly exponentially stable.

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Quadratic Equations in Tropical Regions

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Dedicated to B. V. M.

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Abstract

In this note, the reader is invited to a walk through tropical semifields and the places where they border on "ordinary" algebra. Though mostly neglected in today's lectures on algebra, we point to the places where tropical structures inevitably pervade, and show that they frequently occur in ring theory and classical algebra, touching at least functional analysis, and algebraic geometry. Specifically, it is explained how valuation theory, which plays an essential part in classical commutative algebra and algebraic geometry, is essentially tropical. In particular, it is shown that Eisenstein's well-known irreducibility criterion and other more powerful criteria follow immediately by tropicalization. Some applications to algebraic equations in characteric 1, neat Bézout domains, and rings of continuous functions are given.

Keywords: ℓ-group, projectable, semifield, polynomial, irreducible, Newton polygon, tropical algebra.
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1. Introduction

Mathematical ideas quite often originate from natural sciences where experiments help to understand what happens behind reality. In chemistry, the usual method to analyse a matter is by heating until the components begin to separate. "Tropical" mathematics did not quite emerge in that way, but at least one of its founders (Imre Simon) was working on it in the sunny regions of Brazil.

To illustrate the basic process, consider the function

$$a +_p b := (a^p + b^p)^{1/p}$$

for positive real numbers a, b. At "room temperature" ($p = 1$), the function $a +_1 b$ is just ordinary addition in $\mathbb{R}$. Now turn on the heating - proceed until $p \to \infty$ to get the real number system to

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melt. Recall that F. Riesz (Riesz, 1910) made such an experiment already in 1910, which led him to the invention of Lebesgue spaces $L^p(\mathbb{R})$. If p is replaced by the Planck constant $\hbar := \frac{1}{p}$, the limit process $\hbar \to 0$ is known as a *dequantization* (Litvinov, 2006). Indeed, the passage from L^1 to L^∞ bears a certain analogy to the correspondence principle in quantum mechanics (Bohr, 1920).

Now what remains after melting the real number system? For $p = \infty$, ordinary addition $a + b$ in $\mathbb{R}$ turns into $a \vee b := \max\{a, b\}$. The additive group of $\mathbb{R}$ becomes a semigroup, the field $\mathbb{R}$ of real numbers turns into the semifield $\mathbb{R}^+_{\max}$ of *tropical real numbers*, investigated in the 1987 thesis of Imre Simon (Simon, 1987). A remarkable feature of $\mathbb{R}^+_{\max}$ is that its addition is idempotent:

$$a \vee a = a.$$

Thus, if there would exist additive inverses, the whole system would collapse into the zero ring. So is there any reason to regard the elements of $\mathbb{R}^+_{\max}$ as numbers? Before taking up this question seriously, let us content ourselves for the moment with referring back to F. Riesz' early work on L^p-spaces. Here the connection between $p = 1$ and $p = \infty$ is very tight: $L^\infty(\mathbb{R})$ is just the Banach space dual of $L^1(\mathbb{R})$.

Hilbert once placed the number system between the three-dimensional space and the one-dimensional time, saying that numbers are 'two-dimensional'. Such a statement would still have shocked the mathematical community in the days of Euler who called imaginary numbers "impossible" (Euler, 1911). Nowadays, the two-dimensionality is firmly justified by analytical and algebraic reasons, the latter consisting in the algebraic closedness of $\mathbb{C}$. On the other hand, two-dimensionality would not make sense without reference to the base field $\mathbb{R}$ which is "really" fundamental.

In the tropical world, there is no such distinction: the semifield of tropical reals is "algebraically closed". Making this precise is a good exercise and an invitation to be more careful in stating the 'fundamental theorem of algebra'. To be sure, the latter does not mean that *every* complex polynomial has a root - the non-zero constants have to be excluded. This triviality becomes relevant in the wonderland of tropical algebra: there are tropical semifields where (non-constant) linear equations need not be solvable. Roots and solutions of polynomial equations fall apart, and quadratic equations need not be solvable by radicals. On the other hand, every algebraic equation can be reduced to quadratic ones.

In this paper, classical algebra is revisited with regard to tropical structures, and it is shown that they occur at various places. Apart from a revision of semifields of characteristic 1, we add new characterizations for their algebraic closedness (Theorem 6.1). A connection with neat Bézout domains is given in Corollary 2. As a second application, we show that if the semifield of characteristic 1 corresponding to an ℓ-group $\mathscr{C}(X)$ of contiuous functions on a completely regular space X is algebraically closed, the space X must be an F-space, that is, the corresponding ring $C(X)$ of continuous functions is a Bézout ring (Corollary 3).

Another motivation to study semifields of characteristic 1 comes from a recent, highly conjectural branch of arithmetic geometry. Since André Weil sketched his diagonal argument (Weil, 1940, 1941) to tackle the Riemann hypothesis, some research groups eagerly delve under the surface of $\mathbb{Z}$, searching for its "base field" to make $\mathbb{Z}$ (a ring of Krull dimension one) into an algebra over that field (see, e. g., (Connes & Consani, 2010, 2011; Deitmar, 2008; Soulé, 2011)). The way to this non-existing, mysterious, “field” of characteristic 1 inevitably leads through the tropical region. By Proposition 2.2, this hot region is nothing else than the vast and well-developed theory of lattice-ordered abelian groups.

2. The forgotten characteristic

To include the result of a dequantization, we are advised to consider semifields instead of fields. More generally, a *semiring* is an abelian monoid $(A; +, 0)$ with a multiplicative monoid structure $(A; \cdot, 1)$ satisfying the distributive laws and $a \cdot 0 = 0 \cdot a = 0$ for all $a \in A$. If the group of (multiplicatively) invertible elements, the *unit group* $A^{\times}$, coincides with $A \setminus \{0\}$, we call A a *semi-skewfield*. If, in addition, the multiplicative monoid is commutative, A is said to be a *semi-field*. For example, the above mentioned $\mathbb{R}^{+}_{\max}$ is a semifield.

A *morphism* in the category of semirings is a map $f\colon A \to B$ which satisfies

$$f(a+b) = f(a) + f(b), \qquad f(0) = 0$$
$$f(a \cdot b) = f(a) \cdot f(b), \qquad f(1) = 1.$$

Like in the category of rings, there is an initial object, the semiring $\mathbb{N}$ of non-negative integers: For any semi-ring A there is a unique morphism $c\colon \mathbb{N} \to A$. The image of c is the intersection of all sub-semirings of A, the *prime semiring* of A. Similarly, every semi-skewfield A contains a smallest sub-semi-skewfield. If it coincides with A, we call A a *prime semi-skewfield*.

In general, the kernel $\operatorname{Ker} c := \{n \in \mathbb{N} \mid c(n) = 0\}$ is not of the form $\mathbb{N}p$ for some $p \in \mathbb{N}$. For example, $I := \mathbb{N} \setminus \{1, 2, 4, 7\}$ is an ideal of the semiring $\mathbb{N}$ which occurs, e. g., as the grading of a simple curve singulatity (Greuel & Knörrer, 1985). Thus $\mathbb{N}/I$ is a finite semiring with $\operatorname{Ker}(c) = I$. On the other hand, there exist congruence relations on $\mathbb{N}$ which do not come from an ideal, even if A is a semifield. For example, let $\mathbb{B} := \{0, 1\}$ be the semifield with $1 + 1 = 1$. Then $c\colon \mathbb{N} \twoheadrightarrow \mathbb{B}$ satisfies $c(n) = 1$ for $n \neq 0$. So c has a trivial kernel, while it is far from being a monomorphism.

Note that $\mathbb{B}$ is the prime a sub-semifield of $\mathbb{R}^{+}_{\max}$. Therefore, we write $a \vee b$ for the addition in $\mathbb{B}$. So $\mathbb{B}$ is a Boolean algebra with $a \wedge b := ab$. The reader will notice that $\mathbb{B}$ can be derived from the prime field $\mathbb{F}_2$ via $a \vee b = a + b + ab$, but not vice versa.

Definition 2.1. We define the *characteristic* $\operatorname{char} A$ of a semiring A to be the smallest integer $p > 0$ with $c(n + p) = c(n)$ for some $n \in \mathbb{N}$. If such an integer p does not exist, we set $\operatorname{char} A := 0$.

In analogy to the theory of skew-fields, we have (cf. (Rump, 2015), Proposition 1)

Proposition 2.1. *Every prime semi-skewfield is a semifield. Up to isomorphism, the prime semi-fields are $\mathbb{Q}^{+}$, $\mathbb{B}$, and $\mathbb{F}_p$ for rational primes p. In particular, the prime semifields are determined by their characteristic.*

Proof. Let F be a prime semi-skewfield. Assume first that $\operatorname{char} F = 0$. Then $\mathbb{N}$ can be regarded as a sub-semiring of F. Every non-zero $n \in \mathbb{N}$ has an inverse $\frac{1}{n}$ in F which commutes with all elements of $\mathbb{N}$. Hence $\{\frac{m}{n} \mid m, n \in \mathbb{N}, n > 0\}$ is a sub-semifield isomorphic to the positive cone $\mathbb{Q}^{+}$ of $\mathbb{Q}$.

Now assume that $p := \operatorname{char} F \neq 0$. Then there is an integer $n \in \mathbb{N}$ with $c(n) + c(p) = c(n)$. As this equation holds for almost all n, we can assume that n is a multiple of p. Adding multiples of $c(p)$ on both sides, the equations obtained in this way imply that $c(n) + c(n) = c(n)$. If $c(n) = 0$, then $c(p) = 0$, and the usual argument shows that $c(\mathbb{N}) \cong \mathbb{F}_p$ for a prime p. Otherwise, we obtain $c(1) + c(1) = c(1)$, which yields $c(\mathbb{N}) \cong \mathbb{B}$. □

So the possible prime semifields are

$$\mathbb{Q}^+, \mathbb{B}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \ldots,$$

including the prime fields $\mathbb{F}_p$ and a natural sub-semifield of $\mathbb{Q}$. Note that formally, $\mathbb{Q}^+$ carries more information than $\mathbb{Q}$: The positive cone provides $\mathbb{Q}$ with its natural ordering. Thus, $\mathbb{Q}^+$ connects arithmetic (the semiring $\mathbb{N}$) with algebra and analysis (the ordered field $\mathbb{Q}$ and its completion $\mathbb{R}$), while the newcomer $\mathbb{B}$ bridges the gap between algebra and logic.

Every semifield contains one of the prime semifields according to its characteristic. For fields, this is a well-known piece of algebra. So the question arises how the "logical" semi-skewfields, those containing $\mathbb{B}$, look like. By Proposition 2.1, they are of characteristic 1, which means that they satisfy the equation $1+1 = 1$. Recall that a partially ordered group is said to be *lattice-ordered* or an *ℓ-group* if the partial order is a lattice. For the theory of ℓ-groups, the reader is referred to (Anderson & Feil, 1988; Bigard *et al.*, 1977; Darnel, 1995; Glass, 1999). The commutative case of the following result is due to Weinert and Wiegandt (Weinert & Wiegandt, 1940). Similar ideas have been developed independently by several authors (see (Castella, 2010; Lescot, 2009), and the literature cited there).

Proposition 2.2. *Up to isomorphism, there is a one-to-one correspondence between ℓ-groups and semi-skewfields of characteristic 1.*

Proof. Note first that a semi-skewfield F is of characteristic 1 if and only if $a + a = a$ holds for all $a \in F$. Then it easily checked that

$$a \leqslant b :\Longleftrightarrow a + b = b \tag{2.1}$$

makes F into a $\vee$-semilattice with $a \vee b := a + b$. Furthermore, the distributivity shows that $F^\times$ is an ℓ-group. Conversely, every ℓ-group G can be made into a semi-skewfield $\widetilde{G} := G \sqcup \{0\}$ by adjoining a smallest element 0 with $0a = a0 = 0$ for all $a \in \widetilde{G}$. Since $\widetilde{G}^\times = G$ and $\widetilde{F^\times} = F$, the correspondence is bijective. □

In particular, semifields of characteristic 1 are equivalent to abelian ℓ-groups, and our prime semifield $\mathbb{B}$ corresponds to the ℓ-group of order one. For those who would like to prove the Riemann hypothesis, we should add that $\mathbb{B}$ is not identical with the desperately sought field $\mathbb{F}_1$ - it is still "too big"!

3. Tropical semi-domains

To study field extensions, one has to understand polynomial rings first. Thus, in characteristic 1, we have to deal with polynomials over the semifield $\widetilde{G}$ of an abelian ℓ-group G. For an arbitrary field K, there are many integral domains with quotient field K. If K is an algebraic number field, there is a canonical subring $\mathscr{O}$ - the ring of integers - with quotient field K. Similarly, any semifield $\widetilde{G}$ of characteristic 1 has a canonical sub-semiring $\widetilde{G^-} := G^- \sqcup \{0\}$, where G^- is the negative cone of G. (Since 0 is the smallest element of $\widetilde{G}$, the cone that touches 0 is the negative one.)

Definition 3.1. We define a *semi-domain* to be a commutative semiring A satisfying $ac = bc \Rightarrow a = b$ for $a, b, c \in A$ with $c \neq 0$. We call A *tropical* if there exists an abelian ℓ-group G with $A = \widetilde{G^-}$.

In particular, a semi-domain has no zero-divisors. An intrinsic description of tropical semi-domains is obtained as follows. Recall that a *hoop* (Blok & Ferreirim, 2000) is a commutative monoid H with a binary operation $\rightarrow$ such that the following are satisfied for all $a, b, c \in H$:

$$a \rightarrow a = 1$$
$$ab \rightarrow c = a \rightarrow (b \rightarrow c)$$
$$(a \rightarrow b)a = (b \rightarrow a)b.$$

Every hoop is a $\wedge$-semilattice with respect to the *natural* partial order

$$a \leqslant b \; :\Longleftrightarrow \; \exists\, c \in H\colon a = cb \; \Longleftrightarrow \; a \rightarrow b = 1.$$

A hoop is called *self-similar* (Rump, 2008) if it is cancellative. (For an explanation of the terminology and equational characterizations, see (Rump, 2008), Proposition 5.) Every self-similar hoop H has a group of fractions, the *structure group* $G(H)$ of H, which consists of the fractions $a^{-1}b$ with $a, b \in H$.

Proposition 3.1. *Up to isomorphism, there is a one-to-one correspondence between*
(a) *semifields of charatceristic 1,*
(b) *tropical semi-domains,*
(c) *abelian ℓ-groups, and*
(d) *self-similar hoops.*

Proof. The equivalence between (a) and (c) follows by Proposition 2.2, while the equivalence between (b) and (c) is obvious. For an abelian ℓ-group G, we define

$$a \rightarrow b := ba^{-1} \wedge 1$$

for $a, b \in G^-$. By (Rump, 2008), Section 5, this makes G^- into a self-similar hoop with structure group G. Conversely, the structure group $G(H)$ of a self-similar hoop H is an abelian ℓ-group with $G(H)^- = H$ by (Rump, 2008), Proposition 19. □

Note that Proposition 3.1 implies that a self-similar hoop H is a lattice. Explicitly, the join is given by the formula

$$a \vee b = (a \rightarrow b) \rightarrow b$$

which is well known from the theory of BCK algebras (Iséki & Tanaka, 1978).

The concept of Grothendieck group (Lang, 1965) extends to semirings as follows.

Definition 3.2. Let A be a commutative semiring. We define an *ideal* of A to be an additive submonoid I which satisfies

$$a \in A,\ b \in I \implies ab \in I. \tag{3.1}$$

We say that an ideal P is *prime* if $A \smallsetminus P$ is a submonoid of A.

Let I be an ideal of a commutative semiring. Then

$$a \sim b \ :\Longleftrightarrow\ \exists\, c \in I\colon a + c = b + c$$

is an equivalence relation, and it is easily checked that it is a congruence relation. So the equivalence classes form a commutative semiring A/I, the *factor semiring* modulo I. There is also a concept of localization.

Proposition 3.2. *Let P be a prime ideal of a commutative semiring A. There exists a morphism $q\colon A \to A_P$ of semirings with $q(A \smallsetminus P) \subset A_P^\times$ such that every morphism $f\colon A \to B$ of semirings with $f(A \smallsetminus P) \subset B^\times$ factors uniquely through q.*

Proof. Define an equivalence relation on the multiplicative monoid $A \times (A \smallsetminus P)$:

$$(a, b) \sim (c, d) \ :\Longleftrightarrow\ \exists\, s \in A \smallsetminus P\colon ads = bcs. \tag{3.2}$$

Then $x \sim y$ implies $xz \sim yz$ for all $x, y, z \in A \times (A \smallsetminus P)$. So $\sim$ is a congruence relation on $A \times (A \smallsetminus P)$. As usual, we write $\frac{a}{b}$ for the equivalence class of (a, b). So the equivalence classes form a commutative monoid A_P with a morphism $q\colon A \to A_P$ given by $q(a) := \frac{a}{1}$. Moreover, $q(A \smallsetminus P) \subset A_P^\times$. Furthermore, it is easily checked that

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}$$

is well defined and makes A_P into a commutative semiring such that q becomes a morphism of semirings. Now the universal property is straightforward. □

We call A_P the *localization* of A at P. If the zero ideal is prime, the localization at 0 yields the *quotient semifield* $K(A)$ of A.

Note that there are semirings A where 0 is prime, but A is not a semi-domain. For example, let K be a semifield. We define a *(formal) polynomial* to be an expression

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with $a_i \in K$. If $f \neq 0$, say, $a_n \neq 0$, we call $\deg f := n$ the *degree* of f. Thus, with the usual operations, the formal polynomials make up a semiring $K\langle x \rangle$, and 0 is a prime ideal. To see that $K\langle x \rangle$ need not be a semidomain, consider the case $\operatorname{char} K = 1$, that is, $K = \widetilde{G}$ for an abelian ℓ-group G. Consider two elements $a, b \in G$ with $a \nleqslant b$. Then the two formal polynomials $a \vee bx \vee x^2$ and $a \vee (a \vee b)x \vee x^2$ are distinct. However,

$$(a^2 \vee bx \vee x^2)(a \vee x) = (a^2 \vee (a \vee b)x \vee x^2)(a \vee x),$$

which shows that $\widetilde{G}\langle x\rangle$ fails to be a semi-domain! That is the reason why we speak of *formal* polynomials.

If A is a semidomain, the equivalence (3.2) simplifies to

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc,$$

which implies that all localizations A_P can be regarded as sub-semidomains of $K(A)$.

Example. Let A be a semidomain of characteristic 1. The quotient semifield $K(A)$ is of the form $K(A) = \widetilde{G}$ with an abelian ℓ-group G, and the monoid $A \smallsetminus \{0\} = A \cap G$ is a $\vee$-sub-semilattice. However, $A \cap G$ need not be the negative cone of G. Indeed, this happens if and only if A is tropical. Assume this from now on. By Definition 3.2, an ideal of A is the same as a $\vee$-sub-semilattice which is a downset. So the complement $Q := A \smallsetminus P$ of a prime ideal P of A is a convex submonoid of G^- with the property

$$a \vee b \in Q \implies a \in Q \text{ or } b \in Q,$$

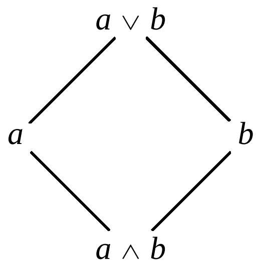
that is, Q is the negative cone of a prime ℓ-ideal in G (see (Darnel, 1995), Definitions 8.1 and 9.1). In other words, there is a one-to-one correspondence between prime ideals of A and prime ℓ-ideals of G. According to (Darnel, 1995), Proposition 14.3, the prime ideals of A can be identified with the prime filters of the negative cone G^- (with the reverse ordering). Note that the zero ideal of A corresponds to G, the "trivial" prime ℓ-ideal of G, which should not be excluded from the prime spectrum of G.

Definition 3.3. Let K be a semifield. The elements of the quotient semifield $K(x)$ of $K\langle x\rangle$ will be called *rational functions* in x. We write $K[x]$ for the image of the natural map $K\langle x\rangle \to K(x)$ and call the elements of $K[x]$ *polynomials* in x.

4. Divisors in characteristic 1

In classical algebraic geometry, divisors are intimately connected with line bundles, invertible sheaves, linear systems, and embeddings into projective spaces. Therefore, they play a decisive rôle. Here we shall study their behaviour in characteristic 1.

Thus, let G be an abelian ℓ-group. As a lattice, G is distributive. So the elements of G can be regarded as functions on a set. Let us take the simplest case where G satisfies the ascending chain condition. By a theorem of Birkhoff (Birkhoff, 1942), this implies that G is a cardinal sum $G = \boxplus_{p\in P}\mathbb{Z}$ with basis P. (Such ℓ-groups naturally arise as groups of fractional ideals of a Dedekind domain.) So each interval $[a,b] := \{c \in G \mid a \leqslant c \leqslant b\}$ has a composition series $a = c_0 < c_1 < \cdots < c_n = b$ with atomic intervals $[c_i, c_{i+1}] = \{c_i, c_{i+1}\}$. For a diagram



(4.1)

with $a, b \in G$, the intervals $[a \wedge b, a]$ and $[b, a \vee b]$ are said to be *isomorphic*, in analogy with the isomorphism theorem in group theory. *Isomorphism* between intervals is then defined by finite sequences of elementary isomorphisms (4.1). So each pair $a, b \in G$ can be connected by a finite chain $a = c_0, c_1, \ldots, c_n = b$ in G, with atomic intervals $[c_i, c_{i+1}]$ or $[c_{i+1}, c_i]$. If we attach a factor -1 to the intervals of the second type, the total count of isomorphism classes of atomic intervals on such a connecting path merely depends on the pair of endpoints a, b. Regarding the isomorphism classes of atomic intervals as "points", every element $a \in G$ is completely determined by the formal $\mathbb{Z}$-linear combination of points encountered on a path between 0 and a which is independent of the chosen path. For algebraic curves, a formal $\mathbb{Z}$-linear combination of points is called a *divisor*.

In general, there are no atomic intervals. So we have to watch out for a substitute. This naturally leads to the following

Definition 4.1. Let G be a (multiplicative) abelian ℓ-group, and let D be the subgroup of the free abelian group $\mathbb{Z}^{(G)}$ generated by the elements

$$(a \vee b) + (a \wedge b) - a - b$$

with $a, b \in G$. The factor group $\mathrm{Div}(G) := \mathbb{Z}^{(G)}/D$ will be called the group of *divisors* of G. The natural map $G \to \mathrm{Div}(G)$ will be denoted by $a \mapsto [a]$.

In the special case of a noetherian group G, it is clear that the homomorphism $G \to \mathrm{Div}(G)$ is injective. In general, this follows since $G \to \mathrm{Div}(G)$ admits a retraction $\mathrm{Div}(G) \to G$, given by the map

$$n_1[a_1] + \cdots + n_r[a_r] \mapsto a_1^{n_1} \cdots a_r^{n_r}.$$

The retraction is well defined by virtue of the equation

$$(a \vee b)(a \wedge b) = ab,$$

which holds in every abelian ℓ-group. However, even for $G = \mathbb{Z}$, the embedding

$$G \hookrightarrow \mathrm{Div}(G)$$

is far from being surjective. Instead, the group $\mathrm{Div}(G)$ tells us much about the polynomial semidomain $\widetilde{G}[x]$.

Let $G(x) := \widetilde{G}(x)^{\times}$ be the abelian ℓ-group which is freely generated by G and a single indeterminate x. Similarly, we set $G[x] := \widetilde{G}[x] \cap G(x)$. The degree of non-zero polynomials extends to a homomorphism

$$\deg\colon G(x) \to \mathbb{Z}.$$

of abelian ℓ-groups. For ordinary fields K, the degree function $\deg\colon K(x)^{\times} \to \mathbb{Z}$ is also important, but it is not a homomorphism of rings. So the degree of a polynomial or rational function in classical algebra signalizes a tropical structure!

The reader may check that

$$\big(x \vee (a \vee b)\big)\big(x \vee (a \wedge b)\big) = (x \vee a)(x \vee b)$$

holds for all $a, b \in G$. To generalize this fact, recall that an abelian ℓ-group G is *divisible* if every $a \in G$ admits an n-th root for each positive integer n, or equivalently, the pure equation

$$x^n = a$$

is solvable for any $a \in G$. (If G is written additively, this just means that G can be regarded as a $\mathbb{Q}$-vector space.) Now we have ((Rump, 2015), Theorem 1):

Fundamental theorem for abelian ℓ-groups. *Let G be a divisible abelian ℓ-group, and let $K := \widetilde{G}$ be the corresponding tropical semifield. Every non-zero polynomial $f \in K[x]$ has a unique factorization*

$$f = a(x \vee d_1)(x \vee d_2)\cdots(x \vee d_n) \tag{4.2}$$

with $a \in G$ and $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ in K.

For $K = \mathbb{R}^+_{\max}$, this theorem is known as the “fundamental theorem of tropical algebra” (see, e. g., (Cuninghame-Green & Meijer, 1980)). Two things are remarkable. First, the *roots* $d_1 \leqslant \cdots \leqslant d_n$ have to be put into linear order - otherwise, they won’t be unique. The roots of a polynomial are in fact nothing else than its divisor. So in contrast to divisors of algebraic curves, tropical divisors are not unique as unordered point sets with multiplicities. For the divisor $[a]+[b]$, the equivalence to $[a \vee b] + [a \wedge b]$ can be seen from the basic relation of Definition 4.1.

Secondly, the roots $d_1 \leqslant \cdots \leqslant d_n$ are not the zeros, because no non-zero polynomial $f \in K[x]$ satisfies $f(a) = 0$ for any $a \in G$. Only equations $f(x) = g(x)$ for a pair of polynomials are sensible! So the question whether polynomial equations can be solved in G is not answered by the fundamental theorem. We will come back to this in Section 5.

By the fundamental theorem, there is a well-defined map

$$\mathrm{div}\colon G[x] \to \mathrm{Div}(G^d) \tag{4.3}$$

for any abelian ℓ-group G with divisible closure G^d, given by

$$\mathrm{div}(f) := [d_1] + [d_2] + \cdots + [d_n]$$

for a non-zero polynomial (4.2). Every rational function $f \in G(x)$ can be written as

$$f = ax^{n_0}(x \vee d_1)^{n_1}(x \vee d_2)^{n_2}\cdots(x \vee d_r)^{n_r} \tag{4.4}$$

with $a, d_1, \ldots, d_r \in G$, and $n_0, \ldots, n_r \in \mathbb{Z}$. In contrast to polynomials where $n_1, \ldots, n_r \in \mathbb{N}$, the d_i cannot be put into linear order, which means that they are not unique! However, a and n_0 are unique. So let $G(x)^0$ denote the subgroup of rational functions $f \in G(X)$ with $a = 1$ and $n_0 = 0$. Then (Rump, 2015), Theorem 2, yields

Theorem 4.1. *Let G be a divisible abelian ℓ-group. The map (4.3) extends uniquely to a group isomorphism*

$$\mathrm{div}\colon G(x)^0 \xrightarrow{\sim} \mathrm{Div}(G)$$

with inverse map $[a] \mapsto (x \vee a)$.

This gives a complete description of the divisor group $\mathrm{Div}(G)$ and its relationship to the unit group of $\widetilde{G}(x)$, namely,

$$G(x) \cong G \times \mathbb{Z} \times G(x)^0.$$

5. Dequantization of Prüfer and Bézout domains

Proposition 3.1 suggests a study of abelian ℓ-groups via semi-domains. A first step of this program has already been taken in Section 3, where a decomposition of polynomials into linear factors has been achieved. Now let us come "back to the roots". The good news is that they are most easily calculated from the coefficients. For an abelian ℓ-group G and a polynomial $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \widetilde{G}[x]$ with $a_0a_n \neq 0$, it is not hard to show that all coefficients a_i can be assumed to be non-zero, that is, they belong to G. (This is of course not true for polynomials over a field, but note that in the tropical case, the zero element is the absolutely smallest one, smaller than every element of G.) By (Rump, 2015), Propositions 3 and 4, we have the following explicit formula for the roots: $d_i = b_{i-1}b_i^{-1}$, where

$$b_j := a_j \vee \bigvee_{i<j<k} (a_i^{k-j}a_k^{j-i})^{\frac{1}{k-i}}.$$

So the roots d_i of each polynomial are expressible in terms of k-th roots, where k does not exceed the degree of the polynomial f. Compared with the efforts of classical algebra up to the final stroke after Ruffini, Abel, and Galois - a quick victory!

However, as already mentioned, roots are not solutions. Nevertheless, the decomposition into linear factors indicates a close relationship to classical solutions. Indeed, here is a point where tropical algebra applies to the classical case.

Recall that a *fractional ideal* of an integral domain R with quotient field K is a non-zero R-submodule I of K such that $I \subset Ra$ for some $a \in K^\times$. A fractional ideal I is said to be *invertible* if there is a (necessarily unique) fractional ideal I^{-1} with $I^{-1}I = R$. Note that every invertible fractional ideal is finitely generated. An integral domain R is said to be a *Prüfer domain* (see (Gilmer, 1992), chap. IV) if the non-zero finitely generated ideals are invertible. If every non-zero finitely generated ideal of R is principal (hence invertible), R is called a *Bézout domain*.

The invertible fractional ideals of a Prüfer domain R form an abelian ℓ-group $G(R)$ with respect to inclusion. Note that

$$(I + J)(I \cap J) = IJ$$

holds for $I, J \in G(R)$, which shows that $G(R)$ is closed under finite intersection. In the special case that R is a Bézout domain, $(G(R); \supset)$ can be identified with $K^\times/R^\times$, the *group of divisibility* of R (see (Gilmer, 1992), section 16).

For a Prüfer domain R, the finitely generated ideals form a tropical semi-domain $A(R)^-$, the *dequantization* of R. By Proposition 3.2 and the Jaffard-Ohm correspondence (Jaffard, 1953; Ohm, 1966), every tropical semi-domain occurs as the dequantization of a Bézout domain. Thus, tropical algebra makes no difference between Prüfer domains and the more special Bézout domains. Since $A(R)^-$ is a semi-domain, we consider its quotient semifield $A(R)$, consisting of all finitely generated R-submodules of K. There is a natural map

$$t \colon K \to A(R) \tag{5.1}$$

from the quotient field K of R to $A(R)$, given by $t(a) := Ra$. Note that t is a monoid homomorphism, but not a morphism of semirings since $R(a + b)$ need not be equal to $Ra + Rb$.

This is by no means an anomaly. To the contrary, here is another point where tropical concepts enter the classical world. Recall that a *valuation* of a field K is a function $v\colon K \to \Gamma$ into a totally ordered abelian group Γ, augmented by an element ∞ with $\alpha + \infty = \infty$ for all $\alpha \in \Gamma \sqcup \{\infty\}$ such that the following are satisfied:

$$v(a) = \infty \iff a = 0 \tag{5.2}$$

$$v(ab) = v(a) + v(b) \tag{5.3}$$

$$v(a+b) \geqslant \min\{v(a), v(b)\}. \tag{5.4}$$

In a time where order-theoretic terms have been almost completely eliminated from the standard curriculum[1], such a function v which is not a morphism in any sense should sting in the eye! Let us rewrite (5.2)-(5.4) as follows. Endow Γ with the opposite order and write it multiplicatively. Then ∞ becomes 0 with $\alpha \cdot 0 = 0$ for all $\alpha \in \Gamma \sqcup \{0\}$, and the inequality (5.4) turns into

$$v(a+b) \leqslant v(a) \vee v(b).$$

So $\tilde{\Gamma} := \Gamma \sqcup \{0\}$ becomes a tropical semifield. The map (5.1) is characterized by the following universal property:

Proposition 5.1. *Let R be a Prüfer domain with quotient field K. Then every valuation $v\colon K \to \tilde{\Gamma}$ with $v(R) \leqslant 1$ factors uniquely through $t\colon K \to A(R)$*

$$\begin{array}{ccc} K & \xrightarrow{\quad t \quad} & A(R) \\ & {}_{v}\searrow & \vdots\, f \\ & & \tilde{\Gamma} \end{array} \tag{5.5}$$

such that $f\colon A(R) \to \tilde{\Gamma}$ is a morphism of semifields.

Proof. Define $f\colon A(R) \to \tilde{\Gamma}$ by $f(I) := \bigvee\{v(a) \mid a \in I\}$. Since every $I \in A(R)$ is of the form $I = Ra_1 + \cdots + Ra_n$, every $a = r_1a_1 + \cdots + r_na_n \in I$ with $r_i \in R$ satisfies $v(a) \leqslant v(a_1) \vee \cdots \vee v(a_n)$, which shows that f is well defined and renders (5.5) into a commutative diagram. The uniqueness of f is obvious. □

For an abelian ℓ-group G, the pure polynomial $1 \vee x^n$ is "purely inseparable":

$$1 \vee x^n = (1 \vee x)^n.$$

Therefore, the Frobenius identity

$$(a \vee b)^n = a^n \vee b^n$$

[1]It seems that Grothendieck's aversion against valuations had its bearing on this. In a letter of October 26, 1961, Serre complained: "You are very harsh on Valuations! I persist nonetheless in keeping them, for several reasons ...". Grothendieck's unrepentant response (October 31, 1961): "Your argument in favor of valuations is pretty funny ..."

holds in G, and (Darnel, 1995), 47.11, implies that G is a subdirect product of linearly ordered abelian groups. Thus, for a Prüfer domain R, the diagram (5.5) can be expressed by a single map

$$K^{\times} \xrightarrow{t} A(R)^{\times} \hookrightarrow \prod \Gamma,$$

where Γ runs through the value groups of all valuations of R. Moreover, t is surjective if and only if R is a Bézout domain. Examples of Bézout domains abound. The most prominent examples are the ring of algebraic integers ((Kaplansky, 1974), Theorem 102) and the ring of entire functions (Helmer, 1940). The ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials $f \in \mathbb{Q}[x]$ is an example of a Prüfer domain which is not a Bézout domain (Brizolis, 1979) (cf. (Narkiewicz, 1995), VII). In contrast to $\mathbb{Z}[x]$, which is not a Prüfer domain, $\mathrm{Int}(\mathbb{Z})$ has an uncountable number of maximal ideals, while both rings have Krull dimension 2.

The valuations $v\colon R \to \widetilde{\Gamma}$ or rather their extensions

$$v\colon K \to \widetilde{\Gamma}$$

to K are just the components of the tropicalization t. Thus, if V is a valuation domain with quotient field K, the corresponding valuation is just the tropicalization

$$t\colon K \to A(V),$$

and $A(V)^{\times}$ is the value group of V. There is a natural extension $t'\colon K[x] \to A(V)[x]$ via $t'(x) := x$. Explicitly:

$$t'(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = t(a_0) \vee t(a_1)x \vee t(a_2)x^2 \vee \cdots \vee t(a_n)x^n.$$

Note that $K[x]$ is even a principal ideal domain. We add a prime to make sure that t' cannot be confused with the restriction of $t\colon K(x) \to A(K[x])$ to $K[x]$.

For higher rank valuations, Hensel's lemma, which roughly states that coprime factorizations of polynomials modulo the maximal ideal can be lifted, is no longer valid (see (Engler & Prestel, 2005), Remark 2.4.6). What remains is that the topology of a field K with a complete valuation extends uniquely to fields L which are finite over K (Roquette, 1958). The proper substitute for complete valuation rings (where Hensel's lemma merely holds in rank 1) are the *Henselian* local rings, introduced by Azumaya (Azumaya, 1951) and developed by Nagata (Nagata, 1962), which satisfy Hensel's lemma by definition. For equivalent characterizations, see (Ribenboim, 1985). The most important characterization of Henselian local integral domains is that every integral extension is local ((Nagata, 1954), Theorem 7). For Henselian valuations of a field K, this means that they uniquely extend to the algebraic closure $\overline{K}$.

Proposition 5.2. *Let V be a Henselian valuation domain with quotient field K. Then $t\colon K \to A(V)$ extends uniquely to the algebraic closure $\overline{K}$ of K, which gives a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{t} & A(V) \\ \downarrow & & \downarrow \\ \overline{K} & \xrightarrow{t} & A(V)^d. \end{array}$$

For every non-zero polynomial $f \in K[x]$ with roots $\alpha_1, \ldots, \alpha_n \in \overline{K}$, the roots of $t'(f)$ are $t(\alpha_1), \ldots, t(\alpha_n)$.

Proof. Since V is Henselian, the integral closure S of V in $\overline{K}$ is local, hence a valuation ring ((Bourbaki, 1972), VI.8.6, Proposition 6). Furthermore, $A(S)$ can be identified with the divisible closure of $A(V)$. If a is the leading coefficient of f, we have $f = a(x - \alpha_1) \cdots (x - \alpha_n)$ in $\overline{K}[x]$. Since t' is multiplicative, this implies that $t'(f) = t(a)(x \vee t(\alpha_1)) \cdots (x \vee t(\alpha_n))$. As $A(V)^d$ is linearly ordered, this proves the claim. □

Proposition 5.2 is the basis for Newton's method, which makes use of the following result. Its first part is essentially due to Ostrowski (Ostrowski, 1935).

Proposition 5.3. *Let V be a Henselian valuation domain with quotient field K, and let $f = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ be a non-zero polynomial. If f is irreducible, $t'(f)$ has a single root in $A(V)^d$. Conversely, if $t'(f)$ has a single root in $A(V)^d$, and there is no divisor $d > 1$ of n such that $A(V)^\times$ contains a d-th root of $t(a_0 a_n^{-1})$, then f is irreducible.*

Proof. Let S be the integral closure of V in the splitting field L of f. Every element σ of the Galois group $G(L|K)$ leaves S invariant: $\sigma(S) = S$. Hence, if f is irreducible, every zero α of f satisfies $t(\sigma(\alpha)) = t(\alpha)$ for all $\sigma \in G(L|K)$. So there is a single root $t(\alpha)$ of $t'(f)$ of multiplicity $\deg f$.

Conversely, assume that $t'(f)$ has a single root in $A(V)^d$, and that there is no divisor $d > 1$ of n such that $A(V)^\times$ contains a d-th root of $t(a_0 a_n^{-1})$. Let g be a monic irreducible factor of f. Without loss of generality, we can assume that $a_n = 1$. Then the single root α of $t'(f)$ satisfies $t'(f) = (x \vee \alpha)^n$ and $\alpha^n = t(a_0)$. If g is of degree m, then $t'(g) = (x \vee \alpha)^m$. Let $d > 0$ be the greatest common divisor of m and n. Then $d = pm + qn$ for some integers $p, q \in \mathbb{Z}$. Hence $h := (x \vee \alpha)^d = t'(g)^p t'(f)^q \in A(V)[x]$, and $d|m$ implies that $t'(g) = h^{m/d}$. Furthermore, the absolute term $a := \alpha^d$ of h belongs to $A(V)^\times$, and $a^{n/d} = t(a_0)$. By assumption, this gives $d = n$. Whence $f = g = h$ is irreducible. □

Proposition 5.3 reduces irreducibility of polynomials over K almost completely to the tropical semifield $A(V)$, where the complete factorization is obtained by straightforward calculation. Contrary to a remark in (Khanduja & Saha, 1997), the condition of the criterion is not necessary, as the trivial example $1 + x + x^2 \in \widehat{\mathbb{Q}}_2[x]$ shows. (The mistake is caused by rewriting the special version of Popescu and Zaharescu (Popescu & Zaharescu, 1995) in a logically different way.) In particular, we have the following

Corollary. *Let V be a Henselian valuation domain with quotient field K, and let $f = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ be a non-zero polynomial. If $t'(f)$ has m distinct roots, f splits into m relatively prime factors.*

Newton's method was applied already in the early days of valuation theory, invented by Hensel (Hensel, 1908), and developed by Kürschák (Kürschák, 1913*a*; Kürschák, 1913*b*), Ostrowski (Ostrowski, 1916, 1917, 1933), and Rychlík (Rump & Yang, 2008; Rychlík, 1924). Newton's method also appears in a paper of Rella (Rella, 1927), but in essence, it can even be traced back to Newton himself via Puiseux's theorem which states, in modern terms, that the field of Puiseux series over

$\overline{\mathbb{C}((x))}$ is the algebraic closure of the field $\mathbb{C}((x))$ of formal Laurent polynomials, the quotient field of $\mathbb{C}[[x]]$.

Here the field $\mathbb{C}((x))$ not only builts a bridge between algebraic curves and complex analysis; in addition, it is maximally close to its tropical shadow: Every finite extension field of $\mathbb{C}((x))$ is isomorphic to $\mathbb{C}((x))$, the extension being just given be extracting some n-th root of x. So if S denotes the the integral closure S of $\mathbb{C}[[x]]$ in the algebraic closure of $\mathbb{C}((x))$, the tropical picture is encoded in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}((x)) & \xrightarrow{t} & A(\mathbb{C}[[x]]) = \mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ \overline{\mathbb{C}((x))} & \xrightarrow{t} & A(S) = \mathbb{Q}. \end{array}$$

A lot of irreducibility criteria can be derived from Proposition 5.3, which seems to be the "true metaphysics"[2] behind polynomial factorization. Eisenstein's criterion is just the first of a series of irreducibility criteria (e. g., (Dumas, 1906; Kürschák, 1923; Ore, 1923, 1924; Rella, 1927; MacLane, 1938; Azumaya, 1951)) which follow the same "tropical" pattern.

6. Algebraic equations in characteristic 1

Now we return to the problem that solutions of equations between tropical polynomials cannot just be read off from the roots. Let us start with linear equations

$$ax \vee b = cx \vee d \tag{6.1}$$

in a tropical semifield K. Looking quite innocent, they already bear a mild challenge. In contrast to classical algebra, such an equation is not always solvable. To avoid trivialities, assume that $a, b, c, d \in G := K^{\times}$. Then x cannot be zero, unless $b = d$. To solve Eq. (6.1), consider the map $p\colon G \to G$ given by

$$p(x) := \big((ad \vee bc)x \vee bd\big)\big(acx \vee (ad \vee bc)\big)^{-1}. \tag{6.2}$$

Note the expression $\Delta := ad \vee bc$ which looks like a determinant! The roots of the left- and right-hand side of Eq. (6.1) are respectively

$$\alpha := a^{-1}b, \qquad \beta := c^{-1}d.$$

Proposition 6.1. *The map* (6.2) *is idempotent and maps G onto the interval*

$$[\alpha \wedge \beta, \alpha \vee \beta]. \tag{6.3}$$

Every solution x of Eq. (6.1) *is mapped into a solution $p(x)$.*

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[2] A common expression of the 18th century (see (Carnot, 1860); or (Speiser, 1956), Chapter 17, concerning Lagrange who considered groups as "la vraie métaphysique" of algebraic equations).

Proof. To verify that $p^2 = p$, note first that $\Delta^2 \geqslant abcd$. Now Eq. (6.2) can be written as

$$p(x) = \frac{\Delta x \vee bd}{acx \vee \Delta}.$$

So we have

$$p(p(x)) = \frac{\Delta(\Delta x \vee bd)(acx \vee \Delta)^{-1} \vee bd}{ac(\Delta x \vee bd)(acx \vee \Delta)^{-1} \vee \Delta} = \frac{\Delta(\Delta x \vee bd) \vee bd(acx \vee \Delta)}{ac(\Delta x \vee bd) \vee \Delta(acx \vee \Delta)}$$
$$= \frac{(\Delta^2 \vee abcd)x \vee \Delta bd}{ac\Delta x \vee (\Delta^2 \vee abcd)} = \frac{\Delta^2 x \vee \Delta bd}{ac\Delta x \vee \Delta^2} = \frac{\Delta x \vee bd}{acx \vee \Delta} = p(x).$$

Furthermore,

$$p(x) = (\Delta x \vee bd)(acx \vee \Delta)^{-1} = (\Delta x \vee bd)(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1})$$
$$= \Delta x(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1}) \vee bd(a^{-1}c^{-1}x^{-1} \wedge \Delta^{-1})$$
$$\leqslant \Delta a^{-1}c^{-1} \vee bd\Delta^{-1} = a^{-1}b \vee c^{-1}d,$$

and similarly, $p(x) = (\Delta x \vee bd)a^{-1}c^{-1}x^{-1} \wedge (\Delta x \vee bd)\Delta^{-1} \geqslant \Delta a^{-1}c^{-1} \wedge bd\Delta^{-1} = a^{-1}b \wedge c^{-1}d$. Thus p maps into the interval $[\alpha \wedge \beta, \alpha \vee \beta]$. For $x \in [\alpha \wedge \beta, \alpha \vee \beta]$, we have $acx \leqslant ac(a^{-1}b \vee c^{-1}d) = \Delta$, and secondly, $bd \leqslant (ad \vee bc)(a^{-1}b \wedge c^{-1}d) \leqslant \Delta x$. Hence $p(x) = (\Delta x \vee bd)\Delta^{-1} = \Delta x \Delta^{-1} = x$.

Finally, if x is a solution of Eq. (6.1), then $(ap(x) \vee b)(acx \vee \Delta) = a(\Delta x \vee bd) \vee b(acx \vee \Delta) = a\Delta x \vee b\Delta = (cx \vee d)\Delta = (c\Delta \vee acd)x \vee d(bc \vee \Delta) = c(\Delta x \vee bd) \vee d(acx \vee \Delta) = (cp(x) \vee d)(acx \vee \Delta)$, which shows that $p(x)$ is a solution of Eq. (6.1). □

By Proposition 6.1, the solutions of Eq. (6.1) are the fibers of the solutions in the interval (6.3) under the projection p. So it remains to consider solutions in the interval (6.3). To solve the equation, we consider another map $s\colon G \to G$ with

$$s(x) := a^{-1}d(ax \vee b)(cx \vee d)^{-1}. \tag{6.4}$$

Proposition 6.2. *The map (6.4) satisfies $s^2 = p$. In particular, s is an involution on the interval (6.3).*

Proof. We have

$$s(s(x)) = a^{-1}d \cdot \frac{a \cdot a^{-1}d(ax \vee b)(cx \vee d)^{-1} \vee b}{c \cdot a^{-1}d(ax \vee b)(cx \vee d)^{-1} \vee d} = a^{-1}d \cdot \frac{d(ax \vee b) \vee b(cx \vee d)}{ca^{-1}d(ax \vee b) \vee d(cx \vee d)}$$
$$= \frac{d(ax \vee b) \vee b(cx \vee d)}{c(ax \vee b) \vee a(cx \vee d)} = \frac{\Delta x \vee bd}{acx \vee \Delta} = p(x).$$ □

Corollary. *The following are equivalent.*
(a) *Eq. (6.1) is solvable.*
(b) $ad \wedge bc \leqslant ab \leqslant ad \vee bc$.

(c) $ad \wedge bc \leqslant cd \leqslant ad \vee bc$.

If Eq. (6.1) *is solvable, the unique solution in the interval* (6.3) *is* $x = (b \vee d)(a \vee c)^{-1}$.

Proof. The equivalence of (b) and (c) follows by symmetry. Condition (c) is equivalent to $a^{-1}d \in [\alpha \wedge \beta, \alpha \vee \beta]$. Furthermore, Eq. (6.4) shows that s maps every solution of Eq. (6.1) to $a^{-1}d$. Hence, if Eq. (6.1) is solvable, there is a solution $x \in [\alpha \wedge \beta, \alpha \vee \beta]$, which yields $a^{-1}d = s(x) = sp(x) = s^3(x) = ps(x) \in [\alpha \wedge \beta, \alpha \vee \beta]$. Thus (c) is necessary for the solvability of Eq. (6.1). Moreover, $x = p(x) = s^2(x) = s(a^{-1}d) = a^{-1}d(d \vee b)(ca^{-1}d \vee d)^{-1} = (d \vee b)(c \vee a)^{-1}$.

Conversely, if $a^{-1}d \in [\alpha \wedge \beta, \alpha \vee \beta]$, then $x := s(a^{-1}d)$ satisfies $s(x) = p(a^{-1}d) = a^{-1}d$. Hence Eq. (6.4) implies that x is a solution. □

Our discussion of linear equations already shows that solutions need not exist, even for non-trivial equations. Therefore, a concept of algebraically closed semifield has to take this into account. So we arrive at the following

Definition 6.1. A semifield K is said to be *algebraically closed* if every equation $f(x) = 1$ with $f \in K(x)$ which is solvable in some extension semifield of K admits a solution in K.

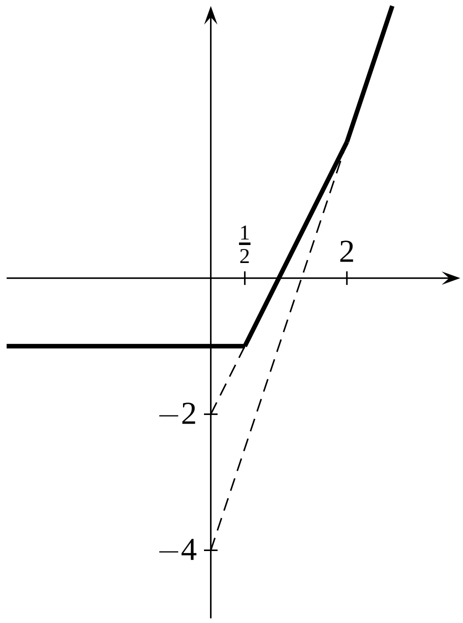
Note that an equation $f(x) = 1$ in $K(x)$ can also be written in the form

$$g(x) = h(x)$$

with polynomials $g, h \in K[x]$. We mention here that polynomials in $K[x]$ can be regarded as functions. Namely, for a non-trivial abelian ℓ-group G, Proposition 5 of (Rump, 2015) implies that $f \in \widetilde{G}[x]$ is uniquely determined by the corresponding function $f\colon G^d \to G^d$ on the divisible closure of G. For $\widetilde{G} = \mathbb{R}^+_{\max}$, it is convenient to write $\mathbb{R}^+$ additively via the logarithm. So G is turned into the additive group of $\mathbb{R}$, and 0 becomes $-\infty$. The graph of a polynomial is then piecewise linear, a classical Newton polygon. For example, the polynomial

$$-1 \vee (-2 + 2x) \vee (-4 + 3x)$$

looks as follows:



Here the coefficients are in $\mathbb{Z}$, while the roots are in $\frac{1}{2}\mathbb{Z}$, because the linear term is missing. The root $\frac{1}{2}$ is of multiplicity 2. Thus, if we add a linear term to get the polynomial into the normal form, the coefficient of x would be $-\frac{3}{2}$.

The corollary of Proposition 6.2 shows that in the tropical case, linear equations are not trivial, and that roots only play a certain rôle with respect to the solutions. In compensation for this initial difficulty of tropical equations, Theorem 4 of (Rump, 2015) states that we don't have to go beyond quadratic equations! Precisely, the theorem says that a tropical semifield K is algebraically closed if and only if the ℓ-group $G := K^{\times}$ is divisible, that is, the pure equations $x^n = a$ are solvable in G, and the quadratic equations

$$(a \vee 1)x^2 \vee (a^2 \vee b \vee 1)x \vee (a^2 \vee a) = ax^2 \vee a \tag{6.5}$$

are solvable for all $a, b \in G$. Note that the solvablity clause (in an extension semifield) of Definition 6.1 is missing. In fact, we have

Proposition 6.3. *The equations (6.5) are solvable in any totally ordered abelian group.*

Proof. For $a \geqslant 1$, Eq. (6.5) becomes $ax^2 \vee (a^2 \vee b)x \vee a^2 = ax^2 \vee a$. We show that this equation holds for all $x \geqslant a \vee a^{-1}b$. Indeed, the latter implies that $ax^2 \geqslant ax(a \vee a^{-1}b) = (a^2 \vee b)x \geqslant a^2$. So the equation (6.5) is solved. For $a \leqslant 1$, the equation becomes $x^2 \vee (b \vee 1)x \vee a = ax^2 \vee a$. Here we choose $x \leqslant a(b \vee 1)^{-1}$. Then $x \leqslant a$ and $(b \vee 1)x \leqslant a$. Hence $ax^2 \leqslant x \leqslant a$, which solves the equation. □

Corollary 1. *For any tropical semifield, there exists a (tropical) extension semifield where Eq. (6.5) is solvable.*

Proof. This follows since every abelian ℓ-group G is a subdirect product of totally ordered abelian groups ((Darnel, 1995), 47.11). □

Furthermore, Theorem 4 of (Rump, 2015) implies

Corollary 2. *Let G be a totally ordered abelian group. Then $\widetilde{G}$ is algebraically closed if and only if the pure equation $x^n = a$ is solvable for all positive integers n and $a \in G$.*

To analyse Eq. (6.5), consider an additive abelian ℓ-group G. The proof of Proposition 6.3 then tells us that in the totally ordered case, solutions of Eq. (6.5) exist, but depending on the sign of a, they must be either large enough if $a > 0$ or small enough if $a < 0$.

It is this point where geometry enters the scene. By the Jaffard-Ohm correspondence, every abelian ℓ-group G occurs as a tropicalized Bézout domain R. By (**?**), Proposition 7, the structure sheaf of R can be transferred to G, which yields a sheaf $\breve{G}$ on a spectral space X with totally ordered stalks such that $\Gamma(X, \breve{G}) \cong G$. In the archimedean case, $\breve{G}$ is a sheaf of germs of continuous functions. Therefore, the sensitivity of Eq. (6.5) against sign change of a is best illustrated by the following

Example. Let G be the ℓ-group $\mathscr{C}[-1, 1]$ of continuous real functions on the closed interval $[-1, 1]$. Multiplying Eq. (6.5) by $a^{-1}x^{-1}$, it takes the symmetric form

$$a^{-}x \vee c \vee a^{+}x^{-1} = |x| \tag{6.6}$$

with $a \in G$ and $c \geqslant |a|$, where $|a| := a \vee a^{-1}$ and

$$a^{+} := a \vee 1, \qquad a^{-} := a^{-1} \vee 1.$$

Writing Eq. (6.6) additively, it becomes

$$(a^{-} + x) \vee c \vee (a^{+} - x) = |x|.$$

Passing to $\mathscr{C}[-1, 1]$, let c be the constant function $t \mapsto 1$, and let a be arbitrary with $|a| \leqslant c$. If $x(t) \geqslant 0$, this implies that $x(t) = |x|(t) \geqslant 1$, while $x(t) \leqslant 0$ gives $-x(t) \geqslant 1$, that is, $x(t) \leqslant -1$. Thus Eq. (6.5) cannot be solvable by a continuous function.

Recall that an element $u \geqslant 1$ of a (multiplicative) abelian ℓ-group G is said to be a *weak order unit* ((Darnel, 1995), 54.3) if $u \wedge a = 1$ implies that $a = 1$. For $a \in G^{+}$, we write $G(a)$ for the ℓ-ideal generated by a. It consists of the elements $x \in G$ with $|x| \leqslant a^n$ for some $n \in \mathbb{N}$.

Definition 6.2. (McGovern, 2005) An abelian ℓ-group G is said to be *weakly complemented* if for any pair $a, b \in G$ with $a \wedge b = 1$, there exist $a', b' \in G$ with $a \leqslant a'$ and $b \leqslant b'$ such that $a' \wedge b' = 1$ and $a'b'$ is a weak order unit of G. If $G(a)$ is weakly complemented for all $a \in G^{+}$, then G is called *locally weakly complemented*.

The following result shows that the solvability of Eq. (6.5) merely depends on the lattice structure of G. To state it, we need a very weak form of projectability. Recall that an abelian ℓ-group G is *strongly projectable* (Darnel, 1995) if the *polar*

$$I^{\perp} := \{a \in G \mid \forall\, b \in I \colon |a| \wedge |b| = 1\}$$

of any ℓ-ideal I is a cardinal summand: $G = I^\perp \boxplus I^{\perp\perp}$. If this holds for principal ℓ-ideals $I = G(a)$, then G is called *projectable*. More generally, G is said to be *semi-projectable*[3] (Bigard *et al.*, 1977) if

$$(a \wedge b)^\perp = a^\perp b^\perp$$

for $a, b \in G^+$. (For a geometric characterization, see (Rump, 2014), corollary of Theorem 1.) Still more generally, we call G *z-projectable* (Rump, 2014) if

$$(ab)^{\perp\perp} = a^{\perp\perp} b^{\perp\perp}$$

holds for $a, b \in G^+$. Thus

strongly projectable $\Longrightarrow$ projectable $\Longrightarrow$ semi-projectable $\Longrightarrow$ z-projectable

All these concepts are pairwise inequivalent. The line of implications could even be enlarged to seven types of projectability (Rump, 2014) which all have their particular relevance (cf. the hierarchy of T_n-spaces in general topology). Now we are ready to prove

Theorem 6.1. *Let G be an abelian ℓ-group. The following are equivalent.*
(a) *The quadratic equations (6.5) are solvable in G.*
(b) *For $a, b, c \in G$ with $a \wedge b = 1$ and $c \geqslant a \vee b$, there exist $a', b' \in G$ with $a' \geqslant a$ and $b' \geqslant b$ such that $a' \wedge b' = 1$ and $a' \vee b' = c$.*
(c) *G is semi-projectable and locally weakly complemented.*
(d) *G is z-projectable and locally weakly complemented.*

Proof. (a) $\Rightarrow$ (b): By assumption, there exists a solution $x \in G$ of Eq. (6.6) with ab^{-1} instead of a. Then $bx \leqslant x^+x^-$ and $ax^{-1} \leqslant x^+x^-$, which gives $(x^-)^2 \geqslant b$ and $(x^+)^2 \geqslant a$. Define $a' := (x^+)^2 \wedge c$ and $b' := (x^-)^2 \wedge c$. Then $a' \wedge b' = 1$ and $a' \vee b' = \big((x^+)^2 \vee (x^-)^2\big) \wedge c = c$.

(b) $\Rightarrow$ (c): Let $P \neq Q$ be minimal prime ℓ-ideals of G. Choose $a \in P \cap G^+ \setminus Q$. Since P is minimal, $a \wedge b = 1$ for some $b \notin P$. For any $c \geqslant a \vee b$, the elements a', b' in (b) satisfy $a' \in b^\perp$ and $b' \in a^\perp$. Hence $a^\perp b^\perp = G$. Since $a^\perp \subset Q$ and $b^\perp \subset P$, we obtain $PQ = G$, which shows that G has stranded primes. By (Bigard *et al.*, 1977), Proposition 7.5.1, this implies that G is semi-projectable. Moreover, (b) implies that G is locally weakly complemented.

(c) $\Rightarrow$ (d): By (Rump, 2014), Proposition 4, every semi-projectable abelian ℓ-group is z-projectable.

(d) $\Rightarrow$ (a): Let $a, b, c \in G$ with $a \wedge b = 1$ and $c \geqslant a \vee b$ be given. By the equivalence of Eq. (6.5) and Eq. (6.6), it is enough to solve the equation

$$ax \vee c \vee bx^{-1} = |x|. \tag{6.7}$$

[3]Some authors replace this term by "having stranded primes", referring to an equivalent form proved in (Bigard *et al.*, 1977), Proposition 7.5.1. Darnel (Darnel, 1995) argues that "semi-projectable" does not come close to "projectable" (referring perhaps to the "projections" of a cardinal sum). Note, however, the equivalent version $a \wedge b = 1 \Rightarrow a^\perp b^\perp = G$, which gives half of a cardinal sum: "semi" $\times$ "projectable".

By assumption, there exist $a', b' \in G$ with $a' \geqslant a$ and $b' \geqslant b$ such that $a' \wedge b' = 1$ and $(a'b')^{\perp} \cap G(c) = \{1\}$. Since G is z-projectable, this yields $c \in (a'b')^{\perp\perp} = (a')^{\perp\perp}(b')^{\perp\perp}$. So there are $p \in (a')^{\perp\perp} \cap G^{+}$ and $q \in (b')^{\perp\perp} \cap G^{+}$ with $c = pq$. In particular, this implies that $p \wedge q = 1$. Hence $a = a \wedge (p \vee q) = (a \wedge p) \vee (a \wedge q) = a \wedge p$. So we have $a \leqslant p$, and similarly, $b \leqslant q$. Thus $x := qp^{-1}$ solves Eq. (6.7). □

By (Rump, 2015), Theorem 4, we obtain

Corollary 1. *Let G be an abelian ℓ-group. The tropical semifield $\widetilde{G}$ is algebraically closed if and only if G is divisible and its underlying lattice satisfies condition (b) of Theorem 6.1.*

Recall that a ring R is said to be *clean* (Nicholson, 1977) if every $a \in R$ is a sum of an idempotent and a unit. Nicholson (Nicholson, 1977) proved that a clean ring R satisfies the *exchange property* (Crawley & Jónsson, 1964; Warfield, 1972), which means that for every decomposition $M = R \oplus N = \bigoplus_{i \in I} M_i$ of modules, there are submodules $M_i' \subset M_i$ with $M = R \oplus \bigoplus_{i \in I} M_i'$. For example, commutative von Neumann regular rings, and semiperfect rings, are clean. For various characterizations, see (McGovern, 2005). If every non-isomorphic homomorphic image of R is clean, the ring R is called *neat* (McGovern, 2005).

Corollary 2. *A Bézout domain is neat if and only if its group of divisibility satisfies the equivalent conditions of Theorem 6.1.*

Proof. By (McGovern, 2005), Theorem 5.7, a Bézout domain is neat if and only if its group of divisibility is semi-projectable and locally weakly complemented. Thus Theorem 6.1 applies. □

Remark. Note that the underlying lattice of an abelian ℓ-group is self-dual via $x \mapsto x^{-1}$. Thus, for a Bézout domain R, Corollary 2 remains valid if the group of divisibility is replaced by the unit group $A(R)^{\times}$ of the tropical semifield $A(R)$. In particular, Corollary 2 gives a characterization of Bézout domains R with $A(R)$ algebraically closed.

Finally, we consider the abelian ℓ-group $\mathscr{C}(X)$ of continuous real valued functions on a topological space X. Note that $\mathscr{C}(X)$ is also a ring. To avoid confusion, let us denote this ring by $C(X)$. By (Gillman & Jerison, 1960), Theorem 3.9, there is no loss of generality if X is assumed to be completely regular. It is known that $C(X)$ is a Bézout ring (that is, every finitely generated ideal is principal) if and only if X is an *F-space*, which originally was just defined by this property (Gillman & Henriksen, 1956). For equivalent characterizations, see (Gillman & Jerison, 1960), Theorem 14.25. One of these characterizations states that for every $f \in C(X)$ there is an element $g \in C(X)$ with $f = g|f|$.

Corollary 3. *Let X be a completely regular space. If the tropical semifield $\widetilde{\mathscr{C}(X)}$ is algebraically closed, then X is an F-space.*

Proof. Let $f \in C(X)$ be given. Then $(f^{+} \wedge 1) \wedge (f^{-} \wedge 1) = 0$ and $(f^{+} \wedge 1) \vee (f^{-} \wedge 1) \leqslant 1$. Thus, by Corollary 1, there exist $g, h \in \mathscr{C}(X)$ with $f^{+} \wedge 1 \leqslant g$ and $f^{-} \wedge 1 \leqslant h$ such that $g \wedge h = 0$ and $g \vee h = 1$. We claim that $f = (g - h)|f|$. If $f(t) > 0$, then $0 < f^{+}(t) \wedge 1 \leqslant g(t)$. Hence $h(t) = 0$, and thus $(g - h)(t) = 1$. Similarly, $f(t) < 0$ implies that $0 < f^{-}(t) \wedge 1 \leqslant h(t)$, which yields $(g - h)(t) = -1$. Thus X is an F-space. □

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New Subclass of p - valent Harmonic Meromorphic Functions

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Abstract

In this paper, we have introduced a new subclass of p-valent harmonic meromorphic and orientation preserving functions in the exterior of the unit disc. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex combination for the functions belonging to this class are obtained.

Keywords: Harmonic functions, p-valent functions, meromorphic functions, convex combination, distortion bounds.
2010 MSC: 30C45, 30C50, 30C55.

1. Introduction

Let C be the field of complex numbers. A continuous function $f(z) = u + iv$ is a complex valued harmonic function in a domain $D \subseteq C$, if both u and v are real harmonic in D. Hengartner and Schober [5], among others, investigated the class of functions of the form $f(z) = h(z) + \overline{g(z)}$, which are harmonic, meromorphic, orientation preserving and univalent in $\widetilde{U} = \{z : |z| > 1\}$ so that $f(\infty) = \infty$. It is known that f(z) admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

where

$$h(z) = az + \sum_{n=0}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=0}^{\infty} b_n z^{-n} \tag{1.2}$$

For $0 \le |\beta| \le |\alpha|$ and $a(z) = \dfrac{\overline{f_{\bar{z}}}}{f_z}$ is analytic and satisfies $|a(z)| < 1$ for $z \in \widetilde{U}$. Since the affine transformation

$$\frac{\overline{\alpha} f - \overline{\beta f} - \overline{\alpha} a_0 + \overline{\beta a_0}}{|\alpha|^2 - |\beta|^2}$$

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is again in the class studied by Hengartner and Schober see (Hengartner & Schober, 1987). Recently, Jahangiri (Jahangiri, 2000) assumed $\alpha = 1, \beta = 0$ and removed the logarithmic singularity by letting $A = 0$ in (1.1) and focused on the study of the family of harmonic meromorphic functions.

For fixed positive integer p, consider the family $\Sigma_H(p)$ consisting of functions

$$f(z) = h(z) + \overline{g(z)} \tag{1.3}$$

which are $p-$valent harmonic meromorphic functions in $\widetilde{U}$, where

$$h(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)},$$
$$g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}, \quad |b_p| < 1 \tag{1.4}$$

we call $h(z)$ the analytic part and $g(z)$ is co-analytic part of $f(z)$. For $0 \le \gamma < 1$, $k \ge 1$ and $0 \le \alpha \le 2\pi$, we define a new subclass as follows: Let $\Sigma_H(p, \gamma, k)$ consist of functions $f(z)$ satisfying the conditions

$$Re\left\{(1 + ke^{i\alpha})\frac{zf'(z)}{z'f(z)} - pke^{i\alpha}\right\} \ge p\gamma, \tag{1.5}$$

where $z' = \dfrac{\partial}{\partial\theta} z$ with $z = re^{i\theta}$, $r > 1$ and θ is real.

Further, let $\Sigma_{\overline{H}}(p, \gamma, k)$ denote the subclass of $\Sigma_H(p, \gamma, k)$ consisting of functions $f(z) = h(z) + \overline{g(z)}$ such that $h(z)$ and $g(z)$ are of the form

$$h(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)},$$
$$g(z) = -\sum_{n=1}^{\infty} |b_{n+p-1}| z^{-(n+p-1)}, \quad |b_p| < 1 \tag{1.6}$$

Note that, various other subclasses of harmonic $p-$valent meromorphic functions have been studied rather extensively by Ahuja and Jahangiri (Ahuja & Jahangiri, 2003) and Murugusundaramoorthy (Murugusundaramoorthy, 2003), we also note that, $\Sigma_H(1, \gamma, 1)$, the class of harmonic meromorphic functions, was studied by Rosy (T. Rosy & Jahangiri, 2001). Among other things, Ahuja and Jahangiri (Ahuja & Jahangiri, 2003), proved that if, $f(z) = h(z) + \overline{g(z)}$ is given by (1.4) and if,

$$\sum_{n=1}^{\infty} (n + p - 1)(|a_{n+p-1}| + |b_{n+p-1}|) \le p, \tag{1.7}$$

then $f(z)$ is harmonic, sense -preserving and $p-$valent in $\widetilde{U}$ and $f \in \Sigma_H(p)$.

In the present paper, we have obtained coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combinations for functions in the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

2. Coefficient Bounds

First we state and prove the coefficient bound for the class $\Sigma_H(p, \gamma, k)$.

Theorem 2.1. *Let* $f(z) = h(z) + \overline{g(z)}$ *with* $h(z)$ *and* $g(z)$ *given by* (1.4) . *If*

$$\sum_{n=1}^{\infty} [(n + p - 1)(1 + k) + p(k + \gamma)] \left|a_{n+p-1}\right| + [(n + p - 1)(1 + k) - p(k + \gamma)] \left|b_{n+p-1}\right| \le p(1 - \gamma), \tag{2.1}$$

then $f(z)$ *is harmonic, orientation preserving and* $p-$*valent in* $\widetilde{U}$ *and* $f \in \Sigma_H(p, \gamma, k)$.

Proof. Suppose that (2.1) holds. Then we have

$$Re\frac{(1 + ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} = \frac{A(z)}{B(z)} \ge p\gamma, \tag{2.2}$$

where $z = re^{i\theta}$, $0 \le r < 1$, $0 \le \gamma < 1$, $k \ge 1$, $0 \le \alpha \le 2\pi$,
here, we let

$$A(z) = (1 + ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)}) \tag{2.3}$$

and

$$B(z) = (h(z) + \overline{g(z)}). \tag{2.4}$$

Using the fact that $Rew \ge p\gamma$, if and only if $|p - \gamma + \omega| \ge |p + \gamma - \omega|$, it suffices to show that

$$|A(z) + p(1 - \gamma)B(z)| - |A(z) - p(1 + \gamma)B(z)| \ge 0. \tag{2.5}$$

Substituting the expressions for $A(z)$ and $B(z)$ in (2.5), we obtain

$$\begin{aligned}
|A(z) + p(1 - \gamma)B(z)| - |A(z) - p(1 + \gamma)B(z)| &= \left|p(1 - \gamma)h(z) + (1 + ke^{i\alpha})zh'(z)) - pke^{i\alpha}h(z)\right| \\
&+ \left|\overline{p(1 - \gamma)g(z) - (1 + ke^{i\alpha})zg'(z) - pke^{i\alpha}g(z)}\right| - |\, p(1 + \gamma)h(z) - (1 + ke^{i\alpha})zh'(z) + pke^{i\alpha}h(z) \\
&+ \overline{p(1 + \gamma)g(z) + (1 + ke^{i\alpha})zg'h(z) + pke^{i\alpha}g(z)}\, | \\
&= \left| p(2 - \gamma)z^p - \sum_{n=1}^{\infty}[(1 + ke^{i\alpha})(n + p - 1) + p(ke^{i\alpha} - p - \gamma)] \right| a_{n+p-1}z^{-(n+p-1)} \\
&- \sum_{n=1}^{\infty}[(1 + ke^{i\alpha})(n + p - 1) + p(1 - ke^{i\alpha} - p - \gamma)] \left|b_{n+p-1}\right| z^{-(n+p-1)}| \\
&\left| \gamma pz^p - \sum_{n=1}^{\infty}[(1 + ke^{i\alpha})(n + p - 1) + p(ke^{i\alpha} + 1 + \gamma)] \left|a_{n+p-1}\right| z^{-(n+p-1)} \right.
\end{aligned}$$

$$\left. - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) - p(ke^{i\alpha}+1+\gamma)]\left|b_{n+p-1}\right| z^{-(n+p-1)} \right|$$

$$2p(1-\gamma)\,|z|^p - \sum_{n=1}^{\infty} 2(n+p-1)(1+k) + 2p(k+\gamma)\left|a_{n+p-1}\right| |z|^{-(n+p-1)}$$

$$- \sum_{n=1}^{\infty} 2(n+p-1)(1+k) - 2p(k+\gamma)\left|b_{n+p-1}\right| |z|^{-(n+p-1)}$$

$$2\,|z|^p \left\{ p(1-\gamma) - \sum_{n=1}^{\infty}(n+p-1)(1+k) + p(k+\gamma)\left|a_{n+p-1}\right| |z|^{-(n+p-2)} \right.$$

$$\left. + \sum_{n=1}^{\infty}(n+p-1)(1+k) - p(k+\gamma)\left|b_{n+p-1}\right| |z|^{-(n+p-2)} \right\}$$

$$\geq 2\{p(1-\gamma) - \sum_{n=1}^{\infty}(n+p-1)(1+k) + p(k+\gamma)\left|a_{n+p-1}\right|$$

$$+ \sum_{n=1}^{\infty}(n+p-1)(1+k) - p(k+\gamma)\left|b_{n+p-1}\right|\} \geq 0,$$

by (2.1). □

Remark 2.2. *It is natural to ask if the condition* (2.1) *is also necessary for functions* $f \in \Sigma_H(p, \gamma, k)$.

In the next theorem we show that the answer to that question which is in affirmative.

Theorem 2.3. *Let* $f(z) = h(z) + \overline{g(z)}$ *be such that* $h(z)$ *and* $g(z)$ *given by* (1.6). *Then* $f(z) \in \sum_{\overline{H}}(p, \gamma, k)$*, if and only if the inequality* (2.1) *holds for the coefficients of* $f(z) = h(z) + \overline{g(z)}$.

Proof. In view of Theorem I, we only need to show that $f(z) \notin \sum_{\overline{H}}(p, \gamma, k)$, if the condition (2.1) does not hold. We note that for $f(z) \in \sum_H(p, \gamma, k)$, we have

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} \right\} \geq p\gamma.$$

This is equivalent to

$$Re\left\{ \frac{(1+ke^{i\alpha})(zh'(z) - \overline{zg'(z)}) - pke^{i\alpha}(h(z) + \overline{g(z)})}{h(z) + \overline{g(z)}} \right\} - p\gamma =$$

$$Re\left\{ \frac{1}{z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} z^{-(n+p-1)}} \Big[p(1-\gamma) z^p \right.$$

$$- \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) + pke^{i\alpha} + p\gamma] a_{n+p-1} z^{-(n+p-1)}$$

$$\left. - \sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1) - pke^{i\alpha} - p\gamma] b_{n+p-1} z^{-(n+p-1)} \Big] \right\} \geq 0$$

The above condition must hold for all values of z such that $|z| = r < 1$. Upon choosing the values of z on the positive real axis, we must have

$$Re\left\{\frac{1}{1+\sum_{n=1}^{\infty}(a_{n+p-1}-b_{n+p-1})r^{-(n-1)}}\left[p(1-\gamma)-\sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)+pke^{i\alpha}+p\gamma]a_{n+p-1}r^{-(n-1)}\right.\right.$$
$$\left.\left.-\sum_{n=1}^{\infty}[(1+ke^{i\alpha})(n+p-1)-pke^{i\alpha}-p\gamma]b_{n+p-1}r^{-(n-1)}\right]\right\}\geq 0.$$

If the condition (2.1) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0 > 1$, for which the quotient in (2.5) is negative. This contradicts the conditions for $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ and this completes the proof. □

3. Distortion Bounds and Extreme Points

The determination of the extreme points of a compact family of harmonic univalent functions enables us to solve many extremal problems for the family. The fundamental reason for considering extreme points for starlike and convex functions is to more easily categorize extremal properties under continuous linear functionals acting on these classes. In this section, we shall obtain distortion bounds for functions in $\Sigma_{\overline{H}}(p, \gamma, k)$ and also determine the extreme points for the class $\Sigma_{\overline{H}}(p, \gamma, k)$.

Theorem 3.1. *If $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ then $r^p - p(1-\gamma)r^{-p} \leq |f(z)| \leq r^p + p(1-\gamma)r^{-p}, |z| = r < 1$.*

Proof. We only prove the inequality on the right. The argument for the inequality on the left is similar. Let $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$. Taking the absolute value of $f(z)$, we obtain

$$|f(z)| \leq \left|z^p + \sum_{n=1}^{\infty} a_{n+p-1}z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1}z^{-(n+p-1)}\right| \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-p} \leq r^p + \sum_{n=1}^{\infty}(a_{n+p-1}+b_{n+p-1})r^{-(n+p-1)}$$

$$\leq r^p + \textstyle\sum_{n=1}^{\infty}[(n+p-1)(1+k)+p(k+\gamma)a_{n+p-1}] + \sum_{n=1}^{\infty}[(n+p-1)(1+k)-p(k+\sum)b_{n+p-1}]$$

$$\leq r^p + (p-\gamma)r^{-p}$$

by (2.1). Our next result shows how $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ looks like. We precisely proved. □

Theorem 3.2. *$f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$, if and only if $f(z)$ can be expressed as*

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}) \tag{3.1}$$

where $z \in \widetilde{U}$,

$$h_{p-1}(z) = z^p,$$
$$h_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)} z^{(n+p-1)} \quad (n = 1,2,3,...).$$

$$g_{p-1}(z) = z^p,$$
$$g_{n+p-1}(z) = z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)} \quad (n = 1,2,3,...)$$

$$\sum_{n=1}^{\infty}(x_{n+p-1}+y_{n+p-1}) = 1, x_{n+p-1} \geq 0 \ \ and \ \ y_{n+p-1} \geq 0.$$

Proof. For the functions $f(z)$ given by (3.1), we may write,

$$f(z) = \sum_{n=1}^{\infty}\left(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1}\right)$$

$$= x_{p-1}h_{p-1} + y_{p-1}g_{p-1} + \sum_{n=1}^{\infty} x_{n+p-1}(z^p + \frac{p(1-\gamma)}{(n+p-1)(1+k)+p(k+\gamma)})z^{(n+p-1)}$$

$$+y_{n+p-1}z^p - \frac{p(1-\gamma)}{(n+p-1)(1+k)-p(k+\gamma)} z^{-(n+p-1)}.$$

Then,

$$= \sum_{n=1}^{\infty}\Bigg[((1+k)(n+p-1)+p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)+p(\gamma+k)}x_{n+p-1}\right)$$
$$+((1+k)(n+p-1)-p(\gamma+k))\left(\frac{p(1-\gamma)}{(1+k)(n+p-1)-p(\gamma+k)}y_{n+p-1}\right).$$
$$= p(1-\gamma)\sum_{n=1}^{\infty} x_{n+p-1} + y_{n+p-1} \leq p(1-\gamma),$$

and so $f(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. Conversely, suppose that $f(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. Set

$$x_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|a_{n+p-1}|$$

and

$$y_n = \frac{(1+k)(n+p-1)+p(\gamma+k)}{p(1-\gamma)}|b_{n+p-1}|, (n = 1,2,3,...)$$

Then note that by Theorem 2, $0 \le x_{p-1} \le 1$.

$$y_{p-1} = 1 - x_{p-1} - \sum_{n=1}^{\infty}(x_{n+p-1} + y_{n+p-1}),$$

we obtain

$$f(z) = \sum_{n=1}^{\infty}(x_{n+p-1}h_{n+p-1} + y_{n+p-1}g_{n+p-1})$$

as required. □

4. Convolution and Convex Linear Combination

In this section, we show that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is invariant under convolution and convex combinations of its members. For harmonic functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} (\overline{z})^{-(n+p-1)}$$

and

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1} (\overline{z})^{-(n+p-1)}$$

we define the convolution of $f(z)$ and $F(z)$ as

$$(f * F)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p-1} A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} b_{n+p-1} B_{n+p-1} (\overline{z})^{-(n+p-1)} \tag{4.1}$$

Using this definition, we show in the next theorem that the class $\Sigma_{\overline{H}}(p, \gamma, k)$ is closed under convolution.

Theorem 4.1. *For* $0 \le \beta \le \gamma \le 1$, *let* $f(z) \in \Sigma_{\overline{H}}(p, \gamma, k)$ *and* $F(z) \in \Sigma_{\overline{H}}(p, \beta, k)$. *Then*

$$f(z) * F(z) \in \Sigma_{\overline{H}}(p, \gamma, k) \subset \Sigma_{\overline{H}}(p, \beta, k). \tag{4.2}$$

Proof. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} |a_{n+p-1}| z^{-(n+p-1)} - \sum_{n=1}^{\infty} |b_{n+p-1}| (\overline{z})^{-(n+p-1)}$$

$$F(z) = z^p + \sum_{n=1}^{\infty} A_{n+p-1} z^{-(n+p-1)} - \sum_{n=1}^{\infty} B_{n+p-1} (\overline{z})^{-(n+p-1)}$$

Note that $A_{n+p-1} \le 1$ and $B_{n+p-1} \le 1$. Obviously, the coefficients of f and F must satisfy conditions similar to the inequality (2.1). So for the coefficients of $f * F$ we can write,

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}A_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}B_{n+p-1}|$$

$$\leq (1+k)(n+p-1)+p(\gamma+k)|a_{n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{n+p-1}|.$$

This right hand side of the above inequality is bounded by 2 because $f(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. By the same token, we can conclude that $f(z) * F(z) \in \Sigma_{\overline{H}}(p,\gamma,k) \subset \Sigma_{\overline{H}}(p,\beta,k)$. Our next result shows that $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination of its members. □

Theorem 4.2. *The family $\Sigma_{\overline{H}}(p,\gamma,k)$ is closed under convex combination*

Proof. For $i = 1,2,3,..$, let $f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$ where $f_i(z)$ is given by

$$f_i(z) = z^p + \sum_{n=1}^{\infty}|a_{i,n+p-1}|(\overline{z})^{(n+p-1)} + \sum_{n=1}^{\infty}|b_{i,n+p-1}|(\overline{z})^{-(n+p-1)}.$$

Then by (2.1),

$$\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)|a_{i,n+p-1}|+(1+k)(n+p-1)-p(\gamma+k)|b_{i,n+p-1}| \leq p(1-\gamma) \quad (4.3)$$

for $\sum_{n=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_i(z)$ may be written as

$$\sum_{n=1}^{\infty} t_i f_i(z) = z^p + \sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)z^{-(n+p-1)} + \sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)(\overline{z})^{-(n+p-1)}.$$

Then by (4.2),

$$\sum_{n=1}^{\infty}[(1+k)(n+p-1)+p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|a_{i,n+p-1}|)|+(1+k)(n+p-1) \\ -p(\gamma+k)|\sum_{n=1}^{\infty}(t_i|b_{i,n+p-1}|)|$$

$$\sum_{n=1}^{\infty} t_i\left\{\sum_{n=1}^{\infty}(1+k)(n+p-1)+p(\gamma+k)a_{i,n+p-1}+(1+k)(n+p-1) \\ -p(\gamma+k)b_{i,n+p-1}\right\} \leq \sum_{n=1}^{\infty} t_i p(1-\gamma) = (1-\gamma).$$

Since this is the condition required by (2.1), we conclude that $\sum_{n=1}^{\infty} t_i f_i(z) \in \Sigma_{\overline{H}}(p,\gamma,k)$. This completes the proof of Theorem (2.1). □

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An Application for Certain Subclasses of p-Valent Meromorphic Functions Associated with the Generalized Hypergeometric Function

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Abstract

The main object of this paper is to give an application of a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ involving the generalized hypergeometric function. We define subclasses of the meromorphic function class $\Sigma_{p,m}$ by means of operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$.

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1. Introduction and definitions

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. We also denote $\Sigma_{p,1-p} = \Sigma_p$.
A function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma S^*_p(\alpha)$ of meromorphically p-valent starlike functions of order α in U if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.2}$$

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Also a function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex of order α in U if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$f(z) \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}$$

For a function $f \in \Sigma_{p,m}$, we say that $f \in \Sigma K_p(\beta,\alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.5}$$

Functions in the class $\Sigma K_p(\beta,\alpha)$ are called meromorphically p-valent close-to-convex functions of order β and type α. We also say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma K_p^*(\beta,\alpha)$ of meromorphically quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C_p(\alpha)$ such that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$f(z) \in \Sigma K_p^*(\beta,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma K_p(\beta,\alpha),$$

where $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ are, respectively, the classes of meromorphically p-valent starlike functions of order α and meromorphically p-valent convex functions of order α $(0 \leqslant \alpha < p)$(see Aouf (Aouf, 2008) and Frasin (Frasin, 2012)).
For a function $f(z) \in \Sigma_{p,m}$, given by (1.1) and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = (f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \quad (p \in \mathbb{N}).$$

For real or complex numbers

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \quad (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s),$$

we consider the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by (see, for example, (Kiryakova, 2011, p.19))

$$ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} $$

$$ (q \leqslant s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}\backslash\{0\} = \mathbb{C}^*), \\ \theta(\theta-1)...(\theta+\nu-1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} $$

Corresponding to the function $\phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ given by

$$ \phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = z^{-p}\ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)\ , $$

we introduce a function $\phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ defined by

$$ \phi_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * \phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = \frac{1}{z^p(1-z)^{\mu+p}} $$

$$ (\mu > -p; z \in U^*). $$

We now define a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s) : \Sigma_{p,m} \to \Sigma_{p,m}$ by

$$ H^{m,\mu}_{p,q,s}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) = \phi_{p,\mu}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * f(z) \tag{1.7} $$

$$ \left(\alpha_i,\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-; i = 1,...,q,\ j = 1,...,s\ ; \mu > -p, f \in \Sigma_{p,m}; z \in U^*\right). $$

For; convenience, we write

$$ H^{m,\mu}_{p,q,s}(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s) = H^{m,\mu}_{p,q,s}(\alpha_1) $$

and

$$ H^{1-p,\mu}_{p,q,s}(\alpha_1) = H^{\mu}_{p,q,s}(\alpha_1) \quad (\mu > -p). $$

If $f(z)$ is given by (1.1), then from (1.7), we deduce that

$$ H^{m,\mu}_{p,q,s}(\alpha_1)f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\mu+p)_{p+k}(\beta_1)_{p+k}...(\beta_s)_{p+k}}{(\alpha_1)_{p+k}...(\alpha_q)_{p+k}} a_k z^k \quad (\mu > -p; z \in U^*). \tag{1.8} $$

It is easily follows from (1.8) that

$$ z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)' = (\mu+p)H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z) - (\mu+2p)H^{m,\mu}_{p,q,s}(\alpha_1)f(z). \tag{1.9} $$

From the identity (1.9), we readily have

$$z\left(H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)\right)' = (\mu+p-1)H_{p,q,s}^{m,\mu}(\alpha_1)f(z) - (\mu+2p-1)H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z) \tag{1.10}$$

and

$$z\left(H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)\right)' = (\mu+p+1)H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z) - (\mu+2p+1)H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z). \tag{1.11}$$

The linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ was introduced by Patel and Palit (Patel & Palit, 2009) .

We note that the linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator (Choi *et al.*, 2002) for analytic functions and is essentially motivated by the operators defined and studied in (Cho & Noor, 2006) (see also, (Dziok & Srivastava, 1999), (Dziok & Srivastava, 2003), (Srivastava, 2007) and (Srivastava & Karlsson, 1985)).

Specializing the parameters $\mu, \alpha_i (i = 1, 2, ...q), \beta_j (j = 1, 2, ...s), q$ and s we obtain the following :

(i) $H_{p,2,1}^{m,0}(p, p; p)f(z) = H_{p,2,1}^{m,1}(p+1, p; p)f(z) = f(z)$;

(ii) $H_{p,2,1}^{m,1}(p, p; p)f(z) = \frac{2pf(z)+zf'(z)}{p}$;

(iii) $H_{p,2,1}^{m,2}(p+1, p; p)f(z) = \frac{(2p+1)f(z)+zf'(z)}{p+1}$;

(iv) $H_{p,1,1}^{m,1-p}(c+1, 1; c)f(z) = J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt$ $(c > 0; z \in U^*)$, this integral operator is defined by

$$J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \quad (c > 0; f \in \Sigma_{p,m}),$$

(v) $H_{p,2,2}^{m,0}(p+1, p; p)f(z) = \frac{p}{z^{2p}}\int_0^z t^{2p-1}f(t)dt$; $(p \in \mathbb{N}; z \in U^*)$;

(vi) $H_{p,2,1}^{1-p,n}(a, 1; a)f(z) = \frac{1}{z^p(1-z)^{n+p}} = D^{n+p-1}f(z)$ $(n > -p)$, the operator D^{n+p-1} studied by Ganigi and Uralegaddi (Ganigi & Uralegaddi, 1989), Yang (Yang, 1995), Aouf (Aouf, 1993), Aouf and Srivastava (Aouf & Srivastava, 1997) and Uralegaddi and Patil (Uralegaddi & Patil, 1989);

(vii) $H_{p,2,1}^{m,\mu}(c, p+\mu; a)f(z) = L_p(a, c)f(z)$ $(a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mu > -p)$ (see Liu (Liu, 2002));

(viii) $H_{1,2,1}^{o,\mu}(\mu+1, n+1; \mu)f(z) = I_{n,\mu}f(z)$ $(\mu > 0; n > -1)$ (see Yuan et al. (Yuan *et al.*, 2008)).

We also observe that, for $m = 0$, $p = 1$ replacing μ by $\mu-1$, we have the operator $H_{1,\mu,q,s}^{0}(\alpha_1)f(z) = H_{\mu,q,s}(\alpha_1)f(z)$ defined by Cho and Kim (Cho & Kim, 2007).

The object of the present paper is to investigate some properties of meromorphic p- valent functions by the above operator $H_{p,q,s}^{m,\mu}(\alpha_1)f(z)$ given by (1.8).

Definition 1.1. *Let $\mathcal{H}$ the set of complex valued functions $h(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ such that*

$$h(r, s, t) \text{ is continuous in a domain } D \subset \mathbb{C}^3;$$

$$(1, 1, 1) \in D \quad \text{and} \quad |h(1, 1, 1)| < 1;$$

$$\left|h\left(e^{i\theta}, \frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}e^{i\theta} + \frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1} + \frac{\mu+p-1}{\mu+p+1}e^{i\theta} + \right.\right.$$

$$\left.\left.\frac{1}{\mu+p+1}\delta + \frac{(\mu+p-1)\delta e^{i\theta} + [\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right| \geqslant 1$$

whenever

$$\left(e^{i\theta}, \frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}e^{i\theta} + \frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1} + \frac{\mu+p-1}{\mu+p+1}e^{i\theta} + \right.$$

$$\left.\frac{1}{\mu+p+1}\delta + \frac{(\mu+p-1)\delta e^{i\theta} + [\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right) \in D$$

with $\Re(\beta \geqslant \delta(\delta-1))$ *for real* θ, $\delta \geqslant 1$ *and* $\lambda > 0$.

2. The Main Result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu (Miller & Mocanu, 1978).

Lemma 2.1. *Let* $w(z) = a + w_n z^n + ...,$*be analytic in* $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ *with* $w(z) \neq a$ *and* $n \geqslant 1$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leqslant r_0} |w(z)|$. *Then*

$$zw'(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$\Re(\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\}) \geqslant \delta, \tag{2.2}$$

where δ *is a real number and*

$$\delta \geqslant n\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2} \geqslant n\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

Theorem 2.1. *Let* $h(r, s, t) \in \mathcal{H}$ *and let* $f \in \Sigma_{p,m}$ *satisfies*

$$\left(\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)}, \frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}, \frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}\right) \in D \subset \mathbb{C}^3 \tag{2.3}$$

and

$$\left| h\left(\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} \right) \right| < 1 \tag{2.4}$$

for all $z \in U$ and for some $m \in \mathbb{N}$.Then we have

$$\left| \frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} \right| < 1 \qquad (z \in U; \mu > -p, 0 \leqslant \alpha < p; p \in \mathbb{N}).$$

Proof. Let

$$\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = w(z). \tag{2.5}$$

Then it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.5) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.6) and (1.10) , we have

$$\frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}. \tag{2.6}$$

Differentiating (2.6) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{z\left[\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}\right]'}{\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}} \tag{2.7}$$

$$= \frac{(\mu+p-1)zw'(z) + \left[\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1 + (\mu+p-1)w(z) + \frac{zw'(z)}{w(z)}}$$

Using the identities (1.9) and (1.11) , we have

$$\begin{aligned}(\mu+p+1)\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= 1+(\mu+p)\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}+\\ &\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}}\\ &= 1+\left[1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}\right]+\\ &\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}}.\end{aligned}$$

We claim that $|w(z)|<1$ for $z\in U$. Otherwise there exists a point $z_0\in U$ such that $\max_{|z|\leqslant r_0}|w(z)|=|w(z_0)|=1$. Letting $w(z_0)=e^{i\theta}$ and using Lemma 2.1 with $a=1$ and $n=1$, we have

$$\begin{aligned}\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)} &= e^{i\theta},\\ \frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} &= \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta,\\ \frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= \frac{2}{(\mu+p+1)}+\frac{(\mu+p-1)}{(\mu+p+1)}e^{i\theta}+\frac{1}{(\mu+p+1)}\delta\\ &\quad +\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p+1)(\mu+p-1)e^{i\theta}+(\mu+p+1)\delta},\end{aligned}$$

where

$$\beta=\frac{z^2w''(z)}{w(z)} \quad \text{and} \quad \delta\geqslant 1.$$

Further, an application of (2.2) in Lemma 2.1 given $\Re(\beta\geqslant\delta(\delta-1)$. Since $h(r,s,t)\in\mathcal{H}$, we have

$$\left|h\left(\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}\right)\right|$$

$$=\left|h\left(e^{i\theta},\frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta,\frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.\right.$$
$$\left.\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right|\geqslant 1$$

which contradicts the condition (2.4) of Theorem 2.1. Therefore, we conclude that

$$\left|\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}\right| < 1 \quad (z \in U).$$

The proof is complete. □

Letting $\mu = 1, q = 2, s = 1, \alpha_1 = p + 1, \alpha_2 = p$ and $\beta_1 = p$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $h(r, s, t) \in \mathcal{H}$ and let $f(z) \in \Sigma_{p,m}$ satisfies*

$$\left(\frac{2pf(z) + zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z) + zf'(z)\right]}{(p+1)\left[2pf(z) + zf'(z)\right]},\right.$$

$$\left.\frac{(2p+2)(2p+1)f(z) + 4(p+1)zf'(z) + z^2f''(z)}{(p+2)(2p+1)f(z) + zf'(z)}\right) \in D \subset \mathbb{C}^3$$

and

$$\left|h\left(\frac{2pf(z) + zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z) + zf'(z)\right]}{(p+1)\left[2pf(z) + zf'(z)\right]},\right.\right.$$

$$\left.\left.\frac{(2p+2)(2p+1)f(z) + 4(p+1)zf'(z) + z^2f''(z)}{(p+2)(2p+1)f(z) + zf'(z)}\right)\right| < 1$$

for all $z \in U$. Then we have

$$\left|\frac{2pf(z) + zf'(z)}{pf(z)}\right| < 1 \quad (z \in U).$$

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Some Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows in Banach Spaces

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Abstract

The exponential dichotomy is one of the most important asymptotic properties for the solutions of evolution equations, studied in the last years from various perspectives. In this paper we study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. Several illustrative examples motivate the approach.

Keywords: Skew-evolution semiflow, invariant projection, uniform exponential dichotomy.
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1. Introduction

The property of exponential dichotomy is a mathematical domain with a substantial recent development as it plays an important role in describing several types of evolution equations. The literature dedicated to this asymptotic behavior begins with the results published in Perron (1930). The ideas were continued by in Massera & Schäffer (1966), with extensions in the infinite dimensional case accomplished in Daleckiĭ & Kreĭn (1974) and in Pazy (1983), respectively in Sacker & Sell (1994). Diverse and important concepts of dichotomy were introduced and studied, for example, in Appell *et al.* (1993), Babuţia & Megan (2015), Chow & Leiva (1995), Coppel (1978), Megan & Stoica (2010), Sasu & Sasu (2006) or Stoica & Borlea (2012).

The notion of skew-evolution semiflow that we sudy in this paper and which was introduced in Megan & Stoica (2008) generalizes the skew-product semiflows and the evolution operators. Several asymptotic properties for skew-evolution semiflows are defined and characterized see Viet Hai (2010), Viet Hai (2011), Stoica & Borlea (2014), Stoica & Megan (2010) or Yue *et al.* (2014).

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In this paper we intend to study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. The definitions of various types of dichotomy are illustrated by examples. We also aim to give connections between them, emphasized by counterexamples.

2. Preliminaries

Let (X, d) be a metric space, V a Banach space and $\mathcal{B}(V)$ the space of all V-valued bounded operators defined on V. Denote $Y = X \times V$ and $T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geqslant t_0\}$.

Definition 2.1. A mapping
$varphi : T \times X \to X$ is said to be *evolution semiflow* on X if the following properties are satisfied:

(es1) $\phi(t, t, x) = x, (\forall)(t, x) \in \mathbb{R}_+ \times X$;

(es2) $\phi(t, s, \phi(s, t_0, x)) = \phi(t, t_0, x), (\forall)(t, s), (s, t_0) \in T, x \in X$.

Definition 2.2. A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow ϕ if:

(ec1) $\Phi(t, t, x) = I, (\forall)t \geqslant 0, x \in X$ (I - identity operator).

(ec2) $\Phi(t, s, \phi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), (\forall)(t, s), (s, t_0) \in T, (\forall)x \in X$.

Let Φ be an evolution cocycle over an evolution semiflow φ. The mapping $C = (\varphi, \Phi)$, defined by:

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v)$$

is called *skew-evolution semiflow* on Y.

Example 2.1. We will denote $C = C(\mathbb{R}, \mathbb{R})$ the set of continous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with uniform convergence topology on compact subsets of $\mathbb{R}$. The set C is metrizable with the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)},\ \text{unde } d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

For every $n \in \mathbb{N}^*$ we consider a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right),\ \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We will denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geqslant 0.$$

Let be X the closure in C of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The application

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{unde } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geqslant 0,$$

is a evolution semiflow on X. Let consider the Banach space $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. Then, the application

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau - s)d\tau} v_2\right),$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is fixed, is a cocycle aplication of evolution over the semiflow φ, and $C = (\varphi, \Phi)$ is a evolution cocycle on Y.

Let us remind the definition of an evolution operator, followed by examples that punctuate the fact that it is generalized by an skew-evolution semiflows.

Definition 2.3. A mapping $E : T \to \mathcal{B}(V)$ is called *evolution operator* on V if following properties hold:
(e_1) $E(t, t) = I,\ \forall t \in \mathbb{R}_+$;
(e_2) $E(t, s)E(s, t_0) = E(t, t_0),\ \forall (t, s), (s, t_0) \in T$.

Example 2.2. One can naturally associate to every evolution operator E the mapping

$$\Phi_E : T \times X \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t, s),$$

which is an evolution cocycle on V over every evolution semiflow φ. Therefore, the evolution operators are particular cases of evolution cocycles.

Example 2.3. Let $X = \mathbb{R}_+$. The mapping

$$\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+,\ \varphi(t, s, x) = t - s + x$$

is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V),\ \Phi_E(t, s, x) = E(t - s + x, x)$$

is an evolution cocycle on V over the evolution semiflow φ. It follows that an evolution operator on V is generating a skew-evolution semiflow on Y.

3. Sequences of Invariant Projections for a Cocycle

Definition 3.1. A continuous map $P : X \to \mathcal{B}(V)$ which satisfies the following relation:

$$P(x)P(x) = P(x), (\forall)x \in X$$

is called projection on V.

Definition 3.2. A projection P on V is called *invariant* for a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

$$P(\varphi(t, s, x))\, \Phi(t, s, x) = \Phi(t, s, x)P(x),$$

for all $(t, s) \in T$ and $x \in X$.

Remark. If P is a projection on V, than the map

$$Q : X \to \mathcal{B}(V),\ Q(x) = I - P(x)$$

is also a projection on V, called complementary projection of P.

Remark. If the projection P is invariant for C then Q is also invariant for C.

Definition 3.3. We will name (C, P) a dichotomy pair where C is a skew-evolution semiflow and P is invariant or C.

4. Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows

Definition 4.1. Let (C, P) be a dichotomy pair. We say that (C, P) is *uniformly strongly exponentially dichotomic* (u.s.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(used1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}$

(used2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}$

for all $(t, s) \in T$ and $x \in X$.

Definition 4.2. We say that (C, P) is *uniformly exponentially dichotomic* (u.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(ued1) $\|\Phi(t, s, x)P(x)v\| \leqslant Ne^{-\nu(t-s)}\|P(x)v\|$

(ued2) $N\|\Phi(t, s, x)Q(x)v\| \geqslant e^{\nu(t-s)}\|Q(x)v\|$

for all $(t, x) \in T \times X$ and for all $v \in V$.

Definition 4.3. We say that (C, P) is *uniformly weakly exponentially dichotomic* (u.w.e.d) if there exist $N \geqslant 1$ and $\nu > 0$ such that:

(uwed1) $\|\Phi(t, s, x)P(x)\| \leqslant Ne^{-\nu(t-s)}\|P(x)\|$

(uwed2) $N\|\Phi(t, s, x)Q(x)\| \geqslant e^{\nu(t-s)}\|Q(x)\|$

for all $(t, x) \in T_x X$ and for all $v \in V$.

Proposition 1. *If (C, P) is (s.u.e.d) then*

$$\sup_{x\in X} \|P(x)\| < +\infty. \tag{4.1}$$

Proof. Consider in (used1) $t = s$. Then we have

$$\|\Phi(t, t, x)P(x)\| = \|P(x)\| = \|P(x)\| \leqslant N \tag{4.2}$$

for all $x \in X$. □

Proposition 2. *If (C, P) is (u.s.e.d) then (C, P) is (u.w.e.d).*

Proof. If (C, P) is (u.s.e.d) then by (used1), for $x \in X$, we have that $\|P(x)\| \leqslant N$ and hence

$$\|Q(x)\| = \|I - P(x)\| \leqslant 1 + \|P(x)\| \leqslant 2N.$$

We have from (used1) and (used2) that:

$$\begin{aligned} \|\Phi(t, s, x)P(x)\| &\leqslant Ne^{-\nu(t-s)} \cdot 1 \leqslant Ne^{-\nu(t-s)}\|P(x)\| \quad (4.3) \\ &\leqslant 2N^2 e^{-\nu(t-s)}\|P(x)\|. \quad (4.4) \end{aligned}$$

$$2N^2\|\Phi(t, s, x)Q(x)\| \geqslant 2Ne^{\nu(t-s)} \geqslant e^{\nu(t-s)}\|Q(x)\|, \quad (4.5)$$

hence (C, P) is (u.w.e.d) □

Proposition 3. *If (C, P) is (u.e.d) then (C, P) is also (u.w.e.d)*

Proof. It follows immediately by taking the supremum over all $v \in V$ with $\|v\| = 1$. □

Definition 4.4. We say that C has a uniform exponential growth (u.e.g) if there exist $M \geqslant 1$, $\omega > 0$ such that

$$\|\Phi(t, s, x)\| \leqslant Me^{\omega(t-s)},$$

for all $(t, s) \in T$ and $x \in X$.

Theorem 4.1. *Assume that a dichotomy pair (C, P) is (u.w.e.d) and C has a uniform exponential growth. Then:*

$$\sup_{x \in X} \|P(x)\| < +\infty.$$

Proof. Let N, ν given by the (u.w.e.d) property of (C, P) and M, ω given by the (u.e.g) of C. Consider $s \geqslant 0$ fixed, $t \geqslant s$ and $x \in X$.

$$\begin{aligned} \left[\frac{1}{2N}e^{\nu(t-s)} - Ne^{-\nu(t-s)}\right]\|P(x)\| &\leqslant \frac{1}{N}e^{\nu(t-s)}\|Q(x)\| - Ne^{-\nu(t-s)}\|P(x)\| \\ &\leqslant \|\Phi(t, s, x)Q(x)\| - \| - \Phi(t, s, x)P(x)\| \\ &\leqslant \|\Phi(t, s, x)\| \leqslant Me^{\omega(t-s)}. \end{aligned}$$

Let $t_0 > 0$ be such that

$$\lambda_0 := \frac{1}{2N}e^{\nu t_0} - Ne^{-\nu t_0} > 0.$$

From the above estimation is follows that for $t = t_0 + s$,

$$\|P(x)\| \leqslant \frac{Me^{\omega t_0}}{\lambda_0}, \ (\forall)x \in X.$$

□

from where the conclusion follows.

Remark. In the following section we will see that for a dichotoomic pair (C, P):

1. (u.s.e.d) does not imply (u.e.d)
2. (u.e.d) does not imply (u.s.e.d)
3. (u.w.e.d) does not imply (u.e.d)
4. (u.w.e.d) does not imply (u.s.e.d)

5. Examples and Counterexamples

Example 5.1. Define, on $\mathbb{R}^3$, the family of projections

$$P(x)(v_1, v_2, v_3) = (v_1, 0, 0)$$

and the evolution cocycle on $\mathbb{R}^3$:

$$\Phi(t, s, x)(v_1, v_2, v_3) = \begin{cases} (v_1, v_2, v_3), \ t = s \\ (e^{s-t}v_1, e^{t-s}v_2, 0), \ t > s, \end{cases}$$

with the following norm:

$$\|x\| = |x_1| + |x_2| + |x_3|, \ x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We have that for all $(t, s) \in T, x \in X$ and $v \in \mathbb{R}^3$

$$\|\Phi(t, s, x)P(x)v\| = e^{s-t}v_1 = e^{s-t}\|P(x)v\|$$

from where we get that

$$\|\Phi(t, s, x)P(x)\| \leqslant e^{s-t}\|P(x)\|$$

and

$$\|\Phi(t, s, x)Q(x)v\| = \begin{cases} \|Q(x)v\|, \ t = s \\ \| (0, e^{t-s}v_2, 0) \|, \ t > s \end{cases} \leqslant e^{t-s}\|Q(x)v\|$$

hence

$$\|\Phi(t, s, x)Q(x)\| \leqslant \|Q(x)\|.$$

Choose $(0, 1, 0) \in \mathbb{R}^3$. Then

$$\|\Phi(t, s, x)Q(x)(0, 1, 0)\| = e^{t-s}\|Q(x)(0, 1, 0)\|$$

from where we finally obtain that:

$$\|\Phi(t, s, x)Q(x)\| = e^{t-s}\|Q(x)\|,$$

hence (C, P) is (u.w.e.d). Assume by a contradiction that (C, P) is (u.e.d). Then there exists, $N \geqslant 1$, $\nu > 0$ such that

$$N\|\Phi(t, s, x)Q(x)(v_1, v_2, v_3)\| \geqslant e^{\nu(t-s)}\|Q(x)(v_1, v_2, v_3)\|. \tag{5.1}$$

Put $t > s$ and $(v_1, v_2, v_3) = (0, 0, 1)$. Then $\|Q(x)(v_1, v_2, v_3)\| = 1$ and

$$e^{\nu(t-s)} \leqslant \|\Phi(t, s, x)(v_1, v_2, v_3)\| = \|\Phi(t, s, x)(0, 0, 1)\| = 0,$$

which is a contradiction.

Example 5.2 (u.e.d does not imply u.s.e.d)**.** On $V = \mathbb{R}^2$ and $(X, d) = (\mathbb{R}_+, d)$ endowed with the max - norm. Consider,

$$P(x) : \mathbb{R}^2 \to \mathbb{R}^2,\ P(x)(v_1, v_2) = (v_1 + xv_2, 0)$$

it follows that

$$\|P(x)\| = 1 + x, (\forall)x \geqslant 0 \tag{5.2}$$

Define the skew - evolutiv cocycle

$$\Phi(t, s, x) = e^{s-t}P(x) + e^{t-s}Q(x).$$

We have that

$$\begin{aligned} &\|\Phi(t, s, x)P(x)\| = e^{s-t}\|P(x)\| \text{ and} \\ &\|\Phi(t, s, x)Q(x)\| \geqslant e^{t-s}\|Q(x)\| \end{aligned} \tag{5.3}$$

Hence (C, P) is (u.e.d). It can not be (u.s.e.d) because of (5.2).

Remark. From the above example, by taking the sup norm in (5.3) over $\|v\| = 1$, we get that (C, P) is also (u.w.e.d). Hence (C, P) is (u.w.e.d) but not (u.s.e.d).

Remark. The connection between the three concepts studied in this paper is summarized in the below diagram

$$(u.s.e.d) \Rightarrow (u.e.d) \Rightarrow (u.w.e.d) \Leftarrow (u.s.e.d)$$

$$(u.s.e.d) \not\Leftarrow (u.e.d) \not\Leftarrow (u.w.e.d) \not\Rightarrow (u.s.e.d).$$

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Rényi Entropy in Measuring Information Levels in Voronoï Tessellation Cells with Application in Digital Image Analysis

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Abstract

This work introduces informative and interesting Voronoï regions through measures utilizing probability density functions and qualities of Voronoï cells of digital image point patterns. Global mesh cell quality exhibits a fairly horizontal behaviour in its range of convergence across several categories of digital images. Simulation results unambiguously show that Shannon entropy does not expose the most information in Voronoï meshes although it's in the range $1 < \beta \leq 2.5$ for which information is maximized. Mesh information is seen to be generally a non-linear, non-decreasing function of image point patterns. Some important mathematical theorems on quantities and optimality conditions are proved.

Keywords: Generator, quality, Voronoï mesh, pattern, entropy, information.
2010 MSC No: Primary 54E05, Secondary 20L05, 35B36.

1. Introduction

This article introduces an approach to measuring the information levels Voronoï tessellation (mesh) cells via Rényi entropy. The focus is on the Rényi entropy of Voronoï meshes with varying quality. Let $p(x_1), \ldots, p(x_i), \ldots, p(x_n)$ be the probabilities of a sequence of events $x_1, \ldots, x_i, \ldots, x_n$ and let $\beta \geq 1$. Then the Rényi entropy (Rényi, 2011) $H_\beta(X)$ of a set of event X is defined by

$$H_\beta(X) = \frac{1}{1-\beta} \ln \sum_{i=1}^{n} p^\beta(x_i) \text{ (Rényi entropy).}$$

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Rényi's entropy is based on the work by R.V.L. Hartley (Hartley, 1928) and H. Nyquist (Nyquist, 1924) on the transmission of information. A proof that $H_\beta(X)$ approaches Shannon entropy as $\beta \longrightarrow 1$ is given in (Bromiley *et al.*, 2010), *i.e.*,

$$\lim_{\beta \longrightarrow 1} \frac{1}{1-\beta} \ln \sum_{i=1}^{n} p^{\beta}(x_i) = -\sum i = 1^n p_i \ln p_i.$$

The information of order β contained in the observation of the event x_i with respect to the random variable X is defined by $H(X)$. In our case, it is information level of the observation of the quality of a Voronoï mesh cell viewed as random event that is considered in this study. The principle application of the proposed approach to measuring the information levels of mesh cells is the tessellation of digital images.

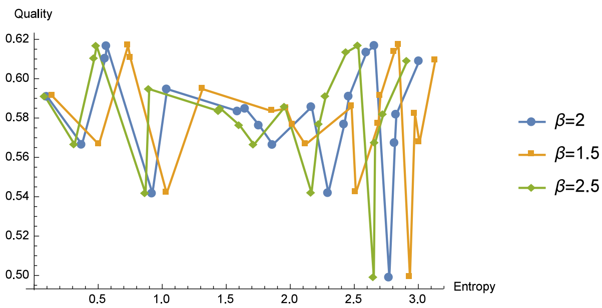


Figure 1. Rényi entropy

A main result reported in this study is the correspondence between image quality and Rényi entropy for different types of tessellated digital images. In other words, the correspondence between the Rényi entropy of mesh cells relative to the quality of the cells varies for different classes of images. For example, with Voronoï tessellations of images of humans, Rényi entropy tends to be higher for higher quality mesh cells (see, *e.g.*, the plot in Fig. 1 for different Rényi entropy levels, ranging from $\beta = 1.5$ to 2.5 in 0.5 increments).

2. Literature Review on Voronoï Diagrams

It is known that generating meshes is a fundamental and necessary step in several domains such as engineering, computing, geometric and scientific applications (Leibon & Letscher, 2000; Owen, 1998; Liu & Liu, 2004). No matter what their domain application and the specific terminology used, the resultant meshes have structures or volumes that result from the geometry of surfaces, dimension of the space and placement or organization of generators(Ebeida & Mitchell, 2012; Mitchell, 1993; Persson, 2004). Meshes may be generated for purposes of image processing and segmentation (Arbeláez & Cohen, 2006), clustering (Ramella *et al.*, 1998), data compression, quantization, analysis of territorial behavior of animals (Persson, 2004; Persson & Strang, 2004; Du *et al.*, 1999) to name a few. Applications of meshes are growing but works in the direction of exploiting pattern nature and information are lacking. We are therefore of the view that understanding the pattern and the underlying process could greatly benefit applications.

Voronoï diagrams were introduced by the Ukrainian mathematician G. Voronoï (Voronoï, 1903, 1907, 1908) (elaborated in the context of proximity and quality spaces spaces in (Peters, 2015*b*,*c*,*a*; A-iyeh & Peters, 2015; Peters, 2016)) provide a means of covering a space with regular polygons. The process allows us to understand fundamental properties of elements of the space by exploiting properties of the meshes. The properties of the space may otherwise have remained inaccessible.

In telecommunications, Voronoï diagrams have furnished a tool for analysis of binary linear block codes (Agrell, 1996) governing regions of block code and performance of Gaussian channels.

In musicology, Voronoï diagrams have demonstrated their utility (McLean, 2007). For example, they have been successfully applied in automatic grouping of polyphony (Hamanaka & Hirata, 2002). Other works bordering on applications of Voronoï meshes are in reservoir modeling (Møller & Skare, 2001) and cancer diagnosis (Demir & Yener, 2005).

The fact that the partitioning algorithm divides the plane into Euclidean neighborhoods permits exploitation of proximity relations while offering the flexibility of modeling the space as a continuous image-like point pattern representing the space. Given the substantial utility of Voronoï tessellations their applicability in additional areas including point pattern detection and image analysis is currently being investigated vigorously.

In this work meshes are generated for the purposes of characterizing a point pattern information using multiple measures for the individual mesh cells. The major focus here goes beyond tessellating a space with meshes. Additionally we search for important cues that may be fundamental for basic pattern understanding which in turn may lead to identifying and understanding the underlying pattern.

3. Preliminaries

In this section, the grounding theory entropy, quality of cells and Voronoï diagrams based on point pattern distributions is set. Some useful definitions are given prior to facilitate the process.

3.1. Notation and Definitions

A subset of points in $\mathbb{R}^n$ is denoted by S. A partition of the space of $S \subseteq \mathbb{R}^n$ according to the Voronoï criterion into contiguous non-overlapping polygons is denoted by the set $\{\mathbb{V} = \mathcal{F}, \mathcal{E}, S = \mathbb{N}\}$ where $\mathcal{F}, \mathcal{E}$ are the faces and edges of graph regions respectively. Also, properties of cells such as length of edges of polygons are represented by l_i, area by A, quality of cells by q_i and entropy by H_R.

Definition 3.1. Given a point pattern set $S \subseteq \mathbb{R}^n$ of three or more non-collinear points and a distance function d_n, the set $\{\mathbb{V}, S = \{\mathbb{N}\}$ is called a Voronoï tessellation of S if $\mathbb{V}_i \cap \mathbb{V}_j \neq \varnothing$ for $i \neq j \subset S$. A Voronoï tessellation is a set of polygons with their edges and vertices that partition a given space of points.

Definition 3.2. The Voronoï region of an image point is a polygon about that site. The set of all regions partition a plane of image points based on a distance function $\| \cdot \|$. This results in a covering of the plane with polygons about the points.

Definition 3.3. Consider the set $S = \{s_1, ..., s_k\}$, a plane (v_i, v_j) is a Voronoï edge of the Voronoï region $\mathbb{V}_i$ if and only if there exists a point x such that the circle centered at x and circumscribing v_i and v_j does not contain in its interior any other point of $\mathbb{V}_i$. A Voronoï edge is a half plane equidistant from two sites and bounds some part of the Voronoï diagram. Every edge is incident upon exactly two vertices and every vertex upon at least three edges.

Definition 3.4. A Voronoï neighborhood of a point p in the vicinity of point q is the locus of bisectors or half planes equidistant from p and q. The union of half planes H_q^p (H_p^q) is the locus of points nearer to p than to q. The intersection of half planes $\bigcap_{q \in S, q \neq p} H_p^q$ defines a region generated at p.

Definition 3.5. A Voronoï vertex is the center of a circumcircle through three sites.

Definition 3.6. A set of points S is a convex set if there is a line connecting each pair of points within S.

Definition 3.7. The convex hull of Voronoï regions about S is the smallest set which contains the Voronoï regions as well as the union of the regions.

Definition 3.8. A point pattern is a set of points of the signal representing locations of signal features. For example sets of corners, keypoints etc. are referred to as point or dot patterns.

Definition 3.9. The quality of a Voronoï cell is a dimensionless real number assigned to the cell based on the extent to which the sides of the cell match.

Definition 3.10. An open pattern point is a point such that a disk centered on it contains the point as an interior point.

Definition 3.11. A closed pattern point is a point such that a disk centered on it contains the point as well a boundary.

Definition 3.12. Let $\mathbb{V}$ be a Voronoï diagram in $\mathbb{R}^2$. The skeleton of $\mathbb{V}_i \in \mathbb{V}$, is the open set $S(\Omega)$ from which the Voronoï diagram is generated.

Definition 3.13. The Voronoï quality of visual information given by a point generator is defined as the aggregate of measure of cells comprising the tessellation. In other words it shows the organization of a point pattern.

Definition 3.14. A point pattern is feasible when there exists a constant $t > 0$ such that at least one quality measure of the Voronoï cells is at least t.

3.2. Voronoï Diagrams

The spatial distribution of point sets informs the nature and organizations of the pattern. This in turn influences the graph geometry of the Voronoï diagram the point set. Assume we have a finite set S of point locations called sites s_i in a space $\mathbb{R}^n$. Computing the Voronoï diagram with respect to S entails partitioning the space of S into Voronoï regions $\mathbb{V}(s_i)$ in such a way that the region $\mathbb{V}(s_i)$ contains all points of S that are closer to s_i than to any other object s_j, $i \neq j$ in S.

More elaborately, given the generator set

$$S = \{s_1, \ldots, s_k : i \in \mathbb{N}\},$$

the Voronoï region $\mathbb{V}(s_i)$ is defined by

$$\mathbb{V}(s_i) = \{x \in \mathbb{R}^n : \|x - s_i\| \leq \|x - s_k\|, s_k \in S, i \neq k\},$$

where $\|.,.\|$ is the Euclidean norm (distance between vectors). The set

$$\mathbb{V}(S) = \bigcup_{s_i \in S} \mathbb{V}(s_i)$$

is called the n-dimensional Voronoï diagram generated by the point set S. In $\mathbb{R}^2$, this effectively covers the plane with convex and non overlapping graphs, one for each generating point in S. By the definition of a Voronoï region above, the region about a site x satisfies

$$d(x, s_i) \leq d(x, s_k) \Leftrightarrow ||x - s_i||^2 \leq ||x - s_k||^2 \forall s_i \in S.$$

Manipulating the expression of a Voronoï region gives

$$\mathbb{V}(s_i) := \{(s_k - s_i)x \leq \frac{||s_k||^2 - ||s_i||^2}{2}, s_k \in S\}.$$

The immediate expression is recognized as an ordinary linear system of equations when S is finite (Goberna *et al.*, 2012). For a partitioned space in which all the individual regions are triangles, the optimal tessellation of the point set which maximizes the minimum angle in each triangular graph is the Delaunay triangulation. The Delaunay triangulation of S is the triangulation DT(S) where the circum-circles of all cells contain only the three points forming the triangle. Since a Delaunay image triangulation can be obtained from the corresponding Voronoï image graph our focus shall be on the latter. Point patterns in Delaunay image triangulations are informative and can be used to study the nature of the underlying tessellated process.

The advantage of Voronoï diagrams in studying patterns is that it associates the local neighborhood of a point with the information in the region inclosed by the point as opposed to point estimates only. Consequently measures may be aggregated for global pattern information gathering.

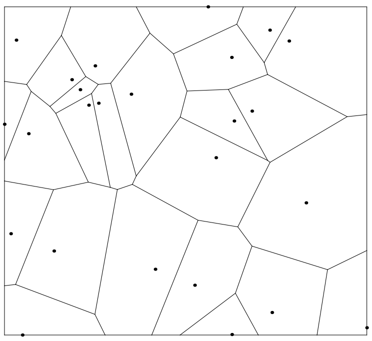


Figure 2. Voronoï mesh pattern

Fig. 3.2 displays a Voronoï diagram generated by a point set (not shown) in $\mathbb{R}^2$. The diagram shows how a space partitioned into regions of influences about the generators in the form convex non-intersecting polygons. The nature of the pattern influences the distribution of the point set as well as the structure of the partitioned space. For example polygons in regions of higher point densities are of smaller sizes or areas compared to polygons of regions with with lower point densities.

4. Patterns and Information Theory

Information in signals and patterns is commonly characterized using information theoretic approaches such as entropy and characteristics of a transformed space of the pattern such as quality measures of Voronoï cells. In the following subsections we present those tools.

4.1. Entropy

Entropy has long been an indicator of information and information content whose utility has since extended to other fields besides thermodynamics where it emerged. In thermodynamics, it was first used for understanding molecular structure. Entropy now finds applications in several other fields including portfolio selection and financial decision making (Zhou *et al.*, 2013), distribution analysis (Chapman, 1970) where it's founded on probability density functions derived from random variables.

Some general observations on entropic information are in order before proceeding. If all the realizations of a random variable have equal chance of being observed, then the variables have equal probabilities. Relating this to Voronoí cells this means we have a simple pattern formed by repetition of a unit. Consequently the same information is contained in all cells of the pattern. This scenario corresponds to maximization of entropy.

When a measure of information in a pattern is maximized the variations of the pattern primitives must be minimal and one variable or cell and its attribute is representative of the pattern. This situation also means there is no other information in the pattern other than the fact that the random variables of the pattern are uniformly distributed. On the contrary variations in a random variable indicates interestingness, disorder, complexity or randomness in the pattern and most importantly a distribution of variables that is anything but uniform.

4.1.1. Renyi Entropy

Renyi entropy is a general information criterion of which Shannon entropy and others are special cases (Xu & Erdogmuns, 2010). This generality is useful in diversity and dissimilarity characterization (Rao, 1982) of pattern structure. Recall that the area of a Voronoï cell satisfies $0 < A_i \leq \infty$ and so the probability $Pr(.)$ of the area random variable assuming a value in the range of areas is defined in $0 \leq Pr(A_i) \leq 1$. Let A_T be the total planar surface area of a Voronoï tessellation $\mathbb{V}$. It follows that the probability of the random variable A_i is defined by

$$Pr(A_i) = \frac{A_i}{A_T},$$

and

$$\sum_i Pr(A_i) = 1.$$

A general entropy criterion utilizing the probability densities of the random variables is defined by:

$$H = \frac{1}{1-\beta} ln \sum_{i=1}^{n} Pr_i^{\beta},$$

where $Pr(A_i) = Pr_i$. A noteworthy property of Renyi entropy is majorization. Assume two finite probability vectors P and Q of length $1 < k \le n$. P is said to majorize Q if

$$P_1 + P_2 + \cdots + P_k \ge Q_1 + Q_2 + \cdots + Q_k.$$

This means that P exhibits a stronger tendency towards uniformity than Q and thus has more entropy. This is an important indicator for understanding the nature of the distribution of a random variable.

4.2. Cell Quality

Mesh quality in the literature is sufficiently developed with guarantees for triangular and tetrahedral elements (Bern & Eppstein, 1995). However this is not so for mesh elements of four or more sides as well as hexahedra. As a result this research is necessitated in the direction of mesh elements from planar Voronoï diagrams which mostly have four or more sides towards their quality guarantees. This is where the potential utility and impact of mesh qualities in this work is directed. The quality of a mesh depicts a way of investigating pattern organization with a measure of geometric structure. The quality q of a cell is defined by the lengths of the sides of the polygon l_i and its area A. To illustrate consider a quadrilateral Voronoï cell. Its quality is defined by

$$q = 4\frac{A}{l_1^2 + l_2^2 + l_3^2 + l_4^2}.$$

Quality factors of different kinds of polygons are adopted to the criteria of (Shewchuk, 2002; Bhatia & Lawrence, 1990; Knupp, 2001). Quality measures are defined to assume values in $0 \le q \le 1$. A quality value of zero corresponding to a degenerate mesh region whilst a value of one corresponds to a region with equal polygonal side lengths.

5. Theorems and Observations on Voronoï Diagrams

Let $\{q_i\}, i = 1, 2, .., n < \infty$ be the set of qualities of cells resulting from a Voronoï tessellation.

Theorem 5.1. *Qualities of cells satisfy the inequality*

$$(q_1 + q_2 + q_3 + ... + q_n)^2 \le n^2.$$

Proof. Without loss of generality assume $n = 4$. Notice that $q_i \in [0, 1]$

$$(q_1 + q_2 + q_3 + q_4)^2 = q_1^2 + 2q_1q_2 + q_2^2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3^2 + 2q_3q_4 + q_4^2q_i^2, q_iq_j \le 1.$$

Each of the individual terms is potentially less than its maximum value since all the qualities may not have $q_i = 1$. So the squared sum of the qualities is equal to n^2 if and only if all cells have a quality of 1. The quality inequality must be as it is to take care of qualities other than the extremes of zero and unity. Thus we must have

$$(q_1 + q_2 + q_3 + ... + q_n)^2 \le n^2,$$

for $n < \infty$. □

Theorem 5.2. *For a Voronoï cell of quality* $q_i = 1$ *there exists a point inside the cell to which all vertices are equidistant.*

Proof. See (A-iyeh & Peters, 2015). □

Theorem 5.3. *For every Voronoï cell with* $q = 1$ *there exists a polygon whose edge lengths are not unequal.*

Proof. See (A-iyeh & Peters, 2015). □

Lemma 5.1. *Let* $A(\mathcal{V}_s)$ *be the area of the smallest polygon in a Voronoï mesh and let* $A(\mathcal{V}_l)$ *be the area of the polygon with the largest area in the same mesh with intermediate polygonal areas* $A(\mathcal{V}_1)\ldots,A(\mathcal{V}_n)$. *Then*

$$A(\mathcal{V}_s) \subseteq A(\mathcal{V}_l)$$

and

$$A(\mathcal{V}_s) \subseteq A(\mathcal{V}_1) \subseteq A(\mathcal{V}_2)\cdots \subseteq A(\mathcal{V}_n) \subseteq A(\mathcal{V}_l)$$

for a mesh with $n+2$ *polygons.*

Lemma 5.2. *The sequence of all ordered elements of the projections of sets* A_1 *and* B_1, *i.e.,* $\{a_n\}$ *and* $\{b_n\}$, $n = 1,2,3,...$ *form a metric space.*

Consider polygonal elements of R^n with elements $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$. Let $\rho(A_1, B_1) = Inf\{\|x-y\| : x \in A_1, y \in B_1\}$ be the distance between functions of bounded elements A_1 and B_1 of the space. Again let $pr_n(A_1) = Inf\{x \in A_1 | \exists x_1, x_2 ..., x_{n-1} \in R : x = (x_1, x_2 ..., x_{n-1}) \in A_1\}$ be the projection of set A_1 onto the nth-coordinate space of R^n and $\Delta_{l_1 \ldots l_{n-1}}$ represents polygons(half-open meshes) of the form $(l_1 h, l_1 h + h] \times \cdots \times (l_{n-1} h, l_{n-1} h + h]$. h is the edge length and $l_1, l_2, ..., l_{n-1}$ are integers.

Theorem 5.4. *If* A_1 *and* B_1 *be bounded polygons in a Voronoï with with a function of the polygons* $\rho(A_1, B_1) = \delta_0 > 0$, *then a family of polygons* $\{\triangle\}_{k=1}^{N}$, $\triangle_k \subseteq \mathbb{R}^{n-1}$ *exists such that*

$$pr_{\mathbb{R}^{n-1}}(A \cup B) \subseteq \prod_{i=1}^{N},$$

for any $\triangle$ *if* $x \in A$, $y \in B$, $pr_{\mathbb{R}^{x-1}x}, pr_{(R)^{n-1}y} \in \triangle_k$, *then* $|x_n - y_n| = |pr_n x - pr_n y| \geq \delta = \frac{\delta_0}{2}$.

Proof. Assume $h \in (0, \delta_0(2n)^{-1/2})$. Let $D_{k_1 \ldots k_{n-1}} = \triangle_{k_1 \ldots k_{n-1}\mathbb{R}}$. Then $D_{k_1 \ldots k_{n-1}}$ possesses the following properties

1. $\bigcup_{k_1,\ldots k_{n-1} \in \mathbb{Z}} D_{k_1,\ldots k_{n-1}} = \mathbb{R}^n$
2. $D_i \cap D_j = \varnothing$
3. $\forall D = D_{k_1,\ldots k_{n-1}}$ and $\forall x, y \in D$ if $\rho(x,y) \geq \delta_0$, then $\rho(pr_n x, pr_y) \geq \delta_0/2$

Consider $x, y \in D$ and assume $|x_n - y_n| = |pr_n x - pr_n y| < \delta_0/2$. Then we have $\rho(x,y) = [(x_1-y_1)^2+\cdots+(x_n-y_n)^2]^{-1/2} \leq (h^2+\cdots+h^2+\delta_0^2/4)$. For $h \in (0, \delta_0(2n)^{-1/2})$, $\rho(x,y) = (\delta_0^2(n-1)/(2n)+\delta_0^2/4)^{-1/2} < \delta_0$. This is untrue. Hence, property 3 is proved.
For property 2, $A \cup B \neq \varnothing$ and the union of the bounded sets is bounded, so $\bigcup_{i=1}^{N} \supseteq A \cup B$. Thus the union of all the polygons covers the space $\mathbb{R}^n$ and that proves property 1. $pr_{\mathbb{R}^{n-1}}(\bigcup_{i=1}^{N} D_i) = pr_{\mathbb{R}^{n-1}}(\bigcup_{k=1}^{N} \triangle_k \mathbb{R}) \supseteq pr_{\mathbb{R}^{n-1}}(A \cup B)$ and $\bigcup_{k=1}^{N} \supseteq pr_{\mathbb{R}^{n-1}}(A \cup B)$. These statements imply that for $x \in A$, $y \in B$ we can find $\triangle_k$ such that $\rho(x,y) \geq \delta_0$ by assumption, so that $\rho(pr_n x, pr_n y) = |x_n - y_n| \geq \delta_0/2 = \delta$. □

Theorem 5.5. *Symmetry is a condition for optimality of Voronoï meshes.*

Proof. Note that $\mathbb{V}$ for a site s can be expressed as $\mathbb{V}(s_i) := \{(s_k - s_i)x \leq \frac{||s_k||^2 - ||s_i||^2}{2}, s_k \in S\}$. To show optimality we need

$$\frac{\partial \mathbb{V}(s_i)}{\partial s_i}.$$

This gives

$$\frac{\partial V(s)}{\partial s} = -x = -\frac{2||s_i||}{2}.$$

The immediate expression is equivalent to

$$x = \begin{cases} s_i, & \text{if } x \geq 0, \\ s_i, & \text{if } x < 0, \end{cases}$$

which is a mathematical expression for symmetry. □

Property 1. Given a measure function $q(.)$ for a Voronoï diagram of an $n \geq 3$ point set the Voronoï tessellation consists of quality functions equal in number to the number of Voronoï cells.

Property 2. The Voronoï diagram of a set S consisting of $n \geq 3$ non-collinear objects with a measure q for the polygons has at most $2n - 5$ vertices and $3n - 6$ edges, respectively.

Theorem 5.6. *The quality of a scaled Voronoï cell is scale invariant.*

Proof. Consider a triangular cell with quality $q = 1$ before scaling. Now assume the edges of the cell have been scaled with a multiplier $m > 0$. The quality before scaling is given by

$$q = 4\sqrt{3}\frac{0.5l^2}{l^2 + l^2 + l^2} = 1.$$

The quality, after scaling, is expressed by

$$q = 4\sqrt{3}\frac{0.5(ml)^2\sqrt{\frac{3}{4}}}{(ml)^2 + (ml)^2 + (ml)^2} = 1.$$

□

6. Applications

The utility of Voronoï tessellations has often been limited to space partitioning and not understanding the pattern as evidenced by numerous articles. Owing to this an abysmal number of works explore the potential of Voronoï diagrams beyond space partitions. Even fewer works examine properties of Voronoï cells with the viewpoint of understanding underlying nature of patterns. We attempt a way of representing part of a signal space from a point set sample distribution that summarizes the pattern by its equivalent Voronoï signature. These points in the pattern form generators for Voronoï diagrams. Keypoint image patterns of buildings, animals, humans and mountains as previously utilized in (A-iyeh & Peters, 2015) were sampled from images of dimensions M by N to summarize the signals. These point patterns consist of 50 units corresponding to the most prominent in the images. To establish a fair basis for cross analysis the same number of point sets is sampled for all images. In addition all the image signals are gray scale of their respective categories from the dataset of (Wang *et al.*, 2001) (Fig. 4).
With the preamble in place we tessellate and cover the pattern spaces with Voronoï polygons. It is expected that since point patterns are distinct their Voronoï diagrams would exhibit discriminatory properties. This could be key in pattern discrimination using the computed quantities.

Upon identifying the subset representing an image space, we apply the Voronoï partition algorithm to the generators in the signal space. The result is a tessellated space of Voronoï polygons. Open polygons are typical of Voronoï partitions as such in the mathematical formulation of some derived features of the tessellated spaces we adopt techniques that allow the infinite polygons as well as the finite ones to be well behaved.
To help examine the nature and bahaviour of patterns, plots of various quantities are given. There are as many qualities as cells so we define a global quality index or fidelity to capture the geometry of the pattern. Using all cell qualities in a tessellation it is defined by

$$q_{all} = \frac{1}{n}\sum_{i=1}^{n} q_i,$$

where n is the total number of cells and q_i is the quality of cell i. This enables a one-to-one correspondence between quantities.

Due to the finite nature of digital image, we limit the geometrical extent of the point patterns to their convex sets. The information content of images are assessed using a general entropy criterion. A special case of the the general entropy criterion H occurs when $\beta = 2$. This is the so called Renyi entropy denoted here H_R. Simulation results are included for $\beta = 2, 1.5, 2.5$. This range of β captures a range of entropies including the Shannon entropy at $\beta = 2$.

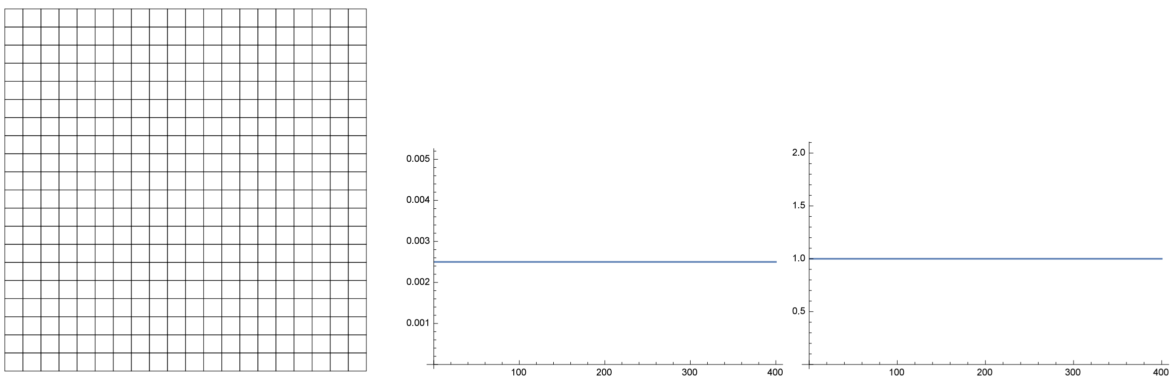
The choice of β in the neighborhood of 2 is not arbitrary. The reasons are two fold; on the one hand we are close to Shannon entropy which enables us to obtain information on the distribution of elements. On the other hand it gives us information on how units of a point pattern influence their distribution. Just as l_0 and l_∞ norms represent extremes of the smallest and largest elements of a set H_0 and H_∞ are the extremes of information measures of which $H_{0<p<\infty}$ gives a tradeoff.

The simulation process is summarized in the following algorithm.

Mesh Quality(q)
for each Voronoï region $\mathbb{V}_i \in \mathbb{V}$ of S **do**
  Access the number of sides and coordinates of the vertices of the polygon.
  Using the coordinates, compute the lengths l_i and Area A of the polygon.
  Use l_i and A_i in the appropriate expression to compute its quality q_i.
end for
$Q = \{q_i\}$
Mesh Entropy(H)
for each Voronoï region $\mathbb{V}_i \in \mathbb{V}$ **do**
  Compute Pr_i
  Use Pr_i to compute H_i
  $H = \{H_i\}$
end for

Remark 6.1. *The assumption made here is that the lengths of the sides of every Voronoï region polygon are measurable. Unfortunately, this is not always the case in, for example, Voronoï tessellations of 2D digital images, since some of the sides of Voronoï region polygons along the borders of an image have infinite length and border polygons have unbounded areas. To cope with this problem, the lengths of all border polygons are a measured relative to one or more image borders.*

Example 1. Consider a completely regular pattern tessellated as shown in Fig. 3.

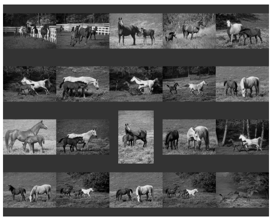


3.1: Mesh 3.2: Probability 3.3: Quality

Figure 3. Perfectly Regular Image Graph Space and Quantities

In Fig. 3 all Voronoï cells have the same area resulting in a uniform distribution of their probabilities. Also all cells have the same quality. Now there are 400 cells in the tessellation and so H attains its maximum value of 5.99146 and the global quality index also attains its maximum value of unity. From the distribution of the probability of cells and their qualities it's straight forward to see that a plot of general entropy against global quality indices would be a straight horizontal line.


4.1: Building set


4.2: Horse set


4.3: Human set

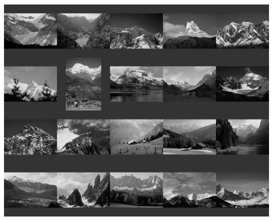
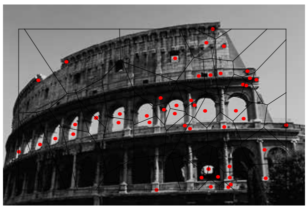
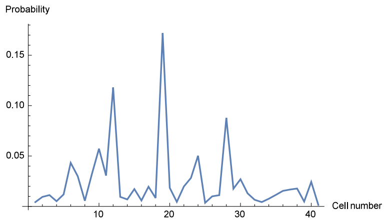
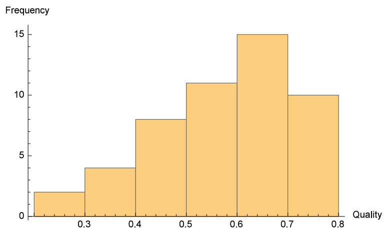

4.4: Mountain set

Figure 4. Data sets


5.1: Mesh



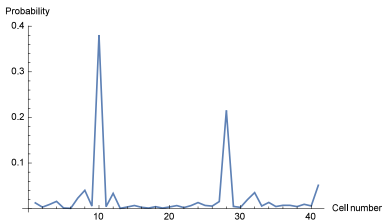
5.2: Probability



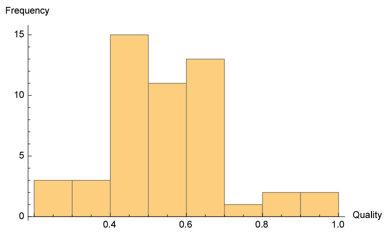
5.3: Quality

Figure 5. Image Graph Spaces


6.1: Mesh



6.2: Probability



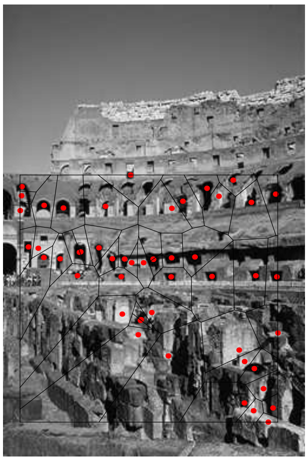
6.3: Quality

Figure 6. Image Graph Spaces

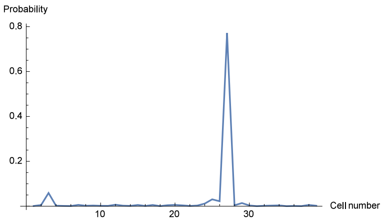
7. Results and Discussion

Most polygons typically have non-zero area so Pr_i is defined for all regions in the plane. In the following image Voronoï graphs, probability functions of cells, image cell qualities and plots of quantities are shown. Also quality of cells and information are studied by examining the nature of the plots. The results of our simulations are shown for only three images per category of the data set given in Fig. 4 for space reasons although the results are presented for the entire data set of 20 images per category amounting to 80 images in total. Corresponding cell area probabilities and distribution of cell qualities are shown next to tessellated spaces in Fig. 5-Fig. 16 in that order.

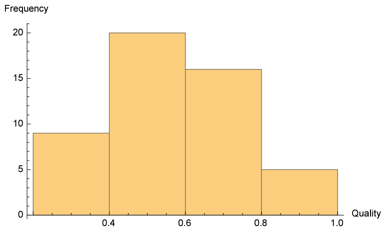
Point patterns consist of a maximum of 50 keypoints and so the resulting cells are usually 50 in number. Notice the nature of the distributions of probabilities and qualities. Probability distri-



7.1: Mesh



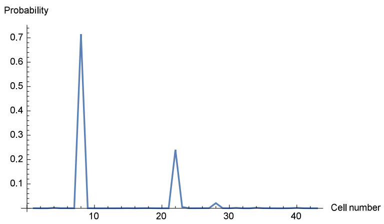
7.2: Probability



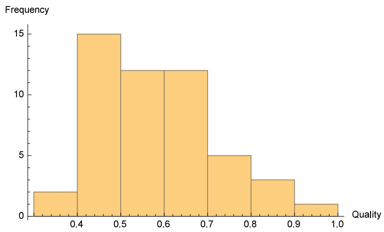
7.3: Quality

Figure 7. Image Graph Spaces

8.1: Mesh



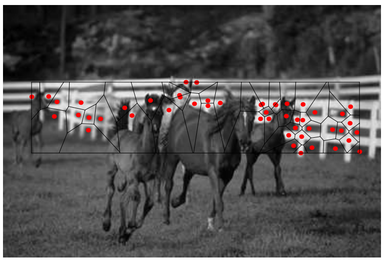
8.2: Probability



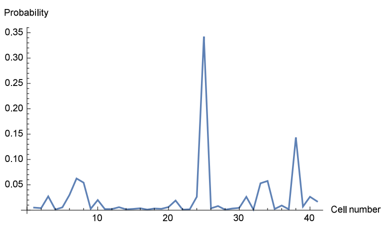
8.3: Quality

Figure 8. Image Graph Spaces

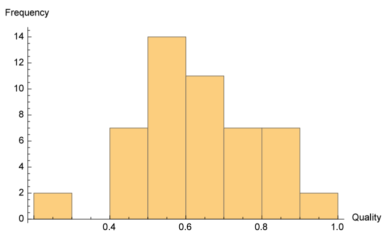
butions range from the extreme of only a few influential cells to cells exhibiting higher tendencies of equal influences. This corresponds to a few large peaks on the probability distributions and a spread out distribution respectively. The qualities of the cells portray the exhibited behaviour.



9.1: Mesh

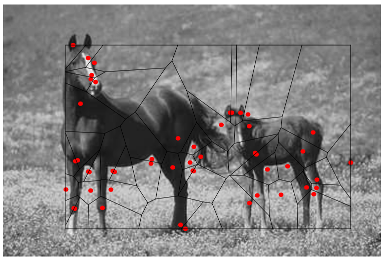


9.2: Probability

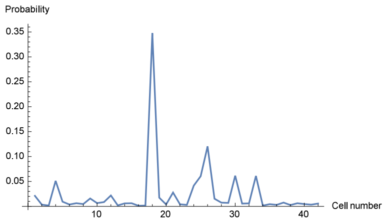


9.3: Quality

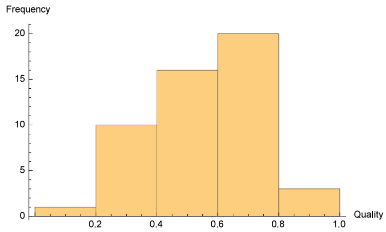
Figure 9. Image Graph Spaces



10.1: Mesh



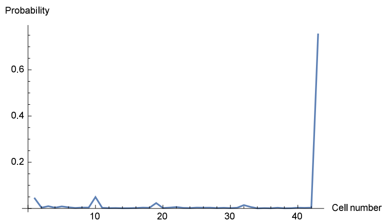
10.2: Probability



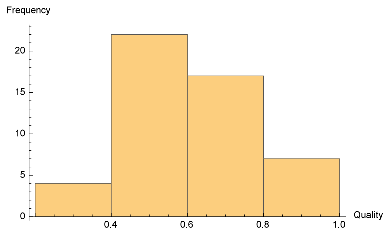
10.3: Quality

Figure 10. Image Graph Spaces

11.1: Mesh



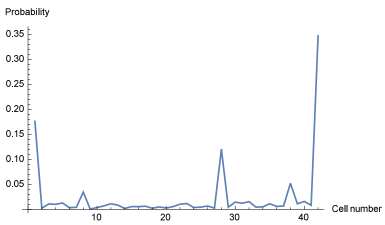
11.2: Probability



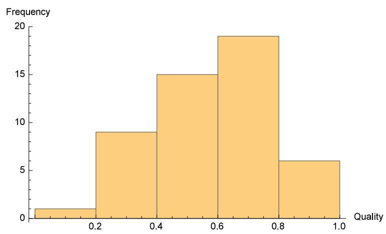
11.3: Quality

Figure 11. Image Graph Spaces

12.1: Mesh



12.2: Probability



12.3: Quality

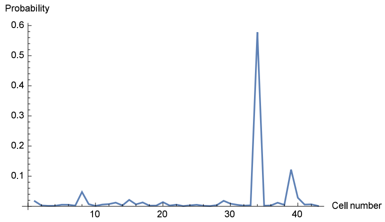
Figure 12. Image Graph Spaces

Entropies of tessellations and global quality indices are condensed into the following plots. For 50 Voronoï cells exihibiting a uniform probability distribution the maximum value possible for Renyi entropy is 3.912. All entropy values fall short of this value. Plots of entropies and global qualities are shown for the buildings, horses, humans and mountain scenery categories in Fig. 17. Notice the flat nature of the global qualities for the images. Renyi entropies as a function of the images is non-decreasing.

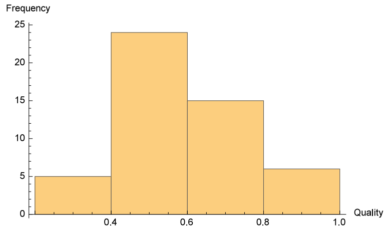
In the following, plots of global qualities, Renyi entropies and plots of entropies against qualities are shown.

Notice the monotonically increasing entropies and global qualities in Fig. 17. Also observe that

13.1: Mesh

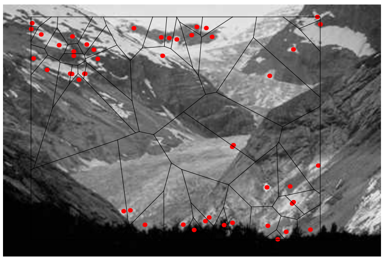


13.2: Probability

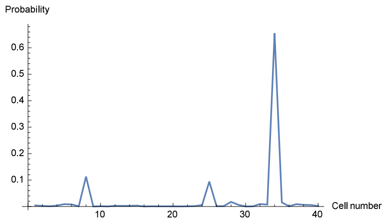


13.3: Quality

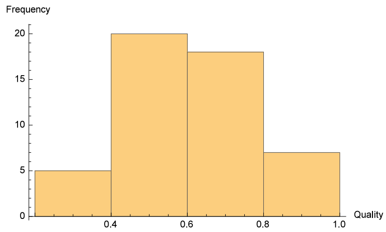
Figure 13. Image Graph Spaces



14.1: Mesh

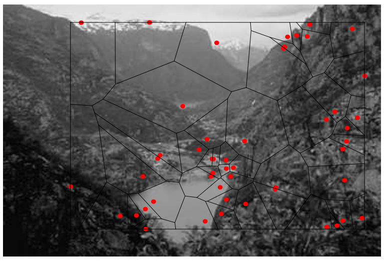


14.2: Probability

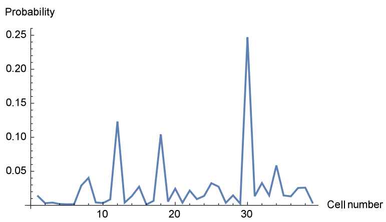


14.3: Quality

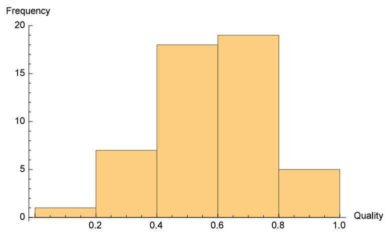
Figure 14. Image Graph Spaces



15.1: Mesh



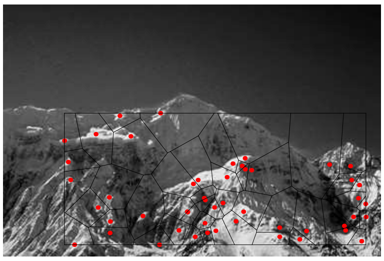
15.2: Probability



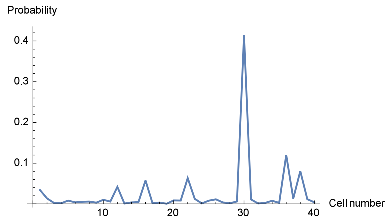
15.3: Quality

Figure 15. Image Graph Spaces

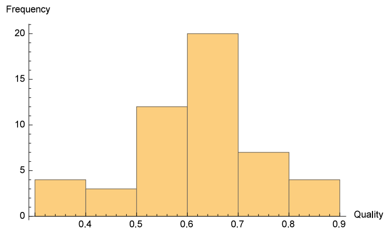
the quantities are distinct across categories. Most importantly entropic information is decreases for $\beta = 1.5, 2.0, 2.5$ in that order. Recall that $\beta = 2$ yields Shannon entropy from the general entropy criterion H. It is interesting to note the oscillating (Fig. 18) as opposed to uniform relationship between entropy and global quality. This confirms the departure of the images from the less interesting case of completely regular patterns.



16.1: Mesh

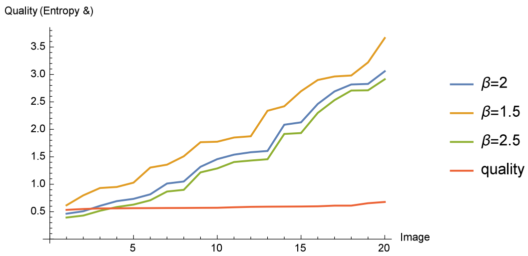


16.2: Probability

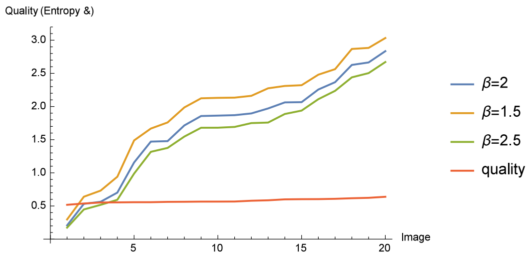


16.3: Quality

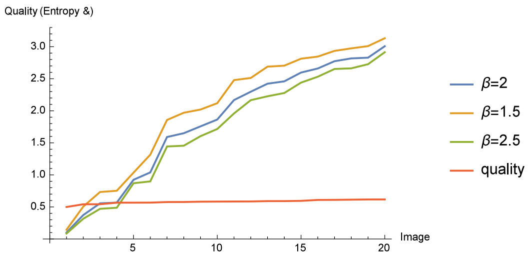
Figure 16. Image Graph Spaces



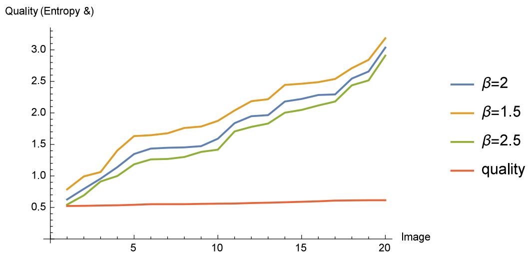
17.1: Buildings



17.2: Horses



17.3: Humans



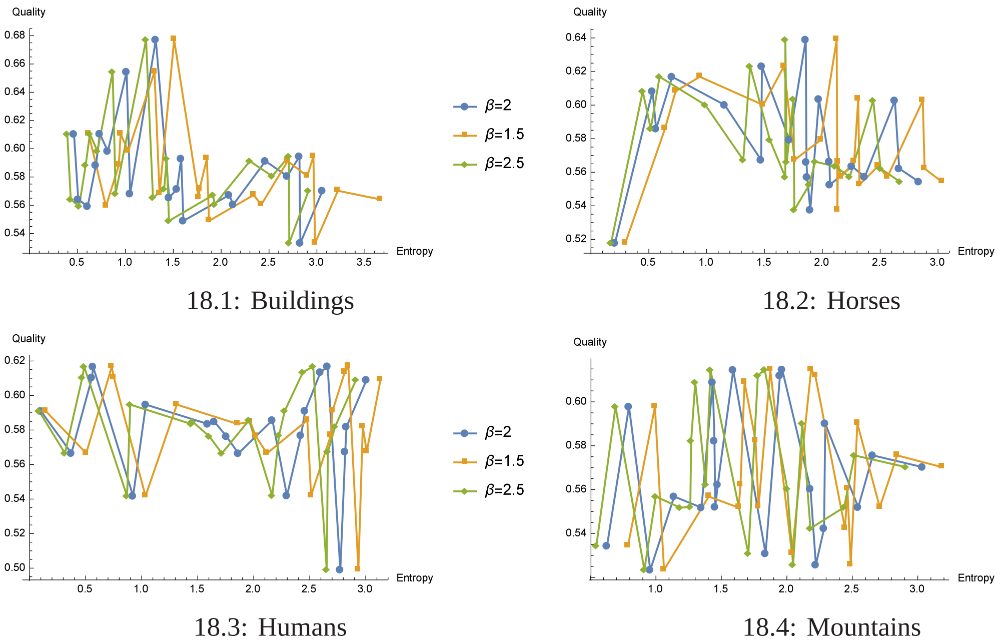
17.4: Mountains

Figure 17. Quantity Relations

8. Conclusion and Future Work

Non-linear probability distribution distribution functions as opposed to uniform ones are observed. However, recall that a uniform distribution maximizes the entropy so that implies that the point patterns are more informative and interesting compared to completely regular patterns. Although the patterns are not uniform the information parameter range of $1 < \beta \leq 2.5$ maximizes the information content of Voronoï cells. This shows that the Renyi entropy is more informative than Shannon entropy. This is due to the variations in pattern structure. Owing to the non-linear relationship between entropy and cell qualities, we see that the patterns are not simple patterns because of the variations.

Notice that the global qualities q_{all} for all image categories practically follow a linear distribution with a gradient close to zero. So given a global quality of a tessellation converging in the



18.1: Buildings

18.2: Horses

18.3: Humans

18.4: Mountains

Figure 18. Quality Signatures

neighborhood of $0.5 \leq q_{all} < 1.0$, the point pattern is not completely regular and could be from a digital image. This range of global qualities observed shows that point pattern primitives of digital images may not be simple and completely regular features.

Image point patterns with global quality coefficients in the range $0.5 \leq q_{all} < 1.0$ are stable. This indicates that the image physical system is sufficiently modeled. This is the so called fidelity of solution of the physical system of differential equations represented by the mesh. A completely regular pattern with a global index or fidelity of unity is the most stable (Fig. 3) so that an unstable system has an index of zero or close to zero.

Since the point patterns are not completely regular they contain more information than regular ones because their global indices are less than unity and their entropies are less than the maximum value.

Notwithstanding this quality guarantees for meshes of four or more sides which is hardly studied and much less developed is seen to be stable and guaranteed in the reported range.

Finally it has been shown that the distribution of digital image point patterns is anything but uniform. Therefore future work should reveal the applicable distribution(s).

It goes without saying that although the method is simple and effective in characterizing pattern information and structure the assignment of zero probabilities to infinite Voronoï cells is a disadvantage. This however is a natural consequence of Voronoï partitioning for which the choice has to be made whether the information is attributed to a few infinite cells or otherwise.

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