Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 1 – 13

Fixed Points of Generalized Kannan Type α-admissible Mappings in Cone Metric Spaces with Banach Algebra

S.K. Malhotra[a], J.B. Sharma[b], Satish Shukla[c,*]

[a]*Department of Mathematics, Govt. S.G.S.P.G. College, Ganj Basoda, Distt Vidisha M.P., India*
[b]*Department of Mathematics, Choithram College of Professional Studies, Dhar Road, Indore, India*
[c]*Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Gram Baroli, Sanwer Road, Indore (M.P.) 453331 India*

Abstract

In this paper, we introduce the generalized Kannan type α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra. Our results generalize and extend the fixed point result for Kannan type mappings in metric and cone metric spaces. An example is presented which illustrates our main result.

Keywords: Cone metric space, α-admissible mapping, Kannan's contraction, Solid cone, Banach algebra, Fixed point.
2010 MSC: 47H10, 54H25.

1. Introduction

Huang and Zhang (Huang & Zhang, 2007) introduced the notion of cone metric spaces as a generalization of metric spaces. They replaced the set of nonnegative real numbers by a subset of a Banach space called the cone; and defined the metric as a vector-valued function. They obtained some fixed point results in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani (Rezapour & Hamlbarani, 2008). Liu and Xu (Liu & Xu, 2013*a*) defined the cone metric spaces with Banach algebra and defined the vector-valued metric into a subset of a Banach algebra. The motivation for the work of Liu and Xu (Liu & Xu, 2013*a*) can be found in (Cakalli *et al.*, 2012; Kadelburg *et al.*, 2011; Du, 2010; Feng & Mao, 2010). The results proved by Liu and Xu (Liu & Xu, 2013*a*) demands the normality of the underlying cone. Later on, Xu and Radenović (Xu

*Corresponding author
Email addresses: jagdishmathematics@gmail.com (J.B. Sharma), satishmathematics@yahoo.co.in (Satish Shukla)

& Radenović, 2014) showed that the condition of normality of cone can be removed, and so, the results of Liu and Xu (Liu & Xu, 2013*a*) are also true in case of a non-normal cone.

Let (X, d) be a metric space and $T: X \to X$ be a mapping satisfying the following condition: there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Then the mapping T is called a Banach contraction. The Banach's contraction principle states that a Banach contraction on a complete metric space has a unique fixed point, i.e., there exists a unique point $x^* \in X$ such that $x^* = Tx^*$. Kannan (Kannan, 1968, 1969) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leqslant \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1.2}$$

Kannan (Kannan, 1968, 1969) showed that the conditions (1.1) and (1.2) are independent of each other, and proved a fixed point result for the mapping satisfying the condition (1.2) instead the condition (1.1).

Samet et al. (Samet *et al.*, 2012) introduced a new type of mappings called α-admissible mappings, and with the help of this new class of mappings they generalized several known results of metric spaces. Very recently, Malhotra et al. (Malhotra *et al.*, 2015) introduced the α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra and solid cones. They generalized and extended several known results of metric and cone metric spaces by proving a fixed point result for generalized Lipschitz contraction over cone metric spaces. The main result of (Malhotra *et al.*, 2015) was a generalization of Banach's fixed point theorem. In this paper, we introduce the notion of generalized Kannan type α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra which extend the concept introduced in (Malhotra *et al.*, 2015) and generalize the result of Kannan (Kannan, 1968, 1969) in cone metric spaces equipped with Banach algebra.

2. Preliminaries

First, we state some known definitions and results which will be used in the sequel.

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in A, a \in \mathbb{R}$

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $a(xy) = (ax)y = x(ay)$;
4. $\|xy\| \leqslant \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details we refer to (Rudin, 1991).

The following proposition is well known (Rudin, 1991).

Proposition 2.1. *Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than one, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if

1. P is non-empty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;
2. $a_1 P + a_2 P \subset P$ for all non-negative real numbers a_1, a_2;
3. $P^2 = PP \subset P$
4. $P \bigcap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in A$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leqslant K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see (Huang & Zhang, 2007)). Note that, for any normal cone P we have $K \geqslant 1$ (see (Rezapour & Hamlbarani, 2008)). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \phi$ (i.e., the cone P is a solid cone) and $\preceq$ is the partial ordering with respect to P.

The following lemmas and remark will be useful in the sequel.

Lemma 2.1 (See (Kadelburg *et al.*, 2010))**.** *If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leqslant \lambda < 1$, then $a = \theta$.*

Lemma 2.2 (See (Radenović & Rhoades, 2009))**.** *If E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.3 (See (Radenović & Rhoades, 2009))**.** *If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.*

Remark (See (Xu & Radenović, 2014)). If $\rho(x) < 1$ then $\|x^n\| \to 0$ as $n \to \infty$.

Definition 2.1 (See (Liu & Xu, 2013*a*,*b*; Huang & Zhang, 2007))**.** Let X be a non-empty set. Suppose that the mapping $d\colon X \times X \to A$ satisfies:

1. $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over the Banach algebra A.

Definition 2.2 (See (Huang & Zhang, 2007))**.** Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

1. The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
2. The sequence $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X.

It is obvious that the limit of a convergent sequence in a cone metric space is unique. A mapping $T: X \to X$ is called continuous at $x \in X$, if for every sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.3 (See (Samet *et al.*, 2012))**.** Let X be a nonempty set and $\alpha: X \times X \to [0, \infty)$ be a function. We say that T is α-admissible if $(x, y) \in X, \alpha(x, y) \geqslant 1 \implies \alpha(Tx, Ty) \geqslant 1$.

Now, we define the generalized Lipschitz contractions on the cone metric spaces with a Banach algebra (see also, (Liu & Xu, 2013*a*)).

Definition 2.4. (Malhotra *et al.*, 2015) Let (X, d) be a complete cone metric space over a Banach algebra A, P the underlying solid cone and $\alpha: X \times X \to [0, \infty)$ be a function. Then the mapping $T: X \to X$ is said to be generalized Lipschitz contraction if there exists $k \in P$ such that $\rho(k) < 1$ and,

$$d(Tx, Ty) \preceq kd(x, y)$$

for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$. Here, the vector k is called the Lipschitz vector of T.

Malhotra et al. (Malhotra *et al.*, 2015) proved a fixed point result for such generalized contraction. Here, we prove a Kannan's version of the result of Malhotra et al. (Malhotra *et al.*, 2015).

Now we can state our main results.

3. Main results

First, we define generalized Kannan type contractions in cone metric spaces with Banach algebra.

Definition 3.1. Let (X, d) be a complete cone metric space over a Banach algebra A, P the underlying solid cone and $\alpha: X \times X \to [0, \infty)$ be a function. Then the mapping $T: X \to X$ is said to be generalized Kannan type contraction if there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and,

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \tag{3.1}$$

for all $x, y \in X$ with $\alpha(x, y) \geqslant 1$. Here, the vector k is called the Kannan-Lipschitz vector of T.

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra A, P be the underlying solid cone and $\alpha\colon X \times X \to [0, \infty)$ be a function. Suppose, $T\colon X \to X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*
(iii) *T is continuous.*

Then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is α-admissible we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant 1 \implies \alpha(Tx_0, T^2x_0) = \alpha(x_1, x_2) \geqslant 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geqslant 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.2}$$

Since T is generalized Kannan type contraction with Kannan-Lipschitz vector k, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq k[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ &= k[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \end{aligned}$$

i.e.,

$$(e - k)d(x_n, x_{n+1}) \preceq kd(x_{n-1}, x_n).$$

Since $\rho(k) < \frac{1}{2} < 1$, $e - k$ is invertible, therefore it follows from the above inequality that

$$d(x_n, x_{n+1}) \preceq k(e-k)^{-1}d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n) \preceq \lambda^n d(x_0, x_1) \tag{3.3}$$

where $\lambda = k(e-k)^{-1}$. Since $(e-k)^{-1} = \sum_{i=0}^{\infty} k^i$ we have

$$\rho((e-k)^{-1}) = \rho\left(\sum_{i=0}^{\infty} k^i\right) \leqslant \sum_{i=0}^{\infty} \rho(k^i) \leqslant \sum_{i=0}^{\infty} [\rho(k)]^i = \frac{1}{1-\rho(k)}.$$

Therefore,

$$\begin{aligned} \rho(\lambda) &= \rho\left(k(e-k)^{-1}\right) \leqslant \rho(k)\rho\left((e-k)^{-1}\right) \\ &\leqslant \frac{\rho(k)}{1-\rho(k)} < 1 \quad (\text{since } \rho(k) < \frac{1}{2}). \end{aligned}$$

Thus, for $n < m$ it follows from the inequality (3.3) that

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &= (e + \lambda + \cdots + \lambda^{m-n-1})\lambda^n d(x_0, x_1) \\ &\preceq \left(\sum_{i=0}^{\infty} \lambda^i\right) \lambda^n d(x_0, x_1) \\ &= (e - \lambda)^{-1} \lambda^n d(x_0, x_1). \end{aligned}$$

Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \to 0$ as $n \to \infty$. Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \preceq (e - \lambda)^{-1} \lambda^n d(x_0, x_1) \ll c$$

for all $n > n_0$. It implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is continuous, it follows that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. By the uniqueness of limit we get $x^* = Tx^*$, that is x^* is a fixed point of T. □

In the above theorem, we use the continuity of the mapping T. We now show that the assumption of continuity can be replaced by another condition.

Theorem 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra A, P be the underlying solid cone and $\alpha: X \times X \to [0, \infty)$ be a function. Suppose, $T: X \to X$ be a generalized Kannan type contraction with Kannan-Lipschitz vector k and the following conditions are satisfied:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*
(iii) *if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point $x^* \in X$.

Proof. By proof of theorem 3.1, we know that the sequence $\{x_n\}$, where $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ is a Cauchy sequence in complete cone metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, from (3.2) and hypothesis (iii), we have

$$\alpha(x_n, x^*) \geqslant 1, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.4}$$

Since T is a generalized Kannan type contraction, using (3.4) we obtain

$$\begin{aligned} d(x^*, Tx^*) &\preceq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\preceq d(x^*, x_{n+1}) + k[d(x_n, Tx_n) + d(x^*, Tx^*)] \end{aligned}$$

i.e.,

$$\begin{aligned} d(x^*, Tx^*) &\preceq (e-k)^{-1}[d(x^*, x_{n+1}) + kd(x_n, Tx_n)] \\ &= (e-k)^{-1}d(x^*, x_{n+1}) + \lambda d(x_n, Tx_n). \end{aligned}$$

By (3.3) we have $d(x_n, Tx_n) = d(x_n, x_{n+1}) \preceq \lambda^n d(x_0, x_1)$, therefore

$$d(x^*, Tx^*) \preceq (e-k)^{-1}d(x^*, x_{n+1}) + \lambda^{n+1}d(x_0, x_1).$$

As $x_n \to x^*$ as $n \to \infty$ and $\rho(\lambda) < 1$, for every $c \in P$ with $\theta \ll c$ and for every $m \in \mathbb{N}$ there exists $n(m)$ such that $d(x_{n+1}, x^*) \ll \frac{(e-k)c}{2m}$ and $\lambda^{n+1}d(x_0, x_1) \ll \frac{c}{2m}$ for all $n > n(m)$. Therefore, it follows from the above inequality that

$$d(x^*, Tx^*) \preceq \frac{c}{2m} + \frac{c}{2m} = \frac{c}{m} \text{ for all } n > n(m), m \in \mathbb{N}.$$

It implies that $\frac{c}{m} - d(x^*, Tx^*) \in P$ for all $m \in \mathbb{N}$. Since P is closed, letting $m \to \infty$ we obtain $\theta - d(x^*, Tx^*) \in P$. By definition, we must have $d(x^*, Tx^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T. □

Next, we give an example which illustrates the above result.

Example 3.1. Let $A = \mathbb{R}^2$ with the norm

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Define the multiplication on A by

$$xy = (x_1y_1, x_1y_2 + x_2y_1) \quad \text{for all} \quad x = (x_1, x_2), y = (y_1, y_2) \in A.$$

Then, A is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1, x_2 \geqslant 0\}$. Then P is a positive cone.

Let $X = [0, 1] \times [0, 1]$ and define the cone metric $d \colon X \times X \to P$ by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then, (X, d) is a complete cone metric space. Let $\mathbb{Q} \cap [0, 1) = \mathbb{Q}_1$ and define the mappings $T \colon X \to X$ and $\alpha \colon X \times X \to [0, \infty)$ by:

$$T(x_1, x_2) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right), & \text{if } x_1, x_2 \in \mathbb{Q}_1; \\ \left(\frac{1}{4}, \frac{1}{4}\right), & \text{if } x_1 = x_2 = 1; \\ (x_1, x_2), & \text{otherwise.} \end{cases}$$

and

$$\alpha((x_1, x_2), (y_1, y_2)) = \begin{cases} 1, & \text{if } (x_1, x_2, y_1, y_2 \in \mathbb{Q}_1) \text{ or } (x_1, x_2 \in \mathbb{Q}_1, y_1 = y_2 = 1); \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is a generalized Kannan type contraction with Kannan-Lipschitz vector $k = \left(\frac{1}{3}, 0\right)$, where $\rho(k) = \frac{1}{3} < \frac{1}{2}$. Indeed, $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ then (3.1) is satisfied trivially. If $x_1, x_2 \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$ then we have

$$\begin{aligned} d(T(x_1, x_2), T(y_1, y_2)) &= d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) \\ &= \left(\frac{1}{4}, \frac{1}{4}\right) \\ &\preceq \left(\frac{1}{3}, 0\right)[d((x_1, x_2), T(x_1, x_2)) + d((y_1, y_2), T(y_1, y_2))]. \end{aligned}$$

T is obviously an α-admissible mapping, and for every $x_1, x_2, y_1, y_2 \in \mathbb{Q}_1$ we have

$$\alpha((x_1, x_2), T(x_1, x_2)) = 1.$$

Therefore, the conditions (i) and (ii) of Theorem 3.2 are satisfied. Finally, one can see that the condition (iii) of Theorem 3.2 is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied and we conclude the existence of at least one fixed point of T. Indeed, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and all the points $(x, 1), x \in \mathbb{Q}_1$ and $(1, x), x \in \mathbb{Q}_1$ are fixed points of T.

Remark. Notice that, in the above example the results of Malhotra et al. (Malhotra *et al.*, 2015) are not applicable. Indeed, if $x_1 = x_2 = \frac{3}{4} \in \mathbb{Q}_1$ and $y_1 = y_2 = 1$, then $\alpha((x_1, x_2), (y_1, y_2)) = 1$ and

$$d(T(x_1, x_2), T(y_1, y_2)) = d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{1}{4}\right)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Now

$$d((x_1, x_2), (y_1, y_2)) = d\left(\left(\frac{3}{4}, \frac{3}{4}\right), (1, 1)\right) = \left(\frac{1}{4}, \frac{1}{4}\right).$$

Therefore, there exists no $k \in P$ such that $\rho(k) < 1$ and the following inequality is satisfied:

$$d(T(x_1, x_2), (y_1, y_2)) \preceq kd((x_1, x_2), (y_1, y_2)).$$

This shows that T is not a generalized Lipschitz contraction, and so, the results of Malhotra et al. (Malhotra *et al.*, 2015) are not applicable here.

In the Example 3.1 we can see that the mapping T may have more than one fixed points. Let us denote the set of all fixed points of T by $\mathrm{Fix}(T)$.

Next, to assure the uniqueness of fixed point of a generalized Kannan type contraction we use the following property (see (Samet *et al.*, 2012)):

$$\forall\ x, y \in \mathrm{Fix}(T)\ \exists\ z \in X: \quad \alpha(x, z) \geqslant 1, \alpha(y, z) \geqslant 1. \tag{H}$$

Theorem 3.3. *Adding condition (H) to the hypothesis of Theorem 3.1 (resp. Theorem 3.2) we obtain the uniqueness of the fixed point of T.*

Proof. Following similar arguments to those in the proof of Theorem 3.1 (resp. Theorem 3.2) we obtain the existence of fixed point. Let the condition (H) is satisfied and $x^*, y^* \in \text{Fix}(T)$ and $x^* \neq y^*$. By (H) there exists $z \in X$ such that

$$\alpha(x^*, z) \geqslant 1 \quad \text{and} \quad \alpha(y^*, z) \geqslant 1. \tag{3.5}$$

Since T is α-admissible and $x^*, y^* \in \text{Fix}(T)$, therefore from (3.5) we obtain

$$\alpha(x^*, T^n z) \geqslant 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geqslant 1. \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.6}$$

Since T is generalized Kannan type contraction, using (3.6), we have

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, T(T^{n-1}z)) \\ &\preceq k[d(x^*, Tx^*) + d(T^{n-1}z, T(T^{n-1}z))] \\ &= kd(T^{n-1}z, T^n z) \\ &\preceq k[d(x^*, T^{n-1}z) + d(x^*, T^n z)] \end{aligned}$$

i.e.,

$$d(x^*, T^n z) \preceq k(e-k)^{-1} d(x^*, T^{n-1}z) = \lambda d(x^*, T^{n-1}z) \ \text{ for all } \ n \in \mathbb{N}.$$

Repetition of this process we obtain

$$d(x^*, T^n z) \preceq \lambda^n d(x^*, Tz) \ \text{ for all } \ n \in \mathbb{N}.$$

where $\lambda = k(e-k)^{-1}$ and $\rho(\lambda) < 1$. Since $\rho(\lambda) < 1$, by Remark 2 we have $\|\lambda^n\| \to 0$ as $n \to \infty$, and so,

$$\|\lambda^n d(x^*, Tz)\| \leqslant \|\lambda^n\| \|d(x^*, Tz)\| \to 0 \text{ as } n \to \infty.$$

Therefore, by Lemma 2.3 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, T^n z) \preceq \lambda^n d(x^*, Tz) \ll c.$$

it implies that

$$T^n z \to x^* \quad \text{as} \quad n \to \infty.$$

Similarly we get

$$T^n z \to y^* \quad \text{as} \quad n \to \infty.$$

Therefore, by uniqueness of the limit we obtain $x^* = y^*$. This finishes the proof. □

4. Some consequences

In this section, we give some consequences of the results of previous section. The following corollary is Theorem 3.3 of Xu and Radenović (Liu & Xu, 2013*a*).

Corollary 4.1 (Theorem 3.3, Xu and Radenović (Liu & Xu, 2013*a*))**.** *Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone with $k \in P$ where $\rho(k) < \frac{1}{2}$. Suppose the mapping $T: X \to X$ satisfies the following condition :*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \quad \text{for all} \quad x, y \in X.$$

Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point of X.

Proof. Define the function $\alpha: X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then, all the conditions of Theorem 3.3 are satisfied, and so, the mapping T has a unique fixed point in X. □

Next, we derive the ordered and cyclic versions of Kannan's contraction principle. In the next theorems, we prove results of Ran and Reurings (Ran & Reurings, 2003), Liu and Xu (Liu & Xu, 2013*a*) and Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) and Kirk et al. (Kirk *et al.*, 2003) for Kannan's mappings.

The following theorem is the Kannan's version of the result of Ran and Reurings (Ran & Reurings, 2003) in cone metric spaces when the cone metric is endowed with a Banach algebra.

Theorem 4.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a continuous nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assumptions hold:*

(i) *there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } x \sqsubseteq y;$$

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.*

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r: X \times X \to [0, \infty)$ by

$$\alpha_r(x, y) = \begin{cases} 1, & \text{if } x \sqsubseteq y; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, the condition (i) implies that the mapping T a generalized Kannan type contraction with Kannan-Lipschitz vector k, where $\rho(k) < \frac{1}{2}$. Since T is nondecreasing it is an α_r-admissible mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $\alpha_r(x_0, Tx_0) = 1$. Therefore, all the conditions of Theorem 3.1 are satisfied, and so, the mapping T has a fixed point in X. □

The following theorem is the Kannan's version of the result of Nieto, Rodríguez-López (Nieto & Rodríguez-López, 2005) when the cone metric is endowed with a Banach algebra.

Theorem 4.2. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Let $T: X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assumptions hold:*

(i) *there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and*

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } x \sqsubseteq y;$$

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*

(iii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r : X \times X \to [0, \infty)$ similar to that as in the proof of Theorem 4.1. Now, the proof follows from the Theorem 3.2. □

Next, we define the cyclic contractions (see (Kirk *et al.*, 2003)) in cone metric spaces.

Let X be a nonempty set, $T: X \to X$ a mapping and $A_1, A_2, \ldots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if

1. A_i, $i = 1, 2, \ldots, m$ are nonempty sets;
2. $T(A_1) \subset A_2, \ldots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset T(A_1)$.

Remark. (See (Kirk *et al.*, 2003)) If $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, then $\text{Fix}(T) \subset \bigcap_{i=1}^{m} A_i$.

A cyclic contraction on a cone metric space is defined as follows.

Definition 4.1. Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $A_1, A_2, \ldots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. A mapping $T: Y \to Y$ is called a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k if following conditions hold:

1. $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;
2. there exists $k \in P$ such that $\rho(k) < \frac{1}{2}$ and

$$d(Tx, Ty) \preceq k[d(x, Tx) + d(y, Ty)] \tag{4.1}$$

for any $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$).

The following theorem is the Kannan's version of the result Kirk et al. (Kirk *et al.*, 2003) when the cone metric is endowed with a Banach algebra.

Theorem 4.3. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space over a Banach algebra A with P the underlying solid cone. Suppose, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ and $T: Y \to Y$ be a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k. Then, T has a unique fixed point in X.*

Proof. Define the mapping $\alpha_c: X \times X \to [0, \infty)$ by:

$$\alpha_c(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_i \times A_{i+1} \ (i = 1, 2, \ldots, m \text{ where } A_{m+1} = A_1); \\ 0, & \text{otherwise.} \end{cases}$$

First, by definition of the function α and the cyclic representation, T is α_c-admissible. Again, by definition of the function α_c, T is a generalized cyclic Kannan type contraction with Kannan-Lipschitz vector k. Suppose, for a sequence $\{x_n\}$ we have $\alpha_c(x_n, x_{n+1}) \geqslant 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then, as $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation with respect to T, we must have $x \in \bigcap_{i=1}^{m} A_i$. Therefore, $\alpha_c(x_n, x) \geqslant 1$ for all $n \in \mathbb{N}$. Now, the proof of existence of fixed point of T follows from Theorem 3.2. For uniqueness, if $x^*, y^* \in \text{Fix}(T)$, then by Remark 4 we have $x^*, y^* \in \bigcap_{i=1}^{m} A_i$. Since each A_i, $i \in \{1, 2, \ldots, m\}$ is nonempty, there exists $z \in Y$ such that $x^*, y^* \in A_i, z \in A_{i+1}$ for some $i \in \{1, 2, \ldots, m\}$, and so $\alpha_c(x^*, z) = \alpha_c(y^*, z) = 1$. Thus, the condition (H) is satisfied and the uniqueness of fixed point follows from Theorem 3.3. □

Acknowledgements. The authors are indebted to the anonymous referee and Editor for his/her careful reading of the text and for suggestions for improvement in several places.

References

Cakalli, H., A. Sonmez and C. Genc (2012). On an equivalence of topological vector space valued cone metric spaces and metric spaces. *Appl. Math. Lett.* **25**, 429–433.

Du, W.S. (2010). A note on cone metric fixed point theory and its equivalence. *Nonlinear Anal.* **72**(5), 2259–2261.

Feng, Y. and W. Mao (2010). The equivalence of cone metric spaces and metric spaces. *Fixed Point Theory* **11**(2), 259–264.

Huang, L.-G. and X. Zhang (2007). Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl.* **332**, 1468–1476.

Kadelburg, Z., M. Pavlović and S. Radenović (2010). Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces. *Comput. Math. Appl.* **59**, 3148–3159.

Kadelburg, Z., S. Radenović and V. Rakočević (2011). A note on the equivalence of some metric and cone metric fixed point results. *Appl. Math. Lett.* **24**, 370–374.

Kannan, R. (1968). Some results on fixed point. *Bull. Calcutta Math. Soc.* **60**, 71–76.

Kannan, R. (1969). Some results on fixed point-ii. *Amer. Math. Monthly.* **76**, 405–408.

Kirk, W.A., P.S. Srinivasan and P. Veeramani (2003). Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory* **4**(1), 79–89.

Liu, H. and S. Xu (2013*a*). Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings. *Fixed Point Theory and Applications* **2013**(1), 320.

Liu, H. and S.-Y. Xu (2013*b*). Fixed point theorems of quasi-contractions on cone metric spaces with Banach algebras. *Abstarct and Applied Analysis* **2013**(Article ID 187348), 5 pages.

Malhotra, S.K., J.B. Sharma and S. Shukla (2015). Fixed points of α-admissible mappings in cone metric spaces with Banach algebra. *International Journal of Analysis and Applications* **9**(1), 9–18.

Nieto, J.J. and R. Rodríguez-López (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order* **22**, 223–239.

Radenović, S. and B.E. Rhoades (2009). Fixed point theorem for two non-self mappings in cone metric spaces. *Comput. Math. Appl.* **57**, 1701–1707.

Ran, A.C.M. and M.C.B. Reurings (2003). A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Amer. Math. Soc.* **132**, 1435–1443.

Rezapour, Sh. and R. Hamlbarani (2008). Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings. *Math. Anal. Appl.* **345**, 719–724.

Rudin, W. (1991). *Functional Analysis*. 2nd ed., McGraw-Hill.

Samet, B., C. Vetro and P. Vetro (2012). Fixed point theorems for α-ψ-contractive type mappings. *Nonlinear Analysis* **75**, 2154–2165.

Xu, S. and S. Radenović (2014). Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality. *Fixed Point Theory Appl.* **2014**, 2014:102.

Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 14 – 26

On the Growth of Solutions of Higher Order Complex Differential Equations with finite $[p, q]$-Order

Mohamed Amine Zemirni[a], Benharrat Belaïdi[a,*]

[a]*Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-Algeria*

Abstract

In this paper, we study the growth of entire solutions of higher order linear complex differential equations with entire coefficients of finite $[p, q]$-order. We give another conditions that generalize some results due to (Belaïdi, 2015), (Liu *et al.*, 2010) and (Li & Cao, 2012).

Keywords: Complex differential equation, Meromorphic solution, Entire solution, $[p, q]$-Order.
2010 MSC: 34M10, 30D35.

1. Introduction

In this article, we use the standard notation and fundamental results of the Nevanlinna value distribution theory of meromorphic functions, see (Hayman, 1964; Laine, 1993; Yang & Yi, 2003). We define, for $r \in [0, +\infty)$, $\exp_0 r := r$, $\exp_1 r := e^r$ and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_0 r := r$, $\log_1 r := \log r$ and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r := \log r$ and $\log_{-1} r := \exp_1 r$.

For a meromorphic function f in complex plane $\mathbb{C}$, the order of growth is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. The exponents of convergence of sequence of the zeros and distinct zeros of f are respectively defined by

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}, \quad \overline{\lambda}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

*Corresponding author

Email addresses: amine.zemirni@univ-mosta.dz (Mohamed Amine Zemirni), benharrat.belaidi@univ-mosta.dz (Benharrat Belaïdi)

where $N\left(r, \frac{1}{f}\right)$ $\left(\text{resp. } \overline{N}\left(r, \frac{1}{f}\right)\right)$ is the integrated counting function of zeros (resp. distinct zeros) of $f(z)$ in the disc $\{z : |z| \leq r\}$.

(Juneja *et al.*, 1976, 1977) have investigated some properties of entire functions of $[p,q]$-order and obtained some results about their growth. In order to maintain accordance with general definitions of the entire function f of iterated p-order[1], (Liu *et al.*, 2010) gave a minor modification of the original definition of the $[p,q]$-order given by (Juneja *et al.*, 1976, 1977).

We recall the following definition,

Definition 1.1. (Kinnunen, 1998) Let $p \geq 1$ be an integer. The iterated p-order $\sigma_p(f)$ of a meromorphic function f is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}.$$

Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order, where p,q are positive integers satisfying $p \geq q \geq 1$ or $2 \leq q = p+1$. In order to keep accordance with Definition 1.1, (Li & Cao, 2012; Belaïdi, 2015) have gave a minor modification to the original definition of $[p,q]$-order (e.g. see, (Juneja *et al.*, 1976, 1977)). We recall the following definitions

Definition 1.2. (Belaïdi, 2015; Li & Cao, 2012; Liu *et al.*, 2010) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order is defined by

$$\sigma_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

It is easy to see that $0 \leq \sigma_{[p,q]}(f) \leq +\infty$. If $f(z)$ is rational, then $\sigma_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By Definition 1.2, we note that $\sigma_{[1,1]}(f) = \sigma(f)$ (order of growth), $\sigma_{[2,1]}(f) = \sigma_2(f)$ (hyper-order), $\sigma_{[1,2]}(f) = \sigma_{\log}(f)$ (logarithmic order) and $\sigma_{[p,1]}(f) = \sigma_p(f)$ (iterated p-order).

Definition 1.3. (Belaïdi, 2015; Li & Cao, 2012) Let $p \geq q \geq 1$ or $2 \leq q = p+1$ be integers. The $[p,q]$ convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.$$

Similarly, the $[p,q]$ convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r}.$$

[1] see (Kinnunen, 1998), for the definition of the iterated p-order.

We recall also the following definitions. The linear measure of a set $E \subset (0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t)dt$$

and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined as

$$\ell m(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt,$$

where $\chi_H(t)$ is the characteristic function of the set H. The upper density of a set $E \subset (0, +\infty)$ is defined by

$$\overline{\text{dens}}\,(E) = \limsup_{r\to+\infty} \frac{m\,(E \cap [0, r])}{r}.$$

The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log\text{dens}}\,(F) = \limsup_{r\to+\infty} \frac{\ell m\,(F \cap [1, r])}{\log r}.$$

Proposition 1.1. (Belaïdi, 2015) *For all $H \subset [1, +\infty)$ the following statements hold :*
(i) If $\ell m\,(H) = \infty$, then $m\,(H) = \infty$,
(ii) if $\overline{\text{dens}}\,(H) > 0$, then $m\,(H) = \infty$,
(iii) if $\overline{\log\text{dens}}\,(H) > 0$, then $\ell m\,(H) = \infty$.

For $a \in \overline{\mathbb{C}}$, the deficiency of a with respect to a meromorphic function f is defined by

$$\delta\,(a, f) = \liminf_{r\to+\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T\,(r, f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T\,(r, f)}.$$

Consider the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0. \quad (1.1)$$

(Liu *et al.*, 2010) studied the growth of solutions of the homogeneous differential equation (1.1) with coefficients that are entire functions of finite $[p, q]$-order and obtained following result

Theorem 1.1. (Liu *et al.*, 2010) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying* $\max\left\{\sigma_{[p,q]}(A_j) : j \neq s\right\} < \sigma_{[p,q]}(A_s) < \infty$. *Then every solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s)$. Furthermore, at least one solution of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.*

Theorem 1.2. (Liu *et al.*, 2010) *Let $A_0, A_1, ..., A_{k-1}$ be entire functions, and let $s \in \{0, ..., k-1\}$ be the largest index for which $\sigma_{[p,q]}(A_s) = \max_{0\leq j\leq k-1} \sigma_{[p,q]}(A_j)$. Then there are at least $k - s$ linearly independent solutions $f(z)$ of (1.1) such that $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$. Moreover, all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f) \leq \rho$ if and only if $\sigma_{[p,q]}(A_j) \leq \rho$ for all $j = 0, 1, ..., k-1$.*

Theorem 1.3. (Liu *et al.*, 2010) *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \alpha.$$

Suppose that there exists a positive constant β satisfying $\beta < \alpha$ such that any given ε $(0 < \varepsilon < \alpha - \beta)$, we have

$$|A_0(z)| \ge \exp_{p+1}\left\{(\alpha - \varepsilon)\log_q r\right\}$$

and

$$\left|A_j(z)\right| \le \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \ldots, k-1)$$

for $z \in H$. Then, every solution $f \not\equiv 0$ of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \alpha$.

Recently, (Belaïdi, 2015) has obtained the following results which generalize and improve Theorem 1.3 and also improve some results due to (Li & Cao, 2012).

Theorem 1.4. (Belaïdi, 2015) *Let H be a set of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \le \beta < \alpha$ such that

$$|A_0(z)| \ge \exp_p\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \tag{1.2}$$

and

$$\left|A_j(z)\right| \le \exp_p\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad (j = 1, \cdots, k-1) \tag{1.3}$$

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold :
(i) If $p \ge q \ge 2$ or $3 \le q = p+1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $p = 1, q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \ge \rho$.

Theorem 1.5. (Belaïdi, 2015) *Let H be a set of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two positive constants α and β such that, we have

$$m(r, A_0) \ge \exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \tag{1.4}$$

and

$$m(r, A_j) \le \exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad (j = 1, \cdots, k-1) \tag{1.5}$$

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold :
(i) If $p \ge q \ge 2$ and $0 \le \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $3 \le q = p+1$, $0 \le \beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) = \rho$.
(iii) If $p = 1, q = 2$, $0 \le (k-1)\beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \ge \rho$.

2. Main results

Now, a natural question is whether somewhat similar results to Theorem 1.4 and Theorem 1.5 could be obtained for the differential equation (1.1), where $A_j(z)$ $(j = 0, 1, \cdots, k)$ are entire functions and the dominant coefficient is some $A_s(z)$ $(0 \le s \le k-1)$ instead of $A_0(z)$? The main purpose of this article is to answer the above question and improving and generalizing the previous results.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}}\,\{|z| : z \in H\} > 0$. Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \le \beta < \alpha$ and let $s \in \{0, ..., k-1\}$ be an integer for which

$$|A_s(z)| \ge \exp_p\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right), \; 0 \le s \le k-1 \tag{2.1}$$

and

$$\left|A_j(z)\right| \le \exp_p\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad j \ne s, \tag{2.2}$$

as $|z| = r \to +\infty$, $z \in H$. Then,
(i) If $p \ge q \ge 1$, then every polynomial solution $f \not\equiv 0$ of equation (1.1) is of $\deg f \le s-1$ $(s \ge 1)$ and every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $2 \le q = p+1, \rho > 1$, then every polynomial solution $f \not\equiv 0$ of equation (1.1) is of $\deg f \le s-1$ $(s \ge 1)$ and every transcendental solution f of equation (1.1) satisfies $\rho \le \sigma_{[p+1,p+1]}(f) \le \rho + 1$.

Corollary 2.1. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}}\,\{|z| : z \in H\} > 0$. Let $F(z) \not\equiv 0, A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions. Suppose that $H, A_j(z)$ $(j = 0, 1, \ldots, k-1)$ satisfy the hypotheses in Theorem 2.1. Consider the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F. \tag{2.3}$$

(i) Let $p \ge q \ge 1$, if $\sigma_{[p+1,q]}(F) \le \rho$, then every transcendental solution f of equation (2.3) satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,q]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $2 \le q = p+1$ and $\rho > 1$, if $\sigma_{[p+1,p+1]}(F) \le \rho$, then every transcendental solution f of equation (2.3) satisfies $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,p+1]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Theorem 2.2. *Let H be a set of complex numbers satisfying $\overline{\log \text{dens}}\,\{|z| : z \in H\} > 0$. Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α *and* β *satisfying* $0 \le \beta < \alpha$ *and let* $s \in \{0, ..., k-1\}$ *be an integer for which*

$$m(r, A_s) \ge \exp_{p-1}\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right),\ 0 \le s \le k-1 \tag{2.4}$$

and

$$m\left(r, A_j\right) \le \exp_{p-1}\left(\beta \left[\log_{q-1} r\right]^{\rho}\right),\ \ j \ne s, \tag{2.5}$$

as $|z| = r \to +\infty$, $z \in H$. *Then the following statements hold :*
(*i*) *If* $p \ge q \ge 1$ *and* $0 \le \beta < \alpha$, *then every polynomial solution* $f \not\equiv 0$ *of* (1.1) *is of* $\deg f \le s-1$ $(s \ge 1)$, *and every transcendental solution* f *satisfies* $\sigma_{[p,q]}(f) \ge \rho \ge \sigma_{[p+1,q]}(f)$.
(*ii*) *If* $2 \le q = p+1$ *and* $0 \le (k-1)\beta < \alpha$, *then every polynomial solution* $f \not\equiv 0$ *of* (1.1) *is of* $\deg f \le s-1$ $(s \ge 1)$, *and every transcendental solution* f *satisfies* $\rho \le \sigma_{[p,p+1]}(f)$ *and* $\sigma_{[p+1,p+1]}(f) \le \rho + 1$.

3. Some preliminary lemmas

Lemma 3.1. (Gundersen, 1988) *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given constant. Then there exists a set* $E_1 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* $s, j (0 \le s < j)$, *such that for all* z *satisfying* $|z| = r \notin E_1 \cup [0,1]$

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{j-s}.$$

Lemma 3.2. (Gundersen, 1988) *Let* f *be a meromorphic function, and let* j *be a given positive integer, and let* $\alpha > 1$ *be a real constant. Then there exists a constant* $R > 0$ *such that for all* $r \ge R$ *we have*

$$T\left(r, f^{(j)}\right) \le (j+2)\, T(\alpha r, f).$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu_f(r)$ be the maximum term, i.e., $\mu_f(r) = \max\{|a_n| r^n;$ $n = 0, 1, \cdots\}$, and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r) = \max\{m;\ \mu_f(r) = |a_m| r^m\}$.

Lemma 3.3. (Hayman, 1974) *Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r, f)$. *Then for all* $|z| = r$ *outside a set* E_2 *of* r *of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)),\ \ j \in \mathbb{N},$$

where $\nu_f(r)$ *is the central index of* $f(z)$.

Lemma 3.4. (Juneja *et al.*, 1976) *Let* $f(z)$ *be an entire function of* $[p, q]$*-order, and let* $\nu_f(r)$ *be the central index of* $f(z)$. *Then*

$$\sigma_{[p,q]}(f) = \limsup_{r\to+\infty}\frac{\log_p \nu_f(r)}{\log_q r}.$$

Lemma 3.5. *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions of finite $[p,q]$-order. Then,*
(i) If $p \geq q \geq 1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,q]}(f) \leq \max\left\{\sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k-1\right\}.$$

(ii) If $2 \leq q = p+1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,p+1]}(f) \leq \max\left\{\sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1\right\} + 1.$$

Proof. We prove only (ii). For the proof of (i) see (Liu *et al.*, 2010). Let $f \not\equiv 0$ be a solution of equation (1.1). By (1.1), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right| + |A_{k-2}|\left|\frac{f^{(k-2)}}{f}\right| + \cdots + |A_1|\left|\frac{f'}{f}\right| + |A_0|. \tag{3.1}$$

Set $\max\left\{\sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1\right\} = \rho$. For any given $\varepsilon > 0$, when r is sufficiently large, we have

$$\left|A_j(z)\right| \leq \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right), \; j = 0, 1, ..., k-1. \tag{3.2}$$

By Lemma 3.3, there exists a set $E_2 \subset [1, +\infty)$ with logarithmic measure $\ell m E_2 < \infty$, we can choose z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$, such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \; j = 1, ..., k \tag{3.3}$$

holds. Substituting (3.2) and (3.3) into (3.1), we obtain

$$\left(\frac{\nu_f(r)}{|z|}\right)^k |1 + o(1)| \leq k \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right)\left(\frac{\nu_f(r)}{|z|}\right)^{k-1} |1 + o(1)|, \tag{3.4}$$

where z satisfies $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r,f)$. By (3.4), we get

$$\nu_f(r)|1 + o(1)| \leq kr|1 + o(1)| \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right). \tag{3.5}$$

So, from (3.5), we obtain

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_f(r)}{\log_{p+1} r} \leq \rho + 1 + \varepsilon. \tag{3.6}$$

Since $\varepsilon > 0$ is arbitrary, by (3.6) and Lemma 3.4 we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. □

Remark. Lemma 3.5 (ii) has been proved for $p = 1$ and $q = 2$ by (Cao *et al.*, 2013).

Lemma 3.6. (Chen & Shon, 2004) *Let $f(z)$ be a transcendental entire function. Then there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0,1] \cup E_3$ and $|f(z)| = M(r,f)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s, \quad s \in \mathbb{N}.$$

Lemma 3.7. *Let f be a transcendental meromorphic function of finite $[p,q]$-order. Then the following statements hold :*
(i) If $p \geq q \geq 1$, then $\rho_{[p,q]}(f') = \rho_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$.

Proof. We prove only (ii). For the proof of (i) see (Belaïdi, 2015). Let f be a transcendental meromorphic function of finite $[p,q]$-order. By lemma of logarithmic derivative [2], we have

$$T(r,f') = m(r,f') + N(r,f') \leq m(r,f) + m\left(r,\frac{f'}{f}\right) + 2N(r,f)$$

$$\leq 2T(r,f) + m\left(r,\frac{f'}{f}\right) \leq 2T(r,f) + O(\log T(r,f) + \log r) \tag{3.7}$$

holds outside of an exceptional set $E_4 \subset (0,+\infty)$ with finite linear measure. By (3.7), it is easy to see that $\rho_{[p,p+1]}(f') \leq \rho_{[p,p+1]}(f)$ if $2 \leq q = p+1$. On the other hand, by (Chuang, 1951), ((Yang & Yi, 2003), p. 35), we have for $r \to +\infty$

$$T(r,f) < O(T(2r,f') + \log r). \tag{3.8}$$

Hence, by using (3.8) we obtain $\rho_{[p,p+1]}(f) \leq \rho_{[p,p+1]}(f')$ if $2 \leq q = p+1$. Thus, $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$ if $2 \leq q = p+1$. □

Remark. Lemma 3.7 (ii) has been proved for $p = 1$ and $q = 2$ by (Chern, 2006).

Lemma 3.8. (Belaïdi, 2015) *Let A_j $(j = 0,1,\ldots,k-1)$, $F \not\equiv 0$ be meromorphic functions. Then the following statements hold :*
(i) If $p \geq q \geq 1$, then every every meromorphic solution f of equation (2.3) such that

$$\max\left\{\sigma_{[p,q]}\left(A_j\right); \sigma_{[p,q]}(F) : j = 0,1,\ldots,k-1\right\} < \sigma_{[p,q]}(f)$$

satisfies $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then every meromorphic solution f of equation (2.3) such that

$$\max\left\{1; \sigma_{[p,q]}\left(A_j\right); \sigma_{[p,q]}(F) : j = 0,1,\ldots,k-1\right\} < \sigma_{[p,q]}(f)$$

satisfies $\overline{\lambda}_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

4. Proofs of main results

Proof of Theorem 2.1 It's should be noticed that the case $s = 0$ returns to Theorem 1.4. So, we will prove Theorem 2.1 in case $s > 0$.

[2] see, (Hayman, 1964; Yang & Yi, 2003).

(i) Case : $p \geq q \geq 1$. Suppose that $f \not\equiv 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ and suppose that $n \geq s$, i.e., $f^{(s)}(z) \not\equiv 0$. Then from (1.1), we have

$$|A_s| A_n^s |a_n| r^{n-s} (1 + o(1)) \leq |A_s| \left|f^{(s)}(z)\right| \leq \sum_{\substack{j=0 \\ j\neq s}}^{k} \left|A_j\right| \left|f^{(j)}(z)\right| \leq \sum_{\substack{j=0 \\ j\neq s}}^{k} \left|A_j\right| A_n^j |a_n| r^{n-j} (1 + o(1)), \quad (4.1)$$

where $A_k \equiv 1$ and $A_n^j = n(n-1)\cdots(n-j+1)$. It follows from (4.1), (2.1) and (2.2) that

$$\exp_p\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right) r^{-s} \leq O\left(\exp_p\left(\beta \left[\log_{q-1} r\right]^{\rho}\right)\right). \quad (4.2)$$

Since $\alpha > \beta$, we see that (4.2) is a contradiction as $r \to +\infty$. Then $\deg f \leq s - 1$.
Now, suppose that f is a transcendental solution of the equation (1.1). From the conditions of Theorem 2.1, there is a set H of complex numbers satisfying $\overline{\log \text{dens}}\{|z| : z \in H\} > 0$, and there exists A_s $(0 \leq s \leq k-1, k \geq 2)$ such that for all $z \in H$ we have (2.1) and (2.2) as $|z| \to +\infty$. Set $H_1 = \{|z| : z \in H\}$, since $\overline{\log \text{dens}}\{|z| : z \in H\} > 0$, then H_1 is a set with $\ell m (H_1) = \infty$.

From (1.1), we have

$$\begin{aligned} -A_s &= \frac{f}{f^{(s)}}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f}\right. \\ &\quad \left. + A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \end{aligned} \quad (4.3)$$

By Lemma 3.1, there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T(2r, f)\right]^{j+1}, \quad j = 1, 2, \ldots, k-1. \quad (4.4)$$

By Lemma 3.6, there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s. \quad (4.5)$$

It follows from (4.3) – (4.5), (2.1) and (2.2) that

$$\exp_p\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right) \leq 2kB\left[T(2r, f)\right]^{k+1} r^s \exp_p\left(\beta \left[\log_{q-1} r\right]^{\rho}\right). \quad (4.6)$$

for all $|z| = r \in H_1 \backslash ([0, 1] \cup E_1 \cup E_3)$ and $|f(z)| = M(r, f)$. Then by (4.6), we obtain $\rho \leq \sigma_{[p+1,q]}(f)$. On the other hand, by Lemma 3.5 (i), we have $\sigma_{[p+1,q]}(f) \leq \rho$. Hence, every transcendental solution f of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) Case : $2 \leq q = p + 1, \rho > 1$. Suppose that $f \not\equiv 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ and suppose that $n \geq s$, i.e. $f^{(s)}(z) \not\equiv 0$. From (4.2), we have

$$\exp_p\left(\alpha \left[\log_p r\right]^{\rho}\right) r^{-s} \leq O\left(\exp_p\left(\beta \left[\log_p r\right]^{\rho}\right)\right). \quad (4.7)$$

Since $\alpha > \beta$, we see that (4.7) is a contradiction as $r \to +\infty$. Then $\deg f \le s-1$.
Now, suppose that f is a transcendental. Then from (4.6) we have

$$\exp_p\left(\alpha\left[\log_p r\right]^{\rho}\right) \le 2kBr^s\left[T(2r,f)\right]^{k+1}\exp_p\left(\beta\left[\log_p r\right]^{\rho}\right) \tag{4.8}$$

holds for all z satisfying $|z| = r \in H_1 \backslash ([0,1] \cup E_1 \cup E_3)$, as $r \to +\infty$. By (4.8), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) \ge \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]}(f) \le \rho + 1$, thus $\rho \le \sigma_{[p+1,p+1]}(f) \le \rho + 1$.

Proof of Corollary 2.1 (i) (a) Let $p \ge q \ge 1$. Let f be a transcendental solution of the equation (2.3) and $\{f_1, f_2, \ldots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (2.3). By Theorem 2.1, we know that for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]}\left(f_j\right) = \rho.$$

Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_k(z) f_k(z), \tag{4.9}$$

where $B_1, B_2, \ldots, B_k$ are suitable meromorphic functions satisfying

$$B_j' = F \cdot G_j(f_1, f_2, \ldots, f_k) \cdot (W(f_1, f_2, \ldots, f_k))^{-1}, \quad j = 1, 2, \ldots, k, \tag{4.10}$$

where $G_j(f_1, f_2, \ldots, f_k)$ are differential polynomials in $f_1, f_2, \ldots, f_k$ and their derivatives with constant coefficients, thus

$$\sigma_{[p+1,q]}\left(G_j\right) \le \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]}\left(f_j\right) = \rho, \quad j = 1, 2, \ldots, k. \tag{4.11}$$

Since the Wronskian $W(f_1, f_2, \ldots, f_k)$ is a differential polynomial in $f_1, f_2, \ldots, f_k$, it is easy to deduce also that

$$\sigma_{[p+1,q]}(W) \le \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]}\left(f_j\right) = \rho. \tag{4.12}$$

Since $\sigma_{[p+1,q]}(F) \le \rho$, then by using Lemma 3.7 (i) and (4.10) – (4.12) we get for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]}\left(B_j\right) = \sigma_{[p+1,q]}\left(B_j'\right) \le \max\left\{\sigma_{[p+1,q]}(F); \rho\right\} = \rho. \tag{4.13}$$

Then by (4.9) and (4.13), we obtain

$$\sigma_{[p+1,q]}(f) \le \max_{j=1,2,\ldots,k}\left\{\sigma_{[p+1,q]}\left(f_j\right); \sigma_{[p+1,q]}\left(B_j\right)\right\} = \rho. \tag{4.14}$$

Now, we assert that every transcendental solution f of (2.3) satisfies $\sigma_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. In fact, if f^* is another transcendental solution with $\sigma_{[p+1,q]}(f^*) < \rho$ of (2.3), then $\sigma_{[p+1,q]}(f_0 - f^*) < \rho$, but $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1), and this is a contradiction with the results of Theorem 2.1. Then, $\sigma_{[p+1,q]}(f) = \rho$ holds for every transcendental solution f of (2.3) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. By Lemma 3.8, every transcendental solution f of (2.3) with $\sigma_{[p+1,q]}(f) = \rho$ satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 3.7 (i), (4.11) and (4.12), we have from (4.10) for $j = 1, 2, \cdots, k$

$$\rho_{[p+1,q]}\left(B_j\right) = \rho_{[p+1,q]}\left(B'_j\right)$$

$$\leq \max\left\{\rho_{[p+1,q]}(F), \rho_{[p+1,q]}\left(f_j\right) : j = 1, 2, \cdots, k\right\} = \rho_{[p+1,q]}(F). \tag{4.15}$$

Then from (4.15) and (4.9), we get

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}\left(f_j\right), \rho_{[p+1,q]}\left(B_j\right) : j = 1, 2, \cdots, k\right\} \leq \rho_{[p+1,q]}(F). \tag{4.16}$$

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (2.3) that a simple consideration of $[p, q]-$order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (4.16) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) For $2 \leq q = p+1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold.

Proof of Theorem 2.2 Suppose that $f \not\equiv 0$ is a solution of the equation (1.1). From the conditions the Theorem 2.2, there is a set H of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$, and there exists A_s $(0 \leq s \leq k-1, k \geq 2)$ such that for all $z \in H$ we have (2.4) and (2.5) as $|z| \to +\infty$. Set $H_1 = \{|z| : z \in H\}$, since $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ then H_1 is a set with $\ell m(H_1) = \infty$.
(i) Let $p \geq q \geq 1$ and $0 \leq \beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$, implies that $\frac{f^{(j)}}{f^{(s)}}$ $(j = 0, 1, \ldots, k)$ is a rational, hence $T\left(r, \frac{f^{(j)}}{f^{(s)}}\right) = O(\log r)$ for r sufficiently large. From (4.3) we have

$$T(r, A_s) \leq \sum_{\substack{j=0 \\ j\neq s}}^{k-1} T\left(r, A_j\right) + O(\log r). \tag{4.17}$$

It follows by (4.17), (2.4) and (2.5) that

$$\exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \leq O\left(\exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right)\right) \tag{4.18}$$

which is a contradiction since $\alpha > \beta$ and $r \to +\infty$. Then, every polynomial solution $f \not\equiv 0$ of (1.1) is of $\deg f \leq s-1$.
Now, suppose that f is a transcendental solution of (1.1). By using the first main theorem of Nevanlinna and properties of the characteristic function, we obtain from (4.3)

$$\begin{aligned} T(r, A_s) \;\; &\leq \;\; T\left(r, f^{(k)}\right) + kT\left(r, f^{(s)}\right) + \sum_{j=0,\, j\neq s}^{k-1} T\left(r, f^{(j)}\right) \\ &\quad + \sum_{j=0,\, j\neq s}^{k-1} T\left(r, A_j\right) + O(1). \end{aligned} \tag{4.19}$$

By Lemma 3.2, there exists a constant $R > 0$ such that for all z satisfying $|z| = r > R$, we rewrite (4.19) as follows

$$\begin{aligned} m(r, A_s) &= T(r, A_s) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0,\ j\neq s}^{k-1} T\left(r, A_j\right) + O(1) \\ &= \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0,\ j\neq s}^{k-1} m\left(r, A_j\right) + O(1). \end{aligned} \tag{4.20}$$

It follows by (4.20), (2.4) and (2.5) that

$$\begin{aligned} \exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) \\ &\quad + (k-1)\exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right) + O(1) \end{aligned} \tag{4.21}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.21), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,q]}(f) \geq \rho$, and by Lemma 3.5 (i), we have $\sigma_{[p+1,q]}(f) \leq \rho$. Thus, $\sigma_{[p,q]}(f) \geq \rho \geq \sigma_{[p+1,q]}(f)$.

(ii) Let $2 \leq q = p+1$ and $0 \leq (k-1)\beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$. By the same reasoning as in the proof in case (i), it is clear that $f(z)$ is a polynomial with $\deg f \leq s - 1$.

Now, suppose that f is a transcendental solution of (1.1). Then by (4.21)

$$\begin{aligned} \exp_{p-1}\left(\alpha\left[\log_{p} r\right]^{\rho}\right) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) \\ &\quad + (k-1)\exp_{p-1}\left(\beta\left[\log_{p} r\right]^{\rho}\right) + O(1) \end{aligned} \tag{4.22}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.22), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. Hence, $\rho \leq \sigma_{[p,p+1]}(f)$ and $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

Acknowledgements

The authors are grateful to the referee for his/her valuable comments which lead to the improvement of this paper. This paper is dedicated to the memory of Omar Bouhenna.

References

Belaïdi, B. (2015). On the [p, q]-order of meromorphic solutions of linear differential equations. *Acta Univ. M. Belii Ser. Math.* **23**, 37–49.

Cao, T. B., K. Liu and J. Wang (2013). On the growth of solutions of complex differentials equations with entire coefficients of finite logarithmic order. *Math. Rep. (Bucur.)* **15**(3), 249–269.

Chen, Z. X. and K. H. Shon (2004). On the growth of solutions of a class of higher order differential equations. *Acta Math. Sci. Ser. B Engl. Ed.* **24**(1), 52–60.

Chern, Peter T. Y. (2006). On meromorphic functions with finite logarithmic order. *Trans. Amer. Math. Soc.* **358**(2), 473–489.

Chuang, C. T. (1951). Sur la comparaison de la croissance d'une fonction méromorphe et de celle de sa dérivée. *Bull. Sci. Math.* **75**(2), 171–190.

Gundersen, G. G. (1988). Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. *J. London Math. Soc.* **37**(2), 88–104.

Hayman, W. K. (1964). *Meromorphic functions*. Vol. 78. Oxford Mathematical Monographs Clarendon Press, Oxford.

Hayman, W. K. (1974). The local growth of power series : A survey of the Wiman–Valiron method. *Canad. Math. Bull.* **17**(3), 317–358.

Juneja, O. P., G. P. Kapoor and S. K. Bajpai (1976). On the $[p, q]$-order and lower $[p, q]$-order of an entire function. *J. Reine Angew. Math.* **282**, 53–67.

Juneja, O. P., G. P. Kapoor and S. K. Bajpai (1977). On the $[p, q]$-type and lower $[p, q]$-type of an entire function. *J. Reine Angew. Math.* **290**, 180–190.

Kinnunen, L. (1998). Linear differential equations with solutions of fnite iterated order. *Southeast Asian Bull. Math.* **22**(4), 385–405.

Laine, I. (1993). *Nevanlinna theory and complex differential equations*. Vol. 15. Walter de Gruyter & Co., Berlin.

Li, L. M. and T. B. Cao (2012). Solutions for linear differential equations with meromorphic coefficients of $[p, q]$-order in the plane. *Electron. J. Differential Equations* **2012**(195), 1–15.

Liu, J., J. Tu and L. Z. Shi (2010). Linear differential equations with entire coefficients of $[p, q]$-order in the complex plane. *J. Math. Anal. Appl.* **372**(1), 55–67.

Yang, C. C. and H. X. Yi (2003). *Uniqueness theory of meromorphic functions*. Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht.

Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 27 – 38

Fuzzy Differential Superordination

Waggas Galib Atshan*, Khudair O. Hussain

Department of Mathematics College of Computer Science and Information Technology University of Al-Qadisiyah, Diwaniya, Iraq

Abstract

S.S. Miller and P.T. Mocanu in (Miller & Mocanu, 2003) the notion of differential superordination as a dual concept of differential subordination (Miller & Mocanu, 2000) . In (Oros & Oros, 2011) The authors define the notion of fuzzy subordination, in (Oros & Oros, 2012*b*) they define the notion of fuzzy differential subordination and in (Oros & Oros, Jun2012*a*) they determine conditions for a function to be a dominant of the fuzzy differential subordination and they also gave the best dominant. In this paper, we introduced the concept of fuzzy differential superordination and we set conditions for a function to be subordinant of fuzzy differential superordination and we also give the best subordinant.

Keywords: fuzzy set, fuzzy differential subordination, fuzzy differential superordination , fuzzy subordinant, fuzzy best subordinant, sandwich theorem.
2010 MSC: 30C45.

1. Introduction and Preliminaries

The general form of differential superordination method can be presented as follows: Let Ω and Δ be any set in C, let p be analytic in the unit disk U and let $\varphi(r, s, t; z) : C^3 \times U \longrightarrow C$. The problem is to study the following implication:

$$\Omega \subset \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow \Delta \subset p(z). \tag{1.1}$$

If Δ is a simply connected domain containing the point a and $\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case, relation (1.1) can be rewritten as

$$\Omega \subset \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.2}$$

*Corresponding author
Email addresses: Waggas.galib@qu.edu.iq; waggashnd@gmail.com (Waggas Galib Atshan), khudair.hussain@qu.edu.iq; khudair_o.hussain@yahoo.com (Khudair O. Hussain)

If Ω is also a simply connected domain and $\Omega \neq C$,then there is conformal mapping h of U onto Ω such that $h(0) = \varphi(a, 0, 0; 0)$.If in addition, the function $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then (1.2) can be rewritten as

$$h(z) \prec \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) \Rightarrow q(z) \prec p(z). \tag{1.3}$$

For further details on the differential superordination method, the valuable monograph (Miller & Mocanu, 2003) can be seen.
Let U denote the unit disc of the complex plane

$$U = \{z \in C : |z| < 1\}, \overline{U} = \{z \in C : |z| \leq 1\}$$

and $H(U)$ denote the class of analytic function in U.For $a \in C$ and $n \in N$, we denote by

$$H[a, n] = \left\{f \in H(U) : f(z) = a + a_{(n+1)}z^{(n+1)} + \ldots, z \in U\right\},$$
$$A_n = \left\{f \in H(U) : f(z) = z + a_{(n+1)}z^{(n+1)} + \ldots, z \in U\right\}$$

with $A_1 = A$. Let $S = \{f \in A : f$ univalent in $U\}$ be the class of analytic and univalent functions in the open unit disk U, with condition $f(0) = 0$, $f^{'}(0) = 1$, that is the analytic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \ldots, z \in U.$$

Denote by
$S^* = \left\{f \in A : Re\left(\frac{zf^{'}(z)}{f(z)}\right) > 0, z \in U\right\}$, the class of normalized starlike functions in U, and
$C = \left\{f \in A : Re\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) > 0, z \in U\right\}$, the class of normalized convex functions in U, and
$K = \left\{f \in A : Re\left(\frac{f^{'}(z)}{g^{'}(z)}\right) > 0, g(z) \in C, z \in U\right\}$, the class of normalized close to convex functions in U.

In order to introduce the notation of fuzzy differential superordination, we use the following definitions and lemmas:

Definition 1.1 (Miller & Mocanu, 2000) We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \left\{\zeta \in \partial U : \lim_{\zeta \to \infty} q(z) = \infty\right\}$, and are such that $q^{'}(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.The set $E(q)$ is called exemption set.

Definition 1.2 (Zadeh, 1965) Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:
A pair(A, F_A), where

$$F_A : X \to [0, 1] \quad \text{and} \quad A = \{x \in X : 0 < F_A \leq 1\} = supp(A, F_A),$$

is called fuzzy subset. The function F_A is called membership function of the fuzzy set (A, F_A).

Definition 1.3 (Zadeh, 1965) Let two fuzzy subsets of X,(M, F_M) and (N, F_N). We say that fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$,$x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fussy subset (N, F_N) if and only if $F_M(x) \leq$

$F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Definition 1.4 (Oros & Oros, 2011) Let $D \subseteq C, z_0 \in D$ be a fixed point, and let the functions $f, g \in H(D)$. The function f is said to be fuzzy subordinate to g, written $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1. $f(z_0) = g(z_0)$,
2. $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), z \in U$.

Definition 1.5 (Oros & Oros, Jun2012*a*) A function $L(z,t)$,$z \in U$,$t \geq 0$, is a fuzzy subordination chain if $L(.,t)$ is analytic and univalent in U. For all $t \geq 0$,$L(z,t)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and

$$F_{L[U\times[0,\infty)]}(L(z,t_1)) \leq F_{L[U\times[0,\infty)]}(L(z,t_2)), \qquad t_1 \leq t_2.$$

Remark (Oros & Oros, 2011) Let the functions $f, g \in H(U)$ and g is an univalent functionthen $f \prec g$ if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. From here if g is an univalent function, then $f \prec_F g$ if and only if $f \prec g$.

Lemma 1.6 (Miller & Mocanu, 2000) Let $q \in Q(a)$ and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, be analytic in U,$q(z) \neq a$ and $n \geq 1$, if q is not subordinate to p, then there exist points $z_0 = r_0 e^i \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$ for which $q(U_{r_0}) \subset p(U)$,

1. $q(z_0) = p(\zeta_0)$,
2. $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$, and
3. $Re\left(\frac{z_0 q''(z_0)}{q'(z_0)} + 1\right) \geq mRe\left(\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1\right)$.

Lemma 1.7 (Oros & Oros, 2012*b*) Let h be a convex function with $h(0) = a$, and let $\gamma \in C^*$ be a complex number with $Re(\gamma) \geq 0$. If $p \in H[a,n]$ with $p(0) = a$ and $\psi : C^2 \times U \to C$, $\psi(p(z), zp'(z)) = p(z) + \frac{1}{\gamma} zp'(z)$, is analytic in U, then

$$F_{\psi(C^2\times U)}[p(z) + \frac{1}{\gamma} zp'(z)] \leq F_{h(U)}h(z), implies, F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n-1}} dt.$$

The function q is convex and is the fuzzy best (a, n) dominant.

Lemma 1.8 (Oros & Oros, 2012*b*) Let h be starlike in U, with $h(0) = 0$. If $p \in H[0, 1] \cap Q$ is univalent in U, then $zp'(z) \prec_F h(z)$, implies $p(z) \prec_F q(z), z \in U$ where q is given by

$$q(z) = \int_0^z h(t) t^{-1} dt.$$

The function q is convex and is the fuzzy best dominant.

Lemma 1.9 (Pascu, 2006) If $L_\gamma : A \to A$ is the integral operator defined by $L_\gamma[f] = F$, given by $L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z h(t) t^{\gamma-1} dt$ and $Re(\gamma) > 0$ then

1. $L_\gamma[S^*] \subset S^*$
2. $L_\gamma[K] \subset K$
3. $L_\gamma[C] \subset C$.

Lemma 1.10 (Oros & Oros, 2012b) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + ...$,with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is fuzzy subordination chain if and only if

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0, z \in U. \tag{1.4}$$

2. Main Results

Let

$$\Omega = supp(\Omega, F_\Omega) = \{z \in C : 0 < F_\Omega(z) \leq 1\},$$

$$\Delta = supp(\Delta, F_\Delta) = \{z \in C : 0 < F_\Delta(z) \leq 1\},$$

$$p(U) = supp(p(U), F_{p(U)}) = \{z \in C : 0 < F_{p(U)}(z) \leq 1\}$$

and

$$\varphi(C^3 \times U) = supp(\varphi(C^3 \times U), F_{\varphi(C^3\times U)}) = \{\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z\}$$

.

Definition 2.1 Let Ω be a set in C and $q \in H[a,n]$ with$q^{'}(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$F_{\varphi(C^3\times U)}(\varphi(r,s,t);\zeta) \leq F_\Omega(z).i.e.F_\Omega(\varphi(r,s,t;\zeta)) > 0, \tag{2.1}$$

Whenever

$$r = q(z), s = \frac{zq^{'}(z)}{m}, Re\left(\frac{t}{s} + 1\right) \geq \frac{1}{m} Re\left(\frac{zq^{''}(z)}{q^{'}(z)} + 1\right),$$

where $\zeta \in \partial U$, $z \in U$ and $m \geq n \geq 1$. When $n = 1$ we write $\Phi_1[\Omega, q]$ as $\Phi[\Omega, q]$.
In the special case when h is an analytic mapping of U onto $\Omega \neq C$, we denote this class $\Phi_n[h(U), q]$ by $\Phi_n[h, q]$.
If $\varphi : C^2 \times U \to C$ and $q \in H[a,n]$, then the admissibility condition (2.1) reduces to

$$F_\Omega(\varphi(q(z), (zq^{'}(z))/m; \zeta)) > 0, \quad \text{when} \quad z \in U, \zeta \in \partial U \quad \text{and} \quad m \geq n \geq 1$$

.
If $\varphi : C \times U \to C$, then the admissibility condition (2.1) reduces to $F_\Omega(\varphi(q(z); \zeta)) > 0$, when $z \in U, \zeta \in \partial U$.
Let (Ω, F_Ω) and(Δ, F_Δ) be any fuzzy sets in C, let p be an analytic function in the unit disc U and let $\varphi(r,s,t;z) : C^3 \times U \to C$. To study the following implication:

$$F_\Omega(z) \leq F_{\varphi(C^3\times U)}(\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)), \Rightarrow F_\Delta(z) \leq F_{p(U)}(p(z)). \tag{2.2}$$

There are there distinct cases to consider in analyzing this implication, which we list as the following Problems.

Problem 2.2 Given (Ω, F_{Ω}) and (Δ, F_{Δ}) any fuzzy sets in C, find condonations on the function φ so that (2.2) holds. We call such a φ an admissible function.
Problem 2.3 Given φ and (Ω, F_{Ω}), find (Δ, F_{Δ}) so that (2.2) holds. Furthermore, find the "largest" such Δ.
Problem 2.4 Given φ and (Δ, F_{Δ}), find (Ω, F_{Ω}) so that (2.2) holds. Furthermore find the "smallest" such Ω.
If either (Ω, F_{Ω}) or (Δ, F_{Δ}) in (2.2) is a simply connected domain. Then it may be possible to rephrase (2.2) in terms of fuzzy differential superordination. If p is univalent in U, and if (Δ, F_{Δ}) is simply connected domain with$\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = p(0)$. In this case (2.2) can be rewritten as $F_{\Omega}(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z))$ implies

$$F_{q(U)}(q(z)) \leq F_{p(U)}p(z), z \in U, i.e. q(z) \prec_F p(z). \tag{2.3}$$

If (Ω, F_{Ω}) is also a simply connected domain and $\Omega \neq C$. Then there is a conformal mapping h of U onto Ω such that $h(0) = (p(0), 0, 0; 0)$, if in addition, the function $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then (2.3) can be rewritten as

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z), \quad \Rightarrow q(z) \prec_F p(z). \tag{2.4}$$

This implication also has meaning if h and q are analytic and not necessarily univalent.
Definition 2.5 Let $\varphi : C^3 \times U \to C$ and let h be analytic in U. If p and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ are univalent in U and satisfy the (second-order) fuzzy differential superordination

$$F_{h(U)}h(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z))$$

.i.e.

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z),$$

then p is called a fuzzy solution of the fuzzy differential superordination. An analytic function q is called fuzzy subordinant of the fuzzy differential superordination, or more simply a fuzzy subordination if $q(z) \prec_F p(z)$,$z \in U$, for all p satisfying (2.4).
A univalent fuzzy subordination $\tilde{q}$ that satisfies $q \prec_F \tilde{q}$ for all fuzzy subordinate q of (2.4) is said to be the fuzzy best subordinate of (2.4).
Note that the fuzzy best subordinant is unique uo to a relation of U. In the special case when the set inclusions of (2.2) can be replaced by the fuzzy superordination of (2.4) we can reinterpret the three problem referred to above as follows:
Problem 2.6 Given analytic functions h and q , find a class of admissible functions $\Phi[h, q]$ such that (2.4) holds.
Problem 2.7 Given the fuzzy differential superordination in (2.4), find a fuzzy subordination q, moreover, find the fuzzy best subordinant.
Problem 2.8 Given φ and fuzzy subordinant q, find the largest class of analytic function h such that holds. The next theorem is key results.

Theorem 2.9 Let $\varphi \in \Phi_n[\Omega, q]$ and let $q \in H[a, n]$ .If $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then

$$F_{\Omega}(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)), \quad z \in U \quad \Rightarrow q(z) \prec_F p(z). \tag{2.5}$$

Proof. Form (2.5) and Definition (1.3). We have

$$\Omega \subset (\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)). \tag{2.6}$$

Assume $q(z) \not\prec p(z)$. By Lemma (1.6), there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$.
That satisfy

$$q(z_0) = p(\zeta_0), z_0q^{'}(z_0) = m\zeta_0p^{'}(\zeta_0) \quad \text{and} \quad Re\left(\frac{zq^{''}(z)}{q^{'}(z)} + 1\right) \geq mRe\left(\frac{\zeta_0p^{''}(\zeta_0)}{p^{'}(\zeta_0)} + 1\right).$$

Using these condition with $r = p(\zeta_0), s = \zeta_0p^{'}(\zeta_0), t = \zeta_0^2p^{''}(\zeta_0)$ and $\zeta = \zeta_0$ in definition (2.1) we obtain

$$F_{\varphi(C^3\times U)}(\varphi(p(\zeta_0), \zeta_0p^{'}(\zeta_0), \zeta_0^2p^{''}(\zeta_0); \zeta_0)) \leq F(\zeta_0) \tag{2.7}$$

Since this contradict (2.6) we must have $q(z) \prec_F p(z)$.
We next consider the special situation when h is analytic on U and $h(U) = \Omega \neq C$. In this case, the class $\Phi_n[h(U), q]$ is written as $\Phi_n[h, q]$ and the following result an immediate consequence of Theorem (2.9).
Theorem 2.10 Let $q \in H[a, n]$, let h be analytic in U and let $\varphi \in \Phi_n[h, q]$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{P''}(z); z) \Rightarrow q(z) \prec_F p(z). \tag{2.8}$$

Theorem (2.9) and Theorem (2.10) can only used to obtain fuzzy subordinates of a fuzzy differential superordination of the form (2.6) or (2.8) the following theorem proves the existence of the fuzzy best subordinate of q for certain φ and also provides a method for finding the fuzzy best subordinant.
Theorem 2.11 Let h be analytic in U and let $\varphi : C^3 \times U \to C$ suppose that the differential equation

$$\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) = h(z), \tag{2.9}$$

has a solution $q \in Q(a)$.If $\varphi \in \Phi_n[h, q], p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) \Rightarrow q(z) \prec_F p(z), \tag{2.10}$$

and q is the best subordinate.
Proof. Since $\varphi \in \Phi[h, q]$, by applying Theorem (2.10) we deduce that q is a fuzzy subordinant, of (2.10), since q also satisfies (2.9), it is also a solution of the fuzzy differential superordination (2.10) and therefore all subordinates of (2.10) will be fuzzy subordinant to q. Hence q will be the fuzzy best subordinant of (2.10). From this theorem we see that the problem of finding the fuzzy best subordinant of (2.10) essentially reduces to showing that differential equation 2.9 has a univalent solution and checking that $\varphi \in \Phi_n[h, q]$. The conclusion of the theorem can written in the symmetric form.

$$\varphi(q(z), zq^{'}(z), z^2q^{''}(z); z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z), \Rightarrow q(z) \prec_F p(z).$$

We can simplify Theorems (2.9), (2.10) and (2.11) for the case of first-order fuzzy differential subordination. The following results are immediately obtained by using these theorems and admissibility condition (2.1).

Theorem 2.12 Let $\Omega \subset C, q \in H[a,n], \varphi : C^2 \times U \to C$ and suppose that $F_{\varphi(C^2\times U)}\varphi(q(z), zq^{'}(z); \zeta) \leq F_{\Omega}(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z); z)$ is univalent in U, then

$$F_{\Omega}(z) \leq F_{\varphi(C^2\times U)}\varphi(p(z), zp^{'}(z); z) \Rightarrow F_{q(U)}q(z) \leq F_{p(U)}p(z) \quad i.e. \quad q(z) \prec_F p(z).$$

Theorem 2.13 Let h be univalent in U, $q \in H[a,n], \varphi : C^2 \times U \to C$ and suppose that $F_{\varphi(C^2\times U)}(\varphi(q(z), zq^{'}(z); z)) \leq F_{h(U)}h(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z); z) \Rightarrow q(z) \prec_F p(z). \tag{2.11}$$

Furthermore if $\varphi(p(z), zp^{'}(z); z) = h(z)$ has a univalent solution $q \in Q(a)$ then q is fuzzy best subordinant.

Georgia and Gheorghe (Oros & Oros, 2012*b*) considered the fuzzy subordination

$$F_{\psi(C^2\times U)}[p(z) + \frac{1}{\gamma}zp^{'}(z)] \leq F_{h_2(U)}h_2(z), \tag{2.12}$$

where h_2 is convex function in U,$h_2(0) = a$,$\gamma \neq 0$ and $Re(\gamma) \geq 0$. They showed if $p \in H[a,1]$ satisfies(2.12), then

$$F_{p(U)}p(z) \leq F_{q_2(U)}q_2(z) \leq F_{h_2(U)}h_2(z), z \in U, \tag{2.13}$$

where

$$q_2(z) = \frac{1}{nz^{\gamma}} \int_0^z h_2(t)t^{\gamma-1}dt.$$

The function q is convex and is the fuzzy best dominant of (2.12).

We next prove an analogous result for the corresponding fuzzy differential subordination.

Theorem 2.14 Let $h_1$be convex in U, with $h_1(0) = a$,$\gamma \neq 0$, with $Re(\gamma) \geq 0$, and $p \in H[a,1] \cap Q$ if $p(z) + \frac{1}{\gamma}zp^{'}(z)$ is univalent in U,

$$h_1(z) \prec_F \frac{1}{\gamma}zp^{'}(z), \tag{2.14}$$

and

$$q_1(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h_1(t)t^{\gamma-1}dt, \tag{2.15}$$

then $q_1(z) \prec_F p(z)$, and the function q_1 is convex and is the fuzzy best subordinate.

Proof. If we let ,$\varphi : C^2 \times U \to C, \varphi(r,s) = r + \frac{1}{\gamma}s$, for $r = p(z), s = zp^{'}(z), z \in U$, then relation (2.14) becomes

$$h_1(z) \prec_F \varphi(p(z), zp^{'}(z); z).$$

The integral given by (2.15), with the exception of a different normalization $q(0) = a$ has the form

$$q_1(z) = \frac{\gamma}{nz^\gamma}\int_0^z h(t)t^{\gamma-1}dt = \frac{\gamma}{nz^\gamma}\int_0^z (a + a_n z^n + \ldots)t^{\gamma-1}dt$$
$$= a + \frac{a_1}{\gamma+1}z + \ldots, z \in U,$$

which gives $q \in [a, 1]$, since h is convex and $Re(\gamma) \geq 0$, we deduce form (2) of Lemma (1.10) that q is convex and univalent. A simple colocation shows that q_1 also satisfies the differential equation.

$$q_1(z) + \frac{1}{\gamma}zq_1^{'}(z) = \varphi(p(z), zp^{'}(z)) = h_1(z). \tag{2.16}$$

Since q_1 is the univalent solution of the differential equation (2.16) associated with fuzzy differential subordination (2.14), we can prove that it is the fuzzy best subordinate of (2.14) by applying Theorem (2.13). Without loss of generality, we can assume that h_1 and q_1 are analytic and univalent on $\overline{U}$ and $q_1^{'}(\zeta)$ for $|\zeta| = 1$. If not, then we could replace h_1 with $h_1(\rho z)$ and q_1 with $q_1(\rho z)$, where $0 < \rho < 1$. These new function would then have the desired properties and we would prove the Theorem by using Theorem (2.14) and then letting $\varphi \to 1$ with our assumptions, to apply Theorem (2.13) only need to show that $\varphi \in \Phi[h_1, q_1]$. This is equivalent to showing that

$$\varphi_0 = \varphi(q_1(z) + tzq_1^{'}(z)) = q_1(z) + \frac{1}{\gamma}zq_1^{'}(z) \in h_1(U),$$

for $z \in U$ and $t \in (0, 1]$. Form (2.16) we see that (2.12) is satisfied with p, h replaced by q_1, h_1. Hence, from (2.11) we obtain

$$q_1(z) \prec_F p(z).$$

Since $h_1(U)$ is convex domain and $t \in (0, 1]$. We conclude that $\varphi_0 \in h_1(U)$ which proves that q_1 is the fuzzy best subordinat.

Theorem 2.15 Let $q \in H[a, 1]$,$\varphi : C^2 \times U \to C$ and set $\varphi(q(z), zq^{'}(z)) = h(z)$. If $L(z, t) = \varphi(q(z), tzq^{'}(z))$ is fuzzy subordination chain and $p \in H[a, 1] \cap Q$, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z)) \Rightarrow (z) \prec_F p(z).$$

Furthermore, if $\varphi(q(z), zq^{'}(z)) = h(z)$, has a univalent solution $q \in Q$,then q is the fuzzy best subordinant.

Proof. Since L is a fuzzy subordination chain $L(z, t) \prec_F L(z, 1)$,or equivalently, $(p(z), zp^{'}(z)) \prec_F h(z)$, for all $z \in U$ and $t \in (0, 1]$. Since this implies that (2.11) is satisfied ,we obtain the desired conclusion by applying of this result by again considering the fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp^{'}(z)), \tag{2.17}$$

with corresponding differential equation

$$q(z) + \frac{1}{\gamma}zq^{'}(z) = h(z). \tag{2.18}$$

In Theorem (2.14), we assumed that h in (2.18) was convex, which implied that solution q was convex. On the other hand if we assuming that q is convex and h is defined by (2.18) and by simple calculation we have

$$Re\left\{\frac{h^{'}(z)}{q^{'}(z)}\right\} = Re\left\{\frac{(\gamma+1)}{\gamma} + \frac{1}{\gamma}\frac{zq^{''}(z)}{q^{'}(z)}\right\} > 0,$$

then h is close to convex, therefore h is univalent function.By using the fuzzy subordination chain as given in Theorem (2.15) to obtain fuzzy best subordinant for (2.17).

In next theorem we introduce an example of a solution of problem (2.4),(2.8) referred to the introduction.

Theorem 2.16 Let q be convex in U and let h be defined by $h(z) = q(z) + \frac{1}{\gamma}zq^{'}(z)$,with $Re(\gamma) > 0$. If $p \in [a, 1] \cap Q$,$p(z) + \frac{1}{\gamma}zp^{'}(z)$ is univalent in U and

$$F_{h(U)}(h(z)) \leq F_{(C^2\times U)}(p(z) + \frac{1}{\gamma}zp^{'}(z))$$

then

$$q(z) \prec_F p(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}}\int_0^z h(t)t^{\gamma-1}dt.$$

The function q is the fuzzy best subordinant.

Proof . Let $L(z,t) = \varphi(q(z), tzq^{'}(z)) = q(z) + \frac{t}{\gamma}zq^{'}(z)$. By simple calculation, we get

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = Re\left\{\gamma + t\frac{zq^{''}(z)}{q^{'}(z)}\right\},$$

since q convex function ,$Re(\gamma) > 0$ and $t \in (0, 1]$,we obtain

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}$$

Using lemma (1.9), we deduce that L is fuzzy subordination chain. By Theorem (2.15), we conclude that q is fuzzy subordinant of fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp^{'}(z)),$$

Furthermore, since q is a univalent solution of (2.18), it also is the fuzzy best subordinant of

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp^{'}(z)).$$

Example 2.17 . Let $h(z) = \frac{1-z}{1+z}$ and $p(z) = 1+z, z \in U$, it is clear to show that $h(0) = 1$, $h^{'}(z) = \frac{-2}{(1+z)^2}$, $h^{''}(z) = \frac{4}{(1+z)^3}$ and $p(z) + zp^{'}(z) = 1 + 2z$. Since

$$Re\left\{\frac{zh^{''}(z)}{h^{'}(z)} + 1\right\} = Re\left\{\frac{1-z}{1+z}\right\} = Re\left\{\frac{(1 - r(\cos\theta + i\sin\theta)}{(1 + r(\cos\theta + i\sin\theta)}\right\}$$

$$= \frac{1-r^2}{1 + 2r\cos\theta + r^2} > 0,$$

where $r = |z| < 1, \theta \in R$. Then the function h is convex in U.
We have

$$q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = \frac{2\ln(1+z)}{z} - 1.$$

Using Theorem (2.14),we obtain $\frac{1-z}{1+z} \prec_F 1+2z$,induce $\frac{2\ln(1+z)}{z} - 1 \prec_F 1+z, \quad z \in U$.
In next result, we introduce fuzzy differential superordination for which the fuzzy subordinant function h is a starlike function.
Theorem 2.18 Let h be starlike in U , with $h(0) = 0$,If $p \in [0,1] \cap Q$ and $zp^{'}(z)$ is univalent in U, then

$$F_{h(U)}(h(z)) \le F_{\varphi(C^2\times U)}(zp^{'}(z)) \Rightarrow F_{q(U)}q(z) \le F_{p(U)}p(z), z \in U. \tag{2.19}$$

Where

$$q(z) = \int_0^z h(t)t^{-1}dt, \tag{2.20}$$

The function q is convex and is the fuzzy best subordinant.
Proof. Differentiating (2.20), we have $zq^{'}(z) = h(z)$, if we let $\varphi : C^2 \times U \to C, \varphi(s) = s$, for $s = zp^{'}(z), z \in U$,relation (2.19) becomes

$$F_{h(U)}h(z) \le F_{\varphi(C^2\times U)}(\varphi(zp^{'}(z))),$$

the function q is the solution of $\varphi(zq^{'}(z)) = zq^{'}(z) = h(z)$. Since h is starlike, we deduce from Alexander's theorem that q is convex and univalent. As in the previous theorem we can assume that h and q are analytic and univalent on$\overline{U}$ and $q^{'}(\zeta) \neq 0$ for$|\zeta| = 1$, the conclusion of this theorem follows form Theorem (2.13), if we show that $\varphi \in \Phi[h, q]$, we get this immediately since $h(U)$ is starlike domain and

$$\varphi(tzq^{'}(z)) = tzq^{'}(z) = th(z) \in h(U), \quad z \in U \quad \text{and} \quad 0 < t \le 1 \tag{2.21}$$

Form (2.21), we have

$$F_{\varphi(C^2\times U)}(tzq^{'}(z)) \le F_{h(U)}(h(z))$$

Using Definition (2.1), we obtain $\varphi \in \Phi[h, q]$,and applying Theorem (2.13), we conclude that q is the fuzzy best subordinant.
Example 2.19 Let $h(z) = z + z^2, z \in U$ It is clear to show that

$$Re\left\{\frac{zh^{'}(z)}{h(z)}\right\} = Re\left\{1 + \frac{z}{1+z}\right\} = Re\left\{1 + \frac{r(\cos\theta + i\sin\theta)}{1 + r(\cos\theta + i\sin\theta)}\right\}$$
$$= 1 + \frac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2} > 0.$$

If $p \in H[0,1] \cap Q$ and $zp^{'}(z)$ is univalent in U, then

$$z + z^2 \prec_F zp^{'}(z) \Rightarrow z + z^2 \prec_F p(z).$$

In the next section, we will combine some theorems for the Sandwich theorem.

3. Sandwich theorem

We can combine Theorem (2.14) to gather with Lemma (1.7) to obtain the following fuzzy differential ”sandwich theorem”.

Theorem 3.1 Let h_1 and h_2 be convex in U, with $h_1(z) = h_2(z) = a$. Let $\gamma \neq 0$, with $Re(\gamma) > 0$ and let the function q_i be defined by

$$q_i = \frac{\gamma}{z^\gamma} \int_0^z h_i(t) t^{\gamma-1} dt,$$

for $i = 1, 2$. If $p \in H[a, 1] \cap Q$ and $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent , then

$$h_1(z) \prec_F p(z) + \frac{1}{\gamma} z p'(z) \prec_F h_2(z) \Rightarrow q_1(z) \prec_F p(z) \prec_F q_2(z), \quad z \in U. \tag{3.1}$$

The function, q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant.

If we set $f(z) = p(z) + \frac{1}{\gamma} z p'(z)$, then (3.1), can be expressed as the following “sandwich theorem”involving fuzzy subordination preserving integral operator.

Corollary 3.2 Let h_1 and h_2 be convex in U and f be univalent function in U, with $h_1(0) = h_2(0) = f(0)$, Let $\gamma \neq 0$, with $Re(\gamma) > 0$. If

$$h_1(z) \prec_F f(z) \prec_F h_2(z),$$

then

$$\frac{\gamma}{z^\gamma} \int_0^z h_1(t) t^{\gamma-1} dt \prec_F \frac{\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \prec_F \frac{\gamma}{z^\gamma} \int_0^z h_2(t) t^{\gamma-1} dt$$

When the middle integral is univalent. If we combine Theorem (2.18), with Lemma (1.8), we obtain the following “sandwich result”.

Theorem 3.3 Let h_1 and h_2 be starlike functions in U, with $h_1(0) = h_1(0) = 0$ and let the function $q_(i)$ be defined by

$$q_i = \int_0^z h_i(t) t^{-1} dt,$$

for $i = 1, 2$. If $p \in H[0, 1] \cap Q$ and $z p'(z)$is univalent in U, then

$$h_1(z) \prec_F z p'(z) \prec_F h_2(z) \Rightarrow q_1(z) \prec_F p(z) \prec_F q_2(z), \quad z \in U.$$

The functions q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant. If we set $f(z) = z p'(z)$, then this last theorem can be expressed as the following “sandwich theorem”involving fuzzy subordination preserving integral operator.

Corollary 3.3 Let h_1 and h_2 be starlike functions in U and f be univalent in U, with $h_1(0) = h_2(0) = 0$. If

$$h_1(z) \prec_F f(z) \prec_F h_2(z), \quad z \in U,$$

then

$$\int_0^z h_1(t) t^{-1} dt \prec_F \int_0^z f(t) t^{-1} dt \prec_F \int_0^z h_2(t) t^{-1} dt \prec_F$$

when the middle integral is univalent.

References

Miller, S.S. and P.T. Mocanu (2000). *Differential Subordinations, Theory and Applications*. Vol. 225. Dekker, New York.

Miller, S.S. and P.T. Mocanu (2003). Subordinants of differential superordinations. *Complex Variables, Theory and Application* **48**, 815–826.

Oros, G. I. and Gh. Oros (2011). The notion of subordination in fuzzy sets theory. *General Mathematics* **19**, 97103.

Oros, G. I. and Gh. Oros (2012*b*). Fuzzy differential subordination. *Acta Universitatis Apulensis* **30**, 55–64.

Oros, G. I. and Gh. Oros (Jun2012*a*). Dominants and best dominants in fuzzy differential subordinations. *Studia Universitatis Babes-Bolyai, Mathematica* **57**, 239–248.

Pascu, N.N. (2006). Alpha-close-to-convex functions. *Romanian Finish Seminar on Complex Analysis* **743**, 331–335.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control* **8**, 338–353.

Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 39 – 46

Evaluation of Classification Models in Machine Learning

Jasmina Dj. Novaković[a,*], Alempije Veljović[b], Siniša S. Ilić[c], Željko Papić[d], Milica Tomović[e]

[a] *Belgrade business school, Higher education institution for applied science, Belgrade, Serbia*
[b] *Faculty of technical sciences Čačak, University of Kragujevac, Čačak, Serbia*
[c] *Faculty of technical sciences K. Mitrovica, University of Priština, K. Mitrovica, Serbia*
[d] *Faculty of technical sciences Čačak, University of Kragujevac, Čačak, Serbia*
[e] *PE Post of Serbia, Belgrade, Serbia*

Abstract

We study the problem of evaluation of different classification models that are used in machine learning. The reason of the model evaluation is to find the optimal solution from various classification models generated in an iterated and complex model building process. Depending on the method of observing, there are different measures for evaluation the performance of the model. To evaluate classification models the most direct criterion that can be measured quantitatively is the classification accuracy. The main disadvantages of accuracy as a measure for evaluation are as follows: neglects the differences between the types of errors and it dependent on the distribution of class in the dataset. In this paper we discussed selection of the most appropriate measures depends on the characteristics of the problem and the various ways it can be implemented.

Keywords: accuracy, confusikon matrix, costs of misclassification, F-measure, ROC graph.
2010 MSC: 68T01, 68T05.

1. Introduction

Machine learning is a field of artificial intelligence that deals with the construction of adaptive computing systems that are able to improve their performance by using information from experience. Machine learning is the discipline that studies the generalization and construction and analysis of algorithms that can generalize. But as much as the applications of machine learning were diverse, there are tasks that are repetitive. Therefore, it is possible to talk about the types of learning tasks that often occur. One of the most common tasks of learning that occurs in practice is classification. Classification is an important recognition of object types, for example whether a particular tissue represents malignant tissue or not.

*Corresponding author

Email addresses: jnovakovic@sbb.rs (Jasmina Dj. Novaković), alempije@beotel.net (Alempije Veljović), sinisa.ilic@pr.ac.rs (Siniša S. Ilić)

Classification is one of the most common tasks of machine learning, and is a problem of classification unknown instance in one of the pre-offered categories - classes. The important observation in classification is that target functions are discrete. In general, the class label can't be meaningfully assigned numerical or some other values. This means that the class attribute, whose value should be determined, categorical attribute.

The classification of an object is based on finding similarities with predetermined objects that are members of different classes, with the similarity of the two objects is determined by analyzing their characteristics. In classifying every object is classified into one of the classes with certain accuracy. The task is that on the characteristics of objects whose classification is known in advance, make a model by which will be performed classification of new objects (Fawcett, 2003; Marzban, 2004; Vardhan *et al.*, 2012). In problem of classification, the number of classes is known in advance and limited.

A wide range of algorithms for classification is available, each with their own strengths and weaknesses. There is no such a learning algorithm which works best with all the problems of supervised learning. Machine learning involves a large number of algorithms such as: artificial neural networks, genetic algorithms, rule induction, decision trees, statistical and pattern recognition methods, k-nearest neighbors, Naïve Bayes classifiers and discriminatory analysis.

The main objective of this paper is to discuss the various classification models that can be used in the problem of classification. This paper presents the advantages and disadvantages of these models. For this purpose we have organized the paper in the following way. In the second part of this paper we present evaluation of classification models, in the third part of the paper we present measures for the evaluation of classification models. In the last part of the paper, we discuss the results and give directions for further research.

2. Evaluation of classification models

For modeling regularity in the data there are a number of methods. Also, methods on the same set of examples for learning result in different models changing the parameters of the method. Due to the same problem and the same set of training data can produce a higher number of different models; it emphasizes the need for the evaluation of the quality model with respect to the given problem. That is why the evaluation of discovered knowledge is one of the essential components of the process of intelligent data analysis. Since this work deals with the classification problems, hereinafter will discuss the evaluation of classification models.

The task of evaluating classification models is to measure the degree to which the classification suggested using the model corresponding to the actual classification of the case. Depending on the method of observing, there are different measures for evaluation the performance of the model. Selection of the most appropriate measures shall be done depending on the char

acteristics of the problem and ways of its implementation.

3. Measures for the evaluation of classification models

In the evaluation of classification models basic concept is the notion of fault. If the application of the classification models in selected case leading to the prediction of a class that is different from the actual class examples then there is an error in classification. If any mistake is equally important, then the total number of errors in the observed set can be an indicator of work a classifier.

This approach is based on accuracy as a measure for evaluating the quality of the classification model. This measure can be defined as the ratio of the number of correctly classified examples according to the total number of classified examples.

$$Accuracy = \frac{number\ of\ correctly\ classified\ examples}{total\ number\ of\ cases} \tag{3.1}$$

The main disadvantages of accuracy as a measure for evaluation are as follows: (1) neglects the differences between the types of errors; (2) dependent on the distribution of class in the dataset.

It is often important in practical problem solving distinguish certain types of errors. It is often the case in medicine, for example detecting the existence of disease in a patient. If system needs to classify breast tissue on malignant and benign based on mammography image, then if the system incorrectly marked diseased tissue as healthy tissue, the error is more important, because it will not notice the existence of the disease and will not apply the appropriate therapy. In case that the system recognizes healthy tissue as sick, error has less importance because it will further surgery and diagnosis to determine that the patient is not diseased.

In cases where it is necessary to distinguish more types of errors result of the classification is shown in the form of two-dimensional matrix, where each row of the matrix corresponds to one class and record number of examples where it is forecasted class, and each column of the matrix is also marked by a class and x h

+ c+aald number of examples where it is an actual class. lllvvvv mI f vf we look for example classification problem with five classes, where we need to classify the emotional state of the person appearing in the video in five different emotional categories: happy, sad, angry, gentle and frightened, then we confusion matrix display as in Figure 1.

		Actual class				
		happy	sad	angry	gentle	frightened
Predicted class	happy	51	2	1	1	1
	sad	3	23	1	1	0
	angry	2	2	17	0	0
	gentle	0	1	2	9	1
	frightened	1	0	1	1	18

Figure 1. Illustration of confusion matrix for the classification problem of recognizing emotional states.

On the diagonal of the matrix is the number of correct classified examples, while other elements of the matrix indicate the number of examples that were incorrectly classified as some of the other classes. Figure 1 shows that the six examples of class *happy* wrongly classified as follows: three are classified as class *sad*, two in class *angry*, zero in class *gentle*, and one in class *frightened*. It can be concluded that the use of a confusion matrix allows better analysis of different types of errors.

The largest number of measures for evaluation of classification models related to classification problems with two classes. This is not a particular limitation for the use of these measures, given that problems with larger number of classes can be displayed as a series of problems with two classes. Each of these measures in particular stands out one of the class as a target class, with the data set is divided into positive and negative examples of the target class. The negative examples include examples of all other classes. That is why below we consider a classification problem with two classes.

Confusion matrix in classification problem with two classes is shown in Figure 2. It can be concluded from the figure that there are possible four different results forecasts. Really positive and really negative outcomes are correct classification, while the false positive and false negative outcomes are two possible types of errors.

False positive example is a negative example class that is wrongly classified as positive and false negative is a positive example of the class who is wrongly classified as negative. In the context of our research entrance to confusion matrix have the following meanings (Kohavi & Provost, 1998):

- a is the number of correct predictions that instances are negative,
- b is the number of incorrect predictions that instances are positive,
- c is the number of incorrect predictions that instances are negative,
- d is the number of correct predictions that instances are positive.

		Predicted class	
		Negatives	Positives
Actual class	Negatives	a	b
	Positives	c	d

Figure 2. Confusion matrix in classification problem with two classes.

A few standard terms are defined in a matrix with two classes: accuracy, true positive rate, false positive rate, true negative rate, false negative rate and precision. The accuracy is the proportion of true results (both true positives and true negatives) among the total number of cases examined. Accuracy may be determined using the equation:

$$Accuracy = \frac{a+b}{a+b+c+d}. \tag{3.2}$$

True positive rate is the proportion of positive cases that are properly identified and can be calculated using equation:

$$True\ positive\ rate = \frac{d}{c+d}. \tag{3.3}$$

The false positive rate is the proportion of negative cases that were incorrectly classified as positive, and calculated with equation:

$$False\ positive\ rate = \frac{b}{a+b}. \tag{3.4}$$

The true negative rate was defined as the proportion of negatives cases which are classified correctly, and is calculated using the equation:

$$\textit{True negative rate} = \frac{a}{a+b}. \tag{3.5}$$

The false negative rate is the proportion of positive cases that were incorrectly classified as negative, and are calculated using equation:

$$\textit{False negative rate} = \frac{c}{c+d}. \tag{3.6}$$

Finally, precision or positive predictive value presents the fraction of predictive positive cases that are accurate, and is calculated using the equation:

$$\textit{Precision} = \frac{d}{b+d}. \tag{3.7}$$

There are cases when accuracy is not adequate measures. The accuracy is determined by the equation (3.2) can't be an adequate measure of performance when the number of negative cases is much higher than the number of positive cases (Kubat *et al.*, 1998). If there are two classes and one is significantly smaller than the other, it is possible to obtain high accuracy if all instances are classified in larger class.

Suppose that there are 1000 cases, 995 negative cases and five cases which are positive. If the system classifies all of them negative, accuracy will be 99.5%, although classifier missed all positive cases. Or, for example, in tests which establish whether the patient is suffering from some disease, and the disease has only 1% of people in the population, a test should always reported that the patient has no disease would have an accuracy of 99%, but is unusable. In such cases, the accuracy as a measure of model quality is not adequate measure. In these cases the sensitivity of the classifier is an important measure and his ability to observe instances that are required, in this case ill patients.

In machine learning, most classifiers assumes equal importance of classes in terms of the number of instances and the level of importance, which means that all classes have the same significance. Standard techniques in machine learning are not successful when predicting a minority class in an unbalanced data set or when the false negatives are considered more important than false positives. In practical terms, unequal costs of inaccurate classifications are common, especially in medical diagnostics, so that the asymmetric misclassification costs must be taken into account as an important factor.

Cost-sensitive classifiers adapting models to costs of misclassification in the learning phase, with the objectives to reduce the costs of misclassification rather than to maximize the accuracy of classification. Because many practical problems of classifications have different costs associated with different types of errors, various algorithms for the evaluation of the sensitivity of classification is used.

Complementarity is one of the important characteristics of the evaluation of classification models. Using the pairs measures can be displayed specific accuracy of classification models with somewhat opposed positions. For example, by varying the parameter selected modeling techniques can be at the expense of one of the specific measures to increase the accuracy of the model shown in another measure. This is an optimization problem in which the selection with the appropriate settings based on the one measure, maximize other measures. In some cases, the quality of the classifier needs expressed by a number, not a pair of dependent measures, which is achieved by using pairs measures. Using the pairs value of measures, one measure is fixed and is observed only second measure. Thus, for example, can be considered measures of accuracy with fixed value of the response to 20% and in this case the derived measure is called the precision of 20%.

Besides derived measure, there are measures that are not based on fixing one component of a pair of measure, for example *F-measure*, which is defined as follows:

$$F\text{-}measure = \frac{2 \times response \times accuracy}{response + accuracy}. \tag{3.8}$$

Another way to test the performance of the classifier is the ROC graph (Swets, 1988). ROC graph is the two-dimensional representation which on the X axis represents the false positive rate and the Y axis represents true positive rate. Item (0,1) is the perfect classifier: classifies all positive and all negative cases correctly. This is (0, 1), because the false positive rate is 0 (zero), a positive real rate is 1 (all). Point (0, 0) is a classifier that predicts all cases to be negative, while point (1, 1) corresponds to the classifier which provides that every case is positive. Point (1, 0) is a classifier that is incorrect for all classifications. In many cases, the classifier has a parameter which can be adjusted increasing the real positive rates at the cost of increasing false positive rates or reducing the false positive rate based on the dropping value of real positive rates.

Each setting parameters gives par value for a false positive rate and positive real rates and the number of such pairs can be used to represent the ROC curves. Nonparametric classifier is presented ROC to one point, which corresponds to the par value of the false positive rate and positive real rate.

Figure 3 shows an example of a ROC graph with two ROC curves and two ROC points marked P1 and P2. Nonparametric algorithms produce a single ROC point for a particular data set. Characteristics of ROC graph are:

- ROC curve or point is independent of the distribution of the class or the cost of errors (Kohavi & Provost, 1998).
- ROC graph contains all the information contained in the matrix of errors (Swets, 1988).
- ROC curve provides a visual tool for testing the ability of the classifier to correctly identify positive cases and negative cases that were incorrectly classified.

The area under of the one ROC curve can be used as a measure of accuracy in many applications, and it is called the measurement accuracy based on the surface (Swets, 1988).

Prevost and Fawcett in 1997 (Provost & Fawcett, 1997) argued that the use of the classification accuracy of the classifier comparison is not adequate measure unless the cost classification and distribution of class unknown, but one classifier must be chosen for each situation. They propose a method of assessing the classifier using the ROC graph, imprecise costs and distribution of class.

Another way of comparing ROC points is the equation that balances accuracy with Euclidean distances from perfect classifier, i.e. from the point (0, 1) on the graph. In this way we include

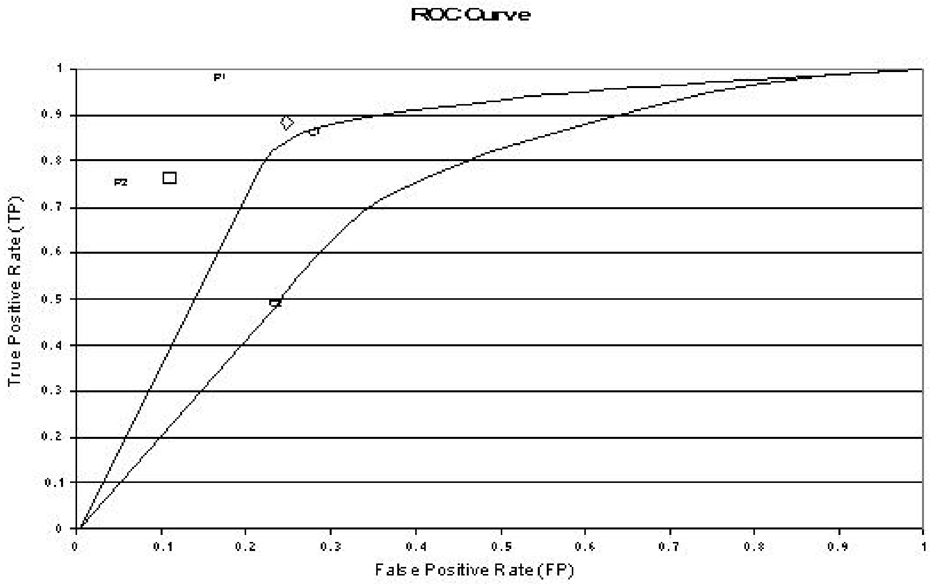


Figure 3. ROC graph
http://www2.cs.uregina.ca/˜dbd/cs831/notes/ROC/ROC.html

weighting factors that allow us to define the relative cost of improper classification, if such data are available.

4. Conclusions

This research discusses the various classification models that can be used in the problem of classification. This research could help in future works, such as the implementation of an adequate classification model in different classification problems. There are many questions and issues that remain to be addressed and that we intend to investigate in future work. These conclusions and recommendations will be used in classification problems in the near future.

Acknowledgements. The authors are grateful to the Ministry of Science and Technological Development of the Republic of Serbia for the support (projects: TR 34009 and TR35026).

References

Fawcett, T. (2003). ROC graphs: Notes and practical considerations for data mining researchers. Technical Report HPL-2003-4. Hewlett Packard, Palo Alto, CA.

Kohavi, R. and F. Provost (1998). *Glossary of terms*. Editorial for the Special Issue on Applications of Machine Learning and the Knowledge Discovery Process.

Kubat, M., R. Holte and S. Matwin (1998). Machine learning for the detection of oil spills in satellite radar images. *Machine Learning* **30**, 195–215.

Marzban, C. (2004). The ROC curve and the area under it as performance measures. *Wea. Forecasting* **19**(6), 1106–1114.

Provost, F. and T. Fawcett (1997). Analysis and visualization of classifier performance: comparison under imprecise class and cost distributions. In: *Proceedings of the Third International Conference on Knowledge Discovery and Data Mining (KDD-97)*.

Swets, J. (1988). Measuring the accuracy of diagnostic systems. *Science* **240**, 1285–1293.

Vardhan, R. V., S. Pundir and G. Sameera (2012). Estimation of area under the ROC curve using exponential and weibull distributions. *Bonfring International Journal of Data Mining* **2**(2), 52–56.

Theory and Applications of
Mathematics & Computer Science
(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 47 – 58

On Threshold Voltage Variation-Tolerant Designs

Valeriu Beiu[a,*], Mihai Tache[b]

[a]*Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania*
[b]*University Politehnica of Bucharest, Bucharest, Romania*

Abstract

Scaling CMOS transistors has been used to achieve *smaller*, *faster*, and *cheaper* integrated circuits. However, with CMOS transistors moving deep towards the nanometer range, the effects *threshold voltage* (V_{TH}) variations (besides other variations and noises) play on their reliabilities and that of the gates they are forming are worrying. For mitigating against this trend, *sizing* can be used to improve on the reliability of the CMOS gates. Simultaneously, sizing can also reduce power or maintain speed while only marginally affecting area. For evaluating the advantages sizing still holds, inverters of different sizings are compared in this paper with reliability enhanced inverters using well-known redundancy schemes like triple modular redundancy and hammock networks. Simulation results show that, *at the same reliability*, *sizing can lead to designs outperforming those obtained by the other methods on any of the design parameters (i.e., area, power or delay)*. These are reinforcing previous reports showing that *space redundancy* applied at the device-level outperform gate-level solutions.

Keywords: CMOS, sizing, reliability, redundancy, area, delay, power.
2010 MSC: 34M10, 30D35.

1. Introduction

Over half a century the semiconductor industry has relied on CMOS scaling as the basis for its growth, implementing *smaller*, *faster*, and *cheaper* integrated circuits (ICs). However, with sizes approaching *10nm*, industry is facing several fundamental limitations. One of these is the randomness of the number and locations of doping atoms (Asenov, 1998), (Asenov *et al.*, 2003), which together with imprecisions in fabrication are leading to device-to-device fluctuations/variations in key parameters, including V_{TH}.

When adding intrinsic and extrinsic noises (on top of variations), reliability looks like one of the greatest threats to the design of future ICs (SIA, 2014). The expected higher *probabilities of failures* (PFs), due to higher sensitivity to noises and variations, could make future ICs

*Corresponding author
Email address: valeriu.beiu@uav.ro (Valeriu Beiu)

prohibitively unreliable. In this context, ITRS (SIA, 2014) predicted that CMOS scaling would become difficult when trying to go beyond *10nm* as more "*errors [will] arise from the difficulty of providing highly precise dimensional control needed to fabricate the devices and also from interference from the local environment.*" That is why VLSI designers should consider reliability as an extra design parameter, in addition to area, power, and delay.

The well-established approach for improving reliability is to add *redundancy* (von Neumann, 1956), (Moore & Shannon, 1956), (Winograd & Cowan, 1963), (Wakerly, 1976). Redundancy can be either in *space*, *time, information*, or a combination of some of these. *Space (hardware) redundancy* can be most easily understood in relation to voting and includes: modular redundancy (von Neumann, 1956), (Wakerly, 1976), (Abraham & Siewiorek, 1974), cascaded modular redundancy (Lee *et al.*, 2007), (Hamamatsu *et al.*, 2010), as well as multiplexing (e.g., von Neumann multiplexing (von Neumann, 1956), enhanced von Neumann multiplexing (Roy & Beiu, 2004), (Roy & Beiu, 2005), and parallel restitution (Sadek *et al.*, 2004)). Still, voters are not necessarily needed. In fact, besides multiplexing, others schemes which do without voting include: quadded logic (Tryon, 1960), (Jensen, 1963); interwoven logic (Pierce, 1964); radial logic (Klaschka, 1967), (Klaschka, 1969); *n*-safe-logic (Mine & Koga, 1967), (Das & Chuang, 1972); dotted logic (Freeman & Metze, 1972); as well as solutions at the device/transistor level. *Time redundancy* is trading space for time (e.g., alternating logic, re-computing with shifted operands or with swapped operands, etc.), while *information redundancy* is based on error detection and error correction codes.

The focus of this paper is on space redundancy. Space redundancy can be applied at the system-, module-, gate-, or device-level. Applying space redundancy at the device-level is much more efficient than applying it at higher levels (as explained in (Moore & Shannon, 1956); see also (Beiu & Ibrahim, 2011)), while the common expectation is that spatial redundancy should always degrade performances, i.e., increase *area, power*, and *delay*. In this paper we will show that redundancy applied at the device-level can improve redundancy without increasing *area*, and even while reducing *power* or *delay.*

Sizing has already been suggested as a way to enhance tolerance to variations (Sulieman *et al.*, 2010), (Ibrahim *et al.*, 2011), (Keller *et al.*, 2011), (Ibrahim & Beiu, 2011). In fact, sizing gives the VLSI designer options for optimizing the trade-offs between reliability and *area-power-delay*, while, in particular, it can enhance reliability and reduce power within the same area. For getting a better understanding of the advantages sizing can bring to reliability, the performances of differently sized inverters will be weighted against those obtained by using reliability improvement schemes including triple modular redundancy (TMR) and four-transistor hammock networks (H_{22}). The paper is organized as follows. The effect sizing plays on tolerating V_{TH} variations is discussed in Section 2. A brief review of space redundancy methods is presented in Section 3. Sizing is revisited in Section 4, followed by simulation results in Section 5 and concluding remarks in Section 6.

2. How Sizing Affects Variations

VLSI designers have normally adjusted the sizing of nMOS and pMOS transistors (i.e., width W and length L) in order to balance the driving currents when either the pMOS (I_{pMOS}) or the nMOS (I_{nMOS}) stacks are switched ON. This requires the balancing of the ON resistances of the pMOS and nMOS stacks (R_{pMOS}, R_{nMOS}), which is achieved by adjusting (sizing) the transistors because pMOS conduction relies on holes which have slower mobility than electrons. Fig. 1 shows four different sizing options for a transistor. Although all of them have the same area $W \times L = 6a^2$, they have different ON resistances. In case of Fig. 1(a) there are 6 squares (a^2) connected in parallel, therefore $R_{ON} = R_\square/6$ (where $R_\square$ is the resistance of a square, e.g., $W = a$, $L = a$). In case of Fig. 1(d), the six squares are connected in series, hence $R_{ON} = 6R_\square$. Similarly, R_{ON} for Fig. 1(b) and 1(c) can be estimated as $3R_\square/2$ and $2R_\square/3$.

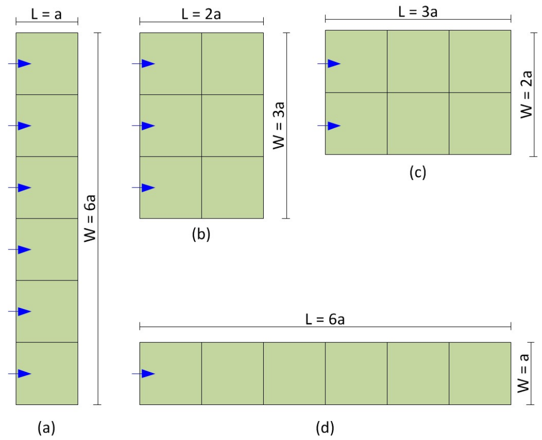


Figure 1. Four different sizing options having the same *area* ($A = 6a^2$).

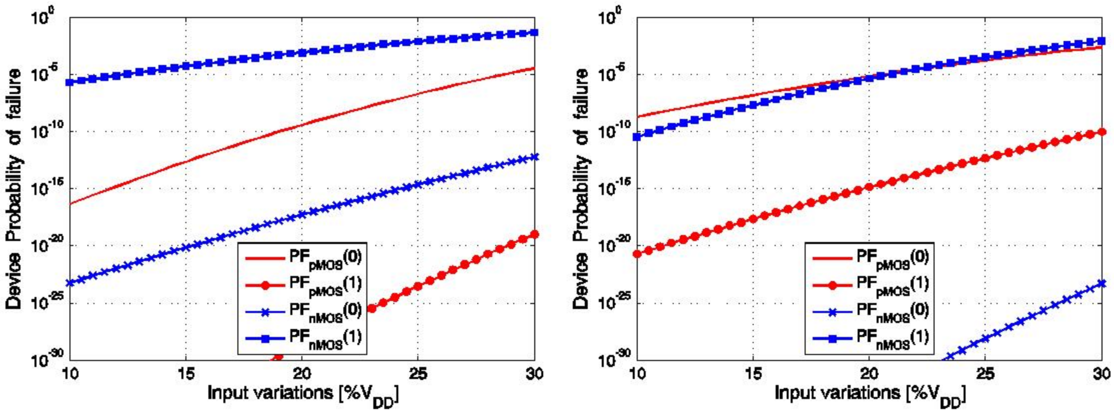
With CMOS scaling approaching *10nm*, it becomes difficult to reproduce V_{TH} over the large number of transistors in a chip. This is due to the random fluctuations of both the number of dopants and of their physical locations. V_{TH} variations can be approximated (see (Asenov *et al.*, 2003)) by a normal distribution with standard deviation:

$$\sigma_{V_{TH}} \simeq 3.19 \times 10^{-8} t_{ox} N_A^{0.4} (L_{eff} \times W_{eff})^{-0.5} [V] \tag{2.1}$$

where t_{ox} is the oxide thickness, N_A is the channel doping, W_{eff} is the effective channel width, and L_{eff} is the effective channel length. In the following we will use normalized dimensions for L and W.

Eq.(2.1) shows that increasing the transistors area (by increasing L and/or W) will always reduce V_{TH} variations. While the four sizing options in Fig. 1 are expected to exhibit similar probabilities of switching (meaning that the transistor fails to open/close, see (Beiu & Ibrahim, 2011), (Ibrahim & Beiu, 2011), (Ibrahim *et al.*, 2012)) as they have the same area, they will lead to very different performances, as their R_{ON} is between $R_\square/6$ and $6R_\square$.

For classical sizing VLSI designers set $L_{nMOS} = L_{pMOS} = min$ and $W_{nMOS} = 2 \times L_{nMOS}$ (i.e., $R_{nMOS} = R_\square/2$). To balance I_{pMOS} and I_{nMOS}, W_{pMOS} is then increased, such as R_{pMOS} matches R_{nMOS}. This also increases the area of the pMOS ($A_{pMOS} = L_{pMOS} \times W_{pMOS}$), improving their reliability. Fig. 2(a) shows that a pMOS transistor is more reliable than an nMOS. As classical sizing increases the area of the pMOS transistors it makes them even more reliable than nMOS transistors.



(a) (b)

Figure 2. PF_{TRS} with respect to variations: (a) classical sized ($L_{nMOS} = L_{pMOS} = 1, W_{nMOS} = 2$, and $W_{pMOS} = 4$); (b) reverse sized ($W_{nMOS} = W_{pMOS} = 1, L_{nMOS} = 4$, and $L_{pMOS} = 2$).

For enhancing PF_{GATE}, it is essential to improve the reliability of the nMOS stack, ideally matching the reliability of the pMOS stack (similar to matching R_{pMOS} to R_{nMOS}). For doing this A_{nMOS} should be enlarged (see eq. (2.1)). Classical sizing is using $W/L > 1$ and $L = min$, so it follows that W_{nMOS} has to be increased. Subsequently, this requires increasing W_{pMOS} (to compensate for the slower mobility of the holes). Hence, relying on classical sizing W_{nMOS} has to be increased, which leads to enlarging all transistors and degrading the gates *area, delay*, and *power* consumption.

3. Space Redundancy

Classical space redundancy schemes start from an unreliable system and use divide-and-conquer in a top-down fashion as follows. The unreliable system is divided into several sub-systems which are interconnected by a *network*. Each sub-system is further divided into several sub-sub-systems, which are also interconnected by a network. This continues down to the elementary transistors, and the level where redundancy will be applied has to be decided. Four levels are well-established: *system, module, gate*, and *device*. Redundancy can be applied simultaneously at more than one level even using different schemes at different levels. This implies that the optimization space is very large. Fundamentally, using space redundancy at any level translates into replicating all of the sub-systems at that level by a *redundancy factor R*. This R-times larger redundant system needs to be connected by a modified network. Most space redundancy methods are done at this point, while some space redundancy methods require additional blocks (e.g., voters) for connecting the original sub-systems. In the following we shall briefly review space redundancy methods by classifying them with respect to the need for voters, while also suggesting how complex is the connectivity pattern (modified network) they use.

3.1. Higher Level Methods

The most well-known high level redundancy methods are *triple modular redundancy* (TMR) and *n-modular redundancy* (NMR). TMR was proposed by von Neumann [4]. It divides a system into modules (sub-systems) and triplicates each module (Fig. 3). A voter is used to combine the outputs of the $R = 3$ modules operating in parallel (Lyons & Vanderkulk, 1962), (Gurzi, 1965), (Longden *et al.*, 1966), (Wakerly, 1975), (Stroud, 1994), Morgan *et al.* (2007). TMR is able to mask failures that affect one module by taking the majority of three modules. The interconnectivity pattern is simple (Fig. 3(a)), while it might get slightly more complex if more voters are used in parallel (Fig. 3(b)).

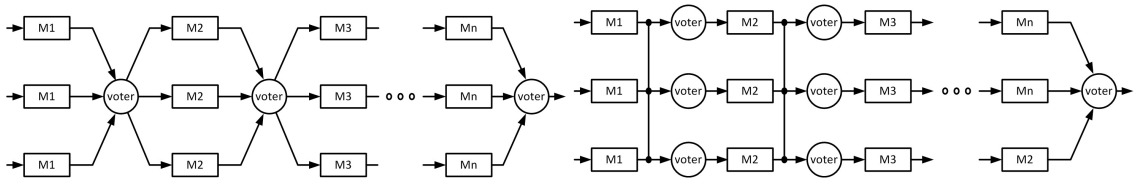


Figure 3. Triple modular redundancy: (a) one voter per stage; (b) three voters per stage.

NMR is an extension of TMR to any odd number n. It requires replicating all the modules n times ($R = n$), and also using larger voters (with n inputs), but it could tolerate $n/2$ module failures (Ness *et al.*, 2007). The connectivity pattern gets more complex and the length of the wires increases as n is increased, and if more voters are being used in parallel. The early analyses have assumed that voters are very reliable. Later it was realized that even assuming that the reliability of a voter is independent of the number of inputs n is unrealistic, and could lead to wrong conclusions. This has motivated research into space redundancy methods which could do without voters.

A gate-level redundancy method without voting was also introduced by von Neumann in [4], and is known as *multiplexing*. Other gate-level space redundancy methods without voting are *quadded* (Tryon, 1960), (Jensen, 1963), *interwoven* (Pierce, 1964), *radial* (Klaschka, 1967), (Klaschka, 1969) and *n-fail-safe* (Mine & Koga, 1967), (Das & Chuang, 1972) logic. All of these exhibit simpler connectivity patterns then multiplexing, as being more regular and local, i.e., having shorter wires. Another gate-level method which does not require voting is *dotted logic* (Freeman & Metze, 1972), which is bridging the gap between gate- and device-level methods. The reason is that dotted logic took advantage of implementations which use wired AND and OR functions. This is not entirely gate-level anymore, but it is not yet device-level either.

3.2. Device-Level Methods

Device-level methods have also been introduced in a seminal article (Moore & Shannon, 1956) (relays in the original paper). The main conclusions of that work have been that:

- redundant relay (device-level) structures are able to outperform redundant gate-level schemes at significantly (orders of magnitude) smaller R; and that
- the modified networks (there are different ways to connect the redundant relays) have a strong influence on reliability.

All the subsequent publications inspired by the original study of Moore and Shannon (Moore & Shannon, 1956) have detailed particular applications of those ideas. They rely on series-and-parallel networks of (a few) devices. The most widespread network used is a series-parallel network of 4 devices/relays/transistors which is the simplest hammock network (Moore & Shannon, 1956). This has been named a *quad configuration* by many of the later papers Suran (1964), Bolchini *et al.* (1996), Abid & El-Razouk (2006), Anghel & Nicolaidis (2007), El-Maleh *et al.* (2008). Here we shall use hammock network (hence the H abbreviation) as we do not want to create any confusion with respect to gate-level *quadded logic* (Tryon, 1960), (Jensen, 1963). A few papers have looked at simpler hammock networks (Djupdal & Haddow, 2007), or at hammock networks having more than four transistors (Anghel & Nicolaidis, 2007), (Aunet *et al.*, 2005). It looks like this trend will be taking up due to developments on carbon nano tubes (Zarkesh-Ha & Shahi, 2010), (Zarkesh-Ha & Shahi, 2011).

4. Transistor Sizing Revisited

While sizing has been used for a very long time to balance driving currents, its use for enhancing reliability has only recently started to be explored for $W/L > 1$ (classical sizing) (Keller *et al.*, 2011). Still, a *reverse sizing* ($W/L < 1$) has been proposed in (Sulieman *et al.*, 2010) for overcoming the problems mentioned in Section 2. This sizing method keeps all W minimum ($W_{nMOS} = W_{pMOS} = min$), and increases L. Normally, this is not used for digital circuits, but has been used in analog circuits as "*better matching can be obtained without consuming additional area, simply by changing the W/L aspect ratio*" (Drennan & McAndrew, 2003). To make A_{nMOS} larger than A_{pMOS}, L_{pMOS} should be kept small ($L_{pMOS} = 2W_{pMOS}$), while L_{nMOS} should be increased. While occupying the same area, the reverse sizing method enhances the gates reliability but diminishes its performances.

This is because increasing L increases R_{ON} and hence the delay, but power is reduced as I_{ON} is reduced. Fig. 2(b) shows PF_{TRS} for reverse sizing. Increasing the area of the nMOS transistors improves their reliability and (more importantly) allows matching the reliability of the pMOS transistors (see $PF_{nMOS}(1)$ and $PF_{pMOS}(0)$ in Fig. 2(b)).

Aiming to simultaneously optimize *reliability* and *power-delay-area*, an exhaustive sizing search was suggested in (Ibrahim *et al.*, 2011). Instead of using L_{min} (classical sizing) or W_{min} (reverse sizing), different sizings are obtained by analyzing all the possible A_{nMOS} and A_{pMOS} combinations, lower than a maximum area A_{max}, and achieving a PF_{GATE} lower than a target PF_{target}. This method is exhaustive as iterating through all the possible nMOS area combinations from $W_{nMOS} \times L_{nMOS} = 1 \times A_{max}$ to $A_{max} \times 1$. For each nMOS sizing combination, the algorithm tries to find all the corresponding pMOS sizing combinations ($W_{pMOS} \times L_{pMOS}$) such that R_{pMOS} matches R_{nMOS} and $A_{pMOS} \leq A_{max}$. If a pMOS sizing combination is found, Gate Reliability EDA (GREDA) (Ibrahim *et al.*, 2012) is used to quickly and accurately estimate PF_{GATE}.

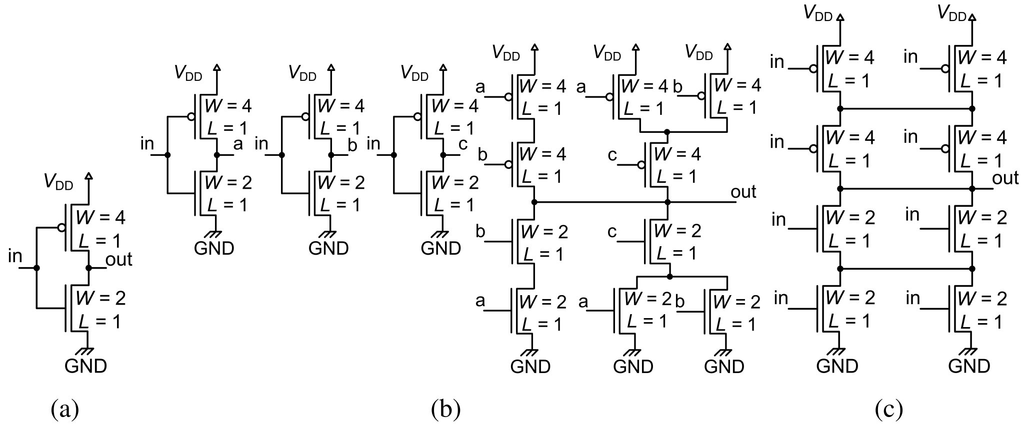


Figure 4. (a) Classical INV; (b) INV-TMR with a MIN-3 as voter; (c) H_{22}-INV.

If $PF_{GATE} \leq PF_{target}$, the method stores the identified nMOS and pMOS sizing combination in a list of candidate combinations. Finally, the method checks the list of candidate combinations. If the list is empty it means that PF_{GATE} cannot achieve PF_{target} with transistors of up to A_{max}. Otherwise, the design process is continued by using Spice to estimate the *delay, power*, and *power-delay-product (PDP)* for each candidate combination. The best combination that optimizes *delay, power* or *PDP* can then be selected. In all cases the *reliability* and the *area* constraints are always going to be satisfied.

5. Simulation Result

We have used $16nm$ PTM v2.1 incorporating high-$k/metal$ gates and stress effects (Zhao & Cao, 2007), (PTM, 2011), as this is strongly affected by variations, and simulated at $V_{DD} = 700mV$ (nominal voltage) and $T = 27°C$. The TMR-INV circuit has three INVs followed by a mirrored MIN-3 gate as voter. This MIN-3 implementation was preferred as it is considered the most reliable one (Sulieman, 2009). All the transistors for TMR-INV (Fig. 4(b)) and H_{22}-INV (Fig. 4(c)) were sized using classical sizing, and the mobility of the electrons was assumed to be twice the mobility of the holes.

5.1. Reliability Results

In the first set of simulations GREDA was used to calculate the reliability of INVs with transistors of different sizings as well as TMR-INV (Fig. 4(b)) and H_{22}-INV (Fig. 4(c)). For all these simulations the input variations were assumed to be 15%, i.e., logic "1" = $0.85V_{DD} = 595mV$ and logic "0" = $0.15V_{DD} = 105mV$.

In the case of a classical INV, the simulation results show $PF_{INV}(0) = 7.25E-21$ and $PF_{INV}(1) = 5.33E-05$. The large difference between these values is due to $PF_{INV}(1)$ being dominated by PF_{nMOS}. For a reverse sized INV the simulation results show that increasing the area of the nMOS by increasing L_{nMOS} reduces $PF_{INV}(1)$ to $1.58E-07$ (i.e., 2 orders of magnitude better than $PF_{INV}(1)$ for classical sizing).

For TMR-INV the simulations show $PF_{TMR-INV}(1) = 8.51E-09$ (4 orders of magnitude better than classical) and $PF_{TMR-INV}(0) = 5.67E-09$, which although 12 orders of magnitude worse then classical, is balanced with respect to $PF_{TMR-INV}(1)$. This is due to the fact that the output of the INV gate ($PF_{INV}(0) = 7.25E-21$) is a logic "1" input for the MIN-3 gate, being significantly degraded (to $5.67E-09$) as affected by $PF_{MIN-3}(1)$, which is determined by PF_{nMOS}.

For H_{22}-INV we have seen $PF_{H_{22}-INV}$ being improved significantly for both logic "0" and logic "1": from $PF_{INV}(0) = 7.25E-21$ and $PF_{INV}(1) = 5.33E-05$ (classical INV) to $PF_{H_{22}-INV}(0) = 2.10E-40$ and $PF_{H_{22}-INV}(1) = 5.67E-09$ respectively.

For a fair comparison of the performances of sizing versus the other space redundancy methods considered, the PF_{target} was set to $1.0E-09$ (range achieved by TMR-INV and H_{22}-INV). The maximum transistor area was limited to $A_{max} = 10a^2$. Table 1 shows the seven different sizing combinations (with W/L aspect ratios above and below 1) satisfying both of these requirements and also matching R_{nMOS} to R_{pMOS}. All of them achieve reliabilities of the order $1E-10$.

5.2. *Performance Results*

The second set of simulations has used Spice to estimate the performances of the different solutions. These are reported in Table 1, starting with the classically sized INV having an average *delay* of 5.54ps and an average *power* consumption of 0.24μW.

Table 1. Reliability enhanced INV circuits as well as differently sized INVs.

	A_{nMOS} ($W\times L$)	A_{pMOS} ($W\times L$)	Area ($\sum A_{trs}$)	Worst PF_{INV}	Delay [ps]	Power [µW]	PDP [aJ]
Classical	2×1	4×1	6	5.33E−05	5.54	0.24	1.34
Reversed	1×4	1×2	6	1.58E−07	30.54	0.06	1.70
TMR	2×1	4×1	48	8.51E−09	20.01	3.55	71.03
H_{22}	2×1	4×1	24	5.67E−09	35.71	0.79	28.34
This paper	1×5	1×3	8	4.47E−10	55.12	0.06	3.49
	1×6	1×3	9	1.89E−10	62.65	0.07	4.20
	3×2	3×1	9	1.89E−10	13.34	0.25	3.39
	1×7	1×3	10	1.89E−10	70.71	0.07	5.01
	5×1	9×1	14	4.47E−10	5.45	0.64	3.49
	5×1	10×1	15	4.47E−10	5.62	0.69	3.89
	1×5	2×5	15	4.47E−10	153.04	0.15	22.57

Table 1 clearly shows that adding more gates (TMR-INV) or adding more transistors (H_{22}-INV), while improving the reliability over the classical INV by 4 orders of magnitude (from $1E-5$ to $1E-9$), significantly degrades both power and delay: TMR-INV increases the average delay by 3.6×, while the average power and PDP are increased by 14.8× and 53× respectively; H_{22}-INV is about 6.5× slower while consuming about 3.3× more power and having a 21× higher PDP.

The reverse sized INV improves redundancy by 2 orders of magnitude while also reducing power by 4×, but degrades delay and PDP by 5.5× and 1.3× respectively. Among other possible sizings, $[3 \times 2, 3 \times 1]$ improves PF_{INV} by more than 5 orders of magnitude (over the classical $[2 \times 1, 4 \times 1]$ sizing) at about the same power, while increasing delay and PDP by only 2.4×. For high-performance applications, one should select $[5 \times 1, 9 \times 1]$ which improves reliability by 5 orders of magnitude while being as fast as a classical INV (in fact it is a shy 2% faster), and consumes about 2.7× more power. Alternatively, $[1 \times 5, 1 \times 3]$ could be selected for low-power applications, with reliability being improved by 5 orders of magnitude, and power being reduced 4×, while delay is increased 10 × .

6. Conclusions

This paper has compared the performances of different sized inverters with classical and reverse sized inverters, as well as with two redundancy methods at the gate-level (TMR) and device-level (H_{22}) (Mukherjee & Dhar, 2015) (Sheikh *et al.*, 2016), (Robinett *et al.*, 2007), (El-Maleh *et al.*, 2009). The main conclusions are:

- Sizing can outperform both TMR and H_{22} methods with respect to reliability.
- Improving the reliability of a CMOS gate can be achieved without increasing *area*.
- Improving the reliability of a CMOS gate should not necessarily lead to penalties in *power* or *delay*, or even on the contrary, i.e., there are reliability enhanced solutions which can achieve either lower *power* or shorter *delays* but not both.

Sizing can be used to improve tolerance to variations, and it is possible to design CMOS gates trading reliability versus *area-power-delay*. The disadvantages are represented by very large libraries of gates and a much more complex design.

Future work will analyze re-sized solutions for other CMOS gates (e.g., NAND, NOR, XOR, etc.), of different *fan-ins* (see (Gemmeke & Ashouei, 2012), (Gemmeke *et al.*, 2013)). These should be compared not only with classical sized gates, TMR, and H_{22}, but also with quadded, interwoven, radial, *n*-safe, and dotted logic solutions, and evaluated jointly with advanced CMOS (Berge & Aunet, 2009) (Liu & Moroz, 2007), (Maly, 2007), (Geppert, 2002), and even beyond-CMOS technologies (Courtland, 2016), (Desai *et al.*, 2016).

Acknowledgements. This work was supported in part by Intel (*ULP-NBA = Ultra Low-Power Application-specific Non-Boolean Architectures, 2011-05-24G*) and in part by the European Union through the European Regional Development Fund under the Competitiveness Operational Program (*BioCell-NanoART = Novel Bio-inspired Cellular Nano-architectures*, POC-A1-A1.1.4-E nr. 30/2016).

References

Abid, Z. and H. El-Razouk (2006). Defect tolerant voter design based on transistor redundancy. *J. Low Power Electr.* **2**, 456–463.

Abraham, J. A. and D. P. Siewiorek (1974). An algorithm for the accurate reliability evaluation of triple modular redundancy networks. *IEEE Trans. Comp.* **23**, 682–692.

Anghel, L. and M. Nicolaidis (2007). *Defect Tolerant Logic Gates for Unreliable Future Technologies*. Sandoval, F., Prieto, A., Cabestany, J., Graa, M. (eds.) Computational and Ambient Intelligence, LNCS. Springer, Heidelberg.

Asenov, A. (1998). Random dopant induced threshold voltage lowering and fluctuations in sub-0.1μm MOSFETs: A 3-D "atomistic" simulation study. *IEEE Trans. Electr. Dev.* **45**, 2505–2513.

Asenov, A., A. R. Brown, J. H. Davies, S. Kaya and G. Slavcheva (2003). Simulation of intrinsic parameter fluctuations in decananometer and nanometer-scale MOSFETs. *IEEE Trans. Electr. Dev.* **50**, 1837–1852.

Aunet, S., Y. Berg and V. Beiu (2005). *Ultra Low Power Redundant Logic Based on Majority-3 Gates*. da Luz Reis, R.A., Osseiran, A., Pfleiderer, H.-J. (eds.) From Systems to Silicon. Springer, Heidelberg.

Beiu, V. and W. Ibrahim (2011). Devices and input vectors are shaping von Neumann multiplexing. *IEEE Trans. Nanotech.* **10**, 606–616.

Berge, H. K. O and S. Aunet (2009). Benefits of decomposing wide CMOS transistors into minimum size gates. In: *Proc. NORCHIP, Trondheim, Norway,* art. 5397795.

Bolchini, C., G. Buonanno, D. Sciuto and R. Stefanelli (1996). Static redundancy techniques for CMOS gates. In: *Proc. Intl. Symp. Citc. and Syst. (ISCAS)*. pp. 576–579.

Courtland, R. (2016). The next high-performance transistor. *IEEE Spectrum* **53**, 11–12.

Das, S. and Y. H. Chuang (1972). Fault restoration using N-fail-safe logic. *Proc. IEEE* **60**, 334–335.

Desai, S. B., S. R. Madhvapathy, A. B. Sachid, J. P. Pablo Llinas, Q. Wang, G. H. Ahn, G. Pitner, M. J. Kim, J. Bokor, C. Hu, H. S. P. Wong and A. Javey (2016). MoS_2 transistors with 1-nanometer gate length. *Science* **354**, 99–102.

Djupdal, A. and P. C. Haddow (2007). Defect tolerant ganged CMOS minority gate. In: *Proc. NORCHIP,* art. 4481060.

Drennan, P. G. and C. C. McAndrew (2003). Understanding MOSFET mismatchfor analog design. *IEEE J. Solid-State Circ.* **38**, 450–456.

El-Maleh, A. H., B. M. Al-Hashimi, A. Melouki and F. Khan (2009). Defect-tolerant n^2-transistor structure for reliable nanoelectronic design. *IET Comput. and Digital Tech.* **3**, 570–580.

El-Maleh, H., B. M. Al-Hashimi and A. Melouki (2008). Transistor-level based defect tolerance for reliable nanoelectronics. In: *Proc. Intl. Conf. Comp. Syst. and Appls. (AICCSA)*. pp. 53–59.

Freeman, H. A. and G. Metze (1972). Fault-tolerant computers using ”dotted logic” redundancy techniques. *IEEE Trans. Comp.* **C-21**, 867–871.

Gemmeke, T. and M. Ashouei (2012). Variability aware cell library optimization for reliable sub-threshold operation. In: *Proc. European Solid-State Circ. Conf. (ESSCIRC), Bordeaux, France*. pp. 42–45.

Gemmeke, T., M. Ashouei, B. Liu, M. Meixner, T. G. Noll and H. de Groot (2013). Cell libraries for robust low-voltage operation in nanometer technologies. *Solid-State Electr.* **84**, 132–141.

Geppert, L. (2002). The amazing vanishing transistor act. *IEEE Spectrum* **39**, 28–33.

Gurzi, K. J. (1965). Estimates for best placement of voters in a triplicated logic network. *IEEE Trans. Electr. Comp.* **EC-14**, 711–717.

Hamamatsu, M., T. Tsuchiya and T. Kikuno (2010). On the reliability of cascaded TMR systems. In: *Proc. Pacific Rim Intl. Symp. Dependable Comp. (PRDC)*. pp. 184–190.

Ibrahim, W. and V. Beiu (2011). Using Bayesian networks to accurately calculate the reliability of complementary metal oxide semiconductor gates. *IEEE Trans. Reliab.* **60**, 538–549.

Ibrahim, W., V. Beiu and A. Beg (2012). GREDA: a fast and more accurate CMOS gates reliability EDA tool. *IEEE Trans. CAD* **31**, 509–521.

Ibrahim, W., V. Beiu and H. Amer (2011). Reliability optimized CMOS gates. In: *Proc. IEEE Intl. Conf. Nanotech. (IEEE-NANO)*. pp. 730–734.

Jensen, P. A. (1963). Quadded NOR logic. *IEEE Trans. Reliab.* **12**, 22–31.

Keller, S., S. S. Bhargav, C. Moore and A. J. Martin (2011). Reliable minimum energy CMOS circuit design. In: *European Workshop CMOS Variability (VARI). Grenoble, France*.

Klaschka, T. F. (1967). Two contributions to redundancy theory. In: *Proc. Annual Symp. Switching and Autom. Th. (SWAT)*. pp. 175–183.

Klaschka, T. F. (1969). *Reliability Improvement by Redundancy in Electronics Systems. Part II: An Efficient New Redundancy SchemeRadial Logic.Tech. Rep. 69045*. Royal Aircraft Establishment, Farnborough, UK.

Lee, S., J. Jung and I. Lee (2007). Voting structures for cascaded triple modular redundant modules. *IEICE Electr. Exp.* **4**, 657–664.

Liu, T. J. King and V. Moroz (2007). Segmented channel MOS transistor. *US Patent* 7,247,887.

Longden, M., L. J. Page and R. A. Scantlebury (1966). An assessment of the value of triplicated redundancy in digital systems. *Microelectr. and Reliab.* **5**, 39–55.

Lyons, R. E. and W. Vanderkulk (1962). The use of triple-modular redundancy to improve computer reliability. *IBM J. R and D* **6**, 200–209.

Maly, W. (2007). Integrated circuit, device, system, and method of fabrication. *WO Patent* 133775.

Mine, H. and Y. Koga (1967). Basic properties and a construction method for fail-safe logical systems. *IEEE Trans. Electr. Comp.* **EC-16**, 282–289.

Moore, F. and C. E. Shannon (1956). Reliable circuits using less reliable relays. *J. Frankl. Inst.* **262**, 191–208 and 281–297.

Morgan, K. S., D. L. McMurtrey, B. H. Pratt and M. J. Wirthlin (2007). A comparison of TMR with alternative fault-tolerant design techniques for FPGAs. *IEEE Trans. Nuclear Sci.* **54**, 2065–2072.

Mukherjee, A. and A. S. Dhar (2015). Fault tolerant architecture design using quad-gate-transistor redundancy. *IET Circ. Dev. and Syst.* **9**, 152–160.

Ness, D. C., C. J. Hescott and D. J. Lilja (2007). Modeling failure reduction for combinational logic using gate level NMR. In: *Proc. Annual Reliab. and Maintain. Symp. (RAMS)*. pp. 208–213.

Pierce, W. H. (1964). Interwoven redundant logic. *J. Frankl. Inst.* **277**, 55–85.

PTM, Predictive Technology Model (2011). http://ptm.asu.edu/.

Robinett, W., P. J. Kuekes and R. S. Williams (2007). Defect tolerance based on coding and series replication in transistor-logic demultiplexer circuits. *IEEE Trans. Circ. and Syst. I* **54**, 2410–2421.

Roy, S. and V. Beiu (2004). Multiplexing schemes for cost-effective fault-tolerance. In: *Proc. IEEE Intl. Conf. Nanotech. (IEEE-NANO)*. pp. 589–592.

Roy, S. and V. Beiu (2005). Majority multiplexing-economical redundant fault-tolerant designs for nanoarchitectures. *IEEE Trans. Nanotech.* **4**, 441–451.

Sadek, S., K. Nikolić and M. Forshaw (2004). Parallel information and computation with restitution for noise-tolerant nanoscale logic networks. *Nanotech* **15**, 192–210.

Sheikh, A. T., A. H. El-Maleh, M. E. S. Elrabaa and S. M. Sait (2016). A fault tolerance technique for combinational circuits based on selective-transistor redundancy. *IEEE Trans. VLSI Syst.* in press.

SIA, Intl. Tech. Roadmap Semicon (2014). http://public.itrs2.net/.

Stroud, C. E. (1994). Reliability of majority voting based VLSI fault-tolerant circuits. *IEEE Trans. VLSI Syst.* **2**, 516–521.

Sulieman, M. H. (2009). *On the Reliability of Interconnected CMOS Gates Considering MOSFET Threshold-Voltage Variations*. Schmid, A., Goel, S., Wang, W., Beiu, V., Carrara, S. (eds.) Nano-Net, LNICST, vol. 20. Springer, Heidelberg.

Sulieman, M. H., V. Beiu and W. Ibrahim (2010). Low-power and highly reliable logic gates: Transistor-level optimizations. In: *Proc. IEEE Intl. Conf. Nanotech. (IEEE-NANO)*. pp. 254–257.

Suran, J. J. (1964). Use of circuit redundancy to increase system reliability. In: *Proc. Intl. Solid-State Circ. Conf. (ISSCC)*. pp. 82–83.

Tryon, J. G. (1960). Redundant logic circuit. *US Patent* 2,942,193.

von Neumann, J. (1956). *Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components*. Shannon, C.E., McCarthy, J. (eds.) Automata Studies, pp. 43-98. Princeton Univ. Press, Princeton, NJ.

Wakerly, J. F. (1975). Transient failures in triple modular redundancy systems with sequential modules. *IEEE Trans. Comp.* **C-24**, 570–573.

Wakerly, J. F. (1976). Microcomputer reliability improvement using triple-modular redundancy. *Proc. IEEE* **64**, 889–895.

Winograd, S. and J. D. Cowan (1963). *Reliable Computation in the Presence of Noise*. MIT Press, Cambridge.

Zarkesh-Ha, P. and A. A. M. Shahi (2010). Logic gates failure characterization for nanoelectronic EDA tools. In: *Proc. Intl. Symp. Defect and Fault Tolerance VLSI Syst. (DFT)*. pp. 16–23.

Zarkesh-Ha, P. and A. A. M. Shahi (2011). Stochastic analysis and design guidelines for CNFETs in gigascale integrated circuits. *IEEE Trans. Electr. Dev.* **58**, 530–539.

Zhao, W. and Y. Cao (2007). Predictive technology model for Nano-CMOS design exploration. *ACM J. Emerg. Tech.* **3**, 1–17.

Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 59 – 71

Matrix Representations of Fuzzy Quaternion Numbers

Lorena Popa[a], Lavinia Sida[a], Sorin Nădăban[a,*]

[a] *Department of Mathematics and Computer Science, "Aurel Vlaicu" University of Arad, Romania*

Abstract

In this paper, firstly we discuss basic arithmetic operations with fuzzy quaternion numbers. Then, we introduce a noncommutative field, formed with 2×2 fuzzy complex matrices which is used for the matrix representation of fuzzy quaternion numbers as elements within this field. Finally, another way of representing fuzzy quaternion numbers is obtained by using 4×4 fuzzy real matrices.

Keywords: fuzzy complex numbers, fuzzy quaternion numbers.
2010 MSC: 08A72.

1. Introduction

The fuzzy quaternion numbers were defined in many ways by many authors. For example, in paper (Moura *et al.*, 2013) the authors proposed an extension for the set of fuzzy real numbers to the set of fuzzy quaternion numbers, defining the notion of fuzzy quaternion numbers by an application $h' : H \to [0,1]$ such that $h'(a + bi + cj + dk) = min\{\widetilde{A}(a), \widetilde{B}(b), \widetilde{C}(c), \widetilde{D}(d)\}$ for some $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ real fuzzy numbers. In paper (Moura *et al.*, 2014), the authors defined the fuzzy quaternion numbers using triangular fuzzy numbers.

Recently, in paper (Sida *et al.*, 2016) a new approach is proposed in order to introduce the fuzzy quaternion numbers concept, similar to the way that Fu and Shen (see (Fu & Shen, 2011)) have introduced the fuzzy complex numbers.

The study of fuzzy quaternion numbers is continued in this paper. More precisely, another approach for multiplication and division operation is presented. Then, following the ideas in (Moţ & Popa, 2014), we will represent fuzzy quaternion numbers as a 2×2 fuzzy complex matrices, as well as 4×4 fuzzy real matrices.

*Corresponding author
Email addresses: lorena.popa@uav.ro (Lorena Popa), lavinia.sida@uav.ro (Lavinia Sida), sorin.nadaban@uav.ro (Sorin Nădăban)

2. Preliminaries

For the concept of fuzzy real numbers and their arithmetic operations we make reference to the following papers: (Das & Mandal, 2002), (Dubois & Prade, 1980), (Dzitac, 2015), (Felbin, 1992), (Janfada *et al.*, 2011), (Kaleva & Seikkala, 1984), (Mizumoto & Tanaka, 1979), (Xiao & Zhu, 2002).

Definition 2.1. (Dzitac, 2015) A fuzzy set in $\mathbb{R}$, namely a mapping

$$\widetilde{A} : \mathbb{R} \to [0,1],$$

with the following properties:

(i) $\widetilde{A}$ is convex, i.e. $\widetilde{A}(y) \geq \min\left(\widetilde{A}(x), \widetilde{A}(z)\right)$, for $x \leq y \leq z$;
(ii) $\widetilde{A}$ is normal, i.e. $(\exists)x_0 \in \mathbb{R}; \widetilde{A}(x_0) = 1$;
(iii) $\widetilde{A}$ is upper semicontinuous, i.e.

$$(\forall)x \in \mathbb{R},\ (\forall)\alpha \in (0,1] :\ \widetilde{A}(x) < \alpha,$$

$$(\exists)\delta > 0 \text{ such that } \mid y\ -\ x \mid < \delta\ \Rightarrow \widetilde{A}(y) < \alpha$$

is called a fuzzy real number.

We will denote by $\mathbb{R}_F$ - the set of all fuzzy real numbers.

Definition 2.2. (Mizumoto & Tanaka, 1979) The basic arithmetic operations $+, -, \cdot, /$ on $\mathbb{R}_F$ are defined by:

1. *Addition*:
$$\left(\widetilde{A} + \widetilde{B}\right)(x) = \bigvee_{y\in\mathbb{R}} min\left\{\widetilde{A}(y), \widetilde{B}(x-y)\right\},\ (\forall)x \in \mathbb{R} \tag{2.1}$$
2. *Subtraction*:
$$\left(\widetilde{A} - \widetilde{B}\right)(x) = \bigvee_{y\in\mathbb{R}} min\left\{\widetilde{A}(y), \widetilde{B}(y-x)\right\},\ (\forall)x \in \mathbb{R} \tag{2.2}$$
3. *Multiplication*:
$$\left(\widetilde{A} \cdot \widetilde{B}\right)(x) = \bigvee_{y\in\mathbb{R}^*} min\left\{\widetilde{A}(y), \widetilde{B}(x/y)\right\},\ (\forall)x \in \mathbb{R} \tag{2.3}$$
4. *Division*:
$$\left(\widetilde{A}/\widetilde{B}\right)(x) = \bigvee_{y\in\mathbb{R}} min\left\{\widetilde{A}(x\cdot y), \widetilde{B}(y)\right\},\ (\forall)x \in \mathbb{R}. \tag{2.4}$$

Remark. A triangular fuzzy number is defined by its membership function

$$x(t) = \begin{cases} 0, & \text{if } \ t < a_1 \\ \frac{t-a_1}{a_2-a_1}, & \text{if } \ a_1 \leq t < a_2 \\ \frac{a_3-t}{a_3-a_2}, & \text{if } \ a_2 \leq t < a_3 \\ 0, & \text{if } \ t > a_3 \end{cases} \quad \text{, where } a_1 \leq a_2 \leq a_3, \tag{2.5}$$

and it is denoted $\widetilde{x} = (a_1, a_2, a_3)$.

Remark. (Chang & Wang, 2009; Elomda & Hefny, 2013; Hanss, 2005; Nădăban *et al.*, 2016) Let $\widetilde{x} = (a_1, a_2, a_3)$, $\widetilde{y} = (b_1, b_2, b_3)$ be two non negative triangular fuzzy numbers and $\alpha \in \mathbb{R}_+$. According to the extension principle, the arithmetic operations are defined as follows:

1. $\widetilde{x} + \widetilde{y} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $\widetilde{x} - \widetilde{y} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$
3. $\alpha\widetilde{x} = (\alpha a_1, \alpha a_2, \alpha a_3)$
4. $\widetilde{x}^{-1} = (1/a_3, 1/a_2, 1/a_1)$
5. $\widetilde{x} \times \widetilde{y} \cong (a_1 b_1, a_2 b_2, a_3 b_3)$
6. $\widetilde{x}/\widetilde{y} \cong (a_1/b_3, a_2/b_2, a_3/b_1)$

We denote that the results of (4) – (6) are not triangular fuzzy numbers, but they can be approximated by triangular fuzzy numbers.

Definition 2.3. (Fu & Shen, 2011) An fuzzy complex number, $\widetilde{z}$, is defined in the form of:

$$\widetilde{z} = \widetilde{A} + i\widetilde{B}, \tag{2.6}$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{R}_F$; $\widetilde{A}$ is the real part of $\widetilde{z}$ while $\widetilde{B}$ represents the imaginary part, i.e. $Re(\widetilde{z}) = \widetilde{A}$ and $Im(\widetilde{z}) = \widetilde{B}$.

We will denote by $\mathbb{C}_F$ - the set of all fuzzy complex numbers. The operations on $\mathbb{C}_F$ are a straightforward extension of those on real complex numbers.

Definition 2.4. (Fu & Shen, 2011) Let $\widetilde{z} = \widetilde{A} + i\widetilde{B}$, $\widetilde{z}_2 = \widetilde{C} + i\widetilde{D} \in \mathbb{C}_F$ where $\widetilde{A}, \widetilde{B}, \widetilde{C}$ and $\widetilde{D}$ are fuzzy real numbers. The basic arithmetic operations are defined as follows:

1. *Addition*:

$$\widetilde{z}_1 + \widetilde{z}_2 = (\widetilde{A} + \widetilde{C}) + i(\widetilde{B} + \widetilde{D}), \tag{2.7}$$

where

$$(\widetilde{A} + \widetilde{C})(y) = \bigvee_{y = x_1 + x_2} \left(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2)\right)$$
$$(\widetilde{B} + \widetilde{D})(y) = \bigvee_{y = x_1 + x_2} \left(\widetilde{B}(x_1) \wedge \widetilde{D}(x_2)\right)$$

2. *Subtraction*:

$$\widetilde{z}_1 - \widetilde{z}_2 = (\widetilde{A} - \widetilde{C}) + i(\widetilde{B} - \widetilde{D}), \tag{2.8}$$

where

$$(\widetilde{A} - \widetilde{C})(y) = \bigvee_{y = x_1 - x_2} \left(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2)\right)$$
$$(\widetilde{B} - \widetilde{D})(y) = \bigvee_{y = x_1 - x_2} \left(\widetilde{B}(x_1) \wedge \widetilde{D}(x_2)\right)$$

3. *Multiplication*:

$$\widetilde{z_1} \times \widetilde{z_2} = (\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D}) + i(\widetilde{B}\widetilde{C} + \widetilde{A}\widetilde{D}), \tag{2.9}$$

where

$$(\widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{D})(y) = \bigvee_{y=x_1x_2-x_3x_4} \big(\widetilde{A}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{B}(x_3) \wedge \widetilde{D}(x_4)\big)$$

$$(\widetilde{B}\widetilde{C} + \widetilde{A}\widetilde{D})(y) = \bigvee_{y=x_1x_2+x_3x_4} \big(\widetilde{B}(x_1) \wedge \widetilde{C}(x_2) \wedge \widetilde{A}(x_3) \wedge \widetilde{D}(x_4)\big)$$

4. *Division*:

$$\widetilde{z_1}/\widetilde{z_2} = \Big(\frac{\widetilde{A}\widetilde{C} + \widetilde{B}\widetilde{D}}{\widetilde{C}^2 + \widetilde{D}^2}\Big) + i\Big(\frac{\widetilde{B}\widetilde{C} - \widetilde{A}\widetilde{D}}{\widetilde{C}^2 + \widetilde{D}^2}\Big), \tag{2.10}$$

where $\frac{\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{D}}{\widetilde{C}^2+\widetilde{D}^2} = \widetilde{t_1}$ and $\frac{\widetilde{B}\widetilde{C}-\widetilde{A}\widetilde{D}}{\widetilde{C}^2+\widetilde{D}^2} = \widetilde{t_2}$ are fuzzy real numbers:

$$\widetilde{t_1}(y) = \bigvee_{y=\frac{x_1x_3+x_2x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} \big(\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)\big)$$

$$\widetilde{t_2}(y) = \bigvee_{y=\frac{x_2x_3-x_1x_4}{x_3^2+x_4^2},\, x_3^2+x_4^2\neq 0} \big(\widetilde{A}(x_1) \wedge \widetilde{B}(x_2) \wedge \widetilde{C}(x_3) \wedge \widetilde{D}(x_4)\big)$$

Remark. Fuzzy complex number $\widetilde{z} = \widetilde{A} + i\widetilde{B}$ admits a matrix representation, namely:

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ -\widetilde{B} & \widetilde{A} \end{pmatrix},$$

where $\widetilde{A}, \widetilde{B} \in \mathbb{R}_F$.

3. On arithmetic operation with fuzzy quaternion numbers

In paper (Sida *et al.*, 2016) it was introduced fuzzy quaternion number as well as basic arithmetic operations with fuzzy quaternion numbers.

Definition 3.1. A fuzzy quaternion number is an element of the form

$$\widetilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k, \tag{3.1}$$

where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F$ and $i^2 = j^2 = k^2 = -1; \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$

We will denote by $\mathbb{H}_F$ - the set of all fuzzy quaternion numbers.

Remark. $\widetilde{A}$ is called the real part of $\widetilde{q}$ and sometimes denoted $\widetilde{A} = Re(\widetilde{q})$, and $\widetilde{B}, \widetilde{C}, \widetilde{D}$, are called imaginary parts of $\widetilde{q}$ and denoted $\widetilde{B} = Im_1(\widetilde{q}), \widetilde{C} = Im_2(\widetilde{q}), \widetilde{D} = Im_3(\widetilde{q})$.

Definition 3.2. If $\widetilde{q_1} = \widetilde{A}_1 + \widetilde{B}_1 i + \widetilde{C}_1 j + \widetilde{D}_1 k$ and $\widetilde{q_2} = \widetilde{A}_2 + \widetilde{B}_2 i + \widetilde{C}_2 j + \widetilde{D}_2 k \in \mathbb{H}_F$ the basic arithmetic operations are defined as follows:

(i)

$$\widetilde{q_1} + \widetilde{q_2} = (\widetilde{A}_1 + \widetilde{A}_2) + (\widetilde{B}_1 + \widetilde{B}_2)i + (\widetilde{C}_1 + \widetilde{C}_2)j + (\widetilde{D}_1 + \widetilde{D}_2)k \tag{3.2}$$

(ii)

$$\widetilde{q_1} - \widetilde{q_2} = (\widetilde{A}_1 - \widetilde{A}_2) + (\widetilde{B}_1 - \widetilde{B}_2)i + (\widetilde{C}_1 - \widetilde{C}_2)j + (\widetilde{D}_1 - \widetilde{D}_2)k \tag{3.3}$$

(iii)

$$\widetilde{q_1} \cdot \widetilde{q_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k. \tag{3.4}$$

where

$$\begin{aligned}
\widetilde{A} &= (\widetilde{A}_1 \cdot \widetilde{A}_2 - \widetilde{B}_1 \cdot \widetilde{B}_2 - \widetilde{C}_1 \cdot \widetilde{C}_2 - \widetilde{D}_1 \cdot \widetilde{D}_2) \\
\widetilde{B} &= (\widetilde{A}_1 \cdot \widetilde{B}_2 + \widetilde{B}_1 \cdot \widetilde{A}_2 + \widetilde{C}_1 \cdot \widetilde{D}_2 - \widetilde{D}_1 \cdot \widetilde{C}_2) \\
\widetilde{C} &= (\widetilde{A}_1 \cdot \widetilde{C}_2 - \widetilde{B}_1 \cdot \widetilde{D}_2 + \widetilde{C}_1 \cdot \widetilde{A}_2 + \widetilde{D}_1 \cdot \widetilde{B}_2) \\
\widetilde{D} &= (\widetilde{A}_1 \cdot \widetilde{D}_2 + \widetilde{B}_1 \cdot \widetilde{C}_2 - \widetilde{C}_1 \cdot \widetilde{B}_2 + \widetilde{D}_1 \cdot \widetilde{A}_2).
\end{aligned}$$

Definition 3.3. If $\widetilde{q_1} = \widetilde{A}_1 + \widetilde{B}_1 i + \widetilde{C}_1 j + \widetilde{D}_1 k$ and $\widetilde{q_2} = \widetilde{A}_2 + \widetilde{B}_2 i + \widetilde{C}_2 j + \widetilde{D}_2 k \in \mathbb{H}_F$, then:

$$\frac{\widetilde{q_1}}{\widetilde{q_2}} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k, \tag{3.5}$$

where

$$\begin{aligned}
\widetilde{A} &= \frac{\widetilde{A}_1 \cdot \widetilde{A}_2 + \widetilde{B}_1 \cdot \widetilde{B}_2 + \widetilde{C}_1 \cdot \widetilde{C}_2 + \widetilde{D}_1 \cdot \widetilde{D}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{B} &= \frac{-\widetilde{A}_1 \cdot \widetilde{B}_2 + \widetilde{B}_1 \cdot \widetilde{A}_2 - \widetilde{C}_1 \cdot \widetilde{D}_2 + \widetilde{D}_1 \cdot \widetilde{C}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{C} &= \frac{-\widetilde{A}_1 \cdot \widetilde{C}_2 + \widetilde{B}_1 \cdot \widetilde{D}_2 + \widetilde{C}_1 \cdot \widetilde{A}_2 - \widetilde{D}_1 \cdot \widetilde{B}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2} \\
\widetilde{D} &= \frac{-\widetilde{A}_1 \cdot \widetilde{D}_2 - \widetilde{B}_1 \cdot \widetilde{C}_2 + \widetilde{C}_1 \cdot \widetilde{B}_2 + \widetilde{D}_1 \cdot \widetilde{A}_2}{\widetilde{A}_2^2 + \widetilde{B}_2^2 + \widetilde{C}_2^2 + \widetilde{D}_2^2}.
\end{aligned}$$

Proposition 3.1. *The expressing of the product $\widetilde{q_1}\widetilde{q_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ can be made as follows:*

(i)

$$\widetilde{A}(y) = \bigvee_{y = x_1x_2 - x_3x_4 - x_5x_6 - x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{A}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{B}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{C}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{D}_2(x_8)\Big)$$

(ii)

$$\widetilde{B}(y) = \bigvee_{y = x_1x_2 + x_3x_4 + x_5x_6 - x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{B}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{A}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{D}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{C}_2(x_8)\Big)$$

(iii)

$$\widetilde{C}(y) = \bigvee_{y=x_1x_2-x_3x_4+x_5x_6+x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{C}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{D}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{A}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{B}_2(x_8)\Big)$$

(iv)

$$\widetilde{D}(y) = \bigvee_{y=x_1x_2+x_3x_4-x_5x_6+x_7x_8} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{D}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{C}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{B}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{A}_2(x_8)\Big).$$

Proposition 3.2. *The expressing of the quotient* $\frac{\widetilde{q}_1}{\widetilde{q}_2} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ *can be made as follows:*

(i)

$$\widetilde{A}(y) = \bigvee_{y=\frac{x_1x_2+x_3x_4+x_5x_6+x_7x_8}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{A}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{B}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{C}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{D}_2(x_8)\Big)$$

(ii)

$$\widetilde{B}(y) = \bigvee_{y=\frac{-x_1x_4+x_2x_3-x_5x_8+x_6x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{B}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{A}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{D}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{C}_2(x_8)\Big)$$

(iii)

$$\widetilde{C}(y) = \bigvee_{y=\frac{-x_1x_6+x_3x_8+x_2x_5-x_4x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{C}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{D}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{A}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{B}_2(x_8)\Big)$$

(iv)

$$\widetilde{D}(y) = \bigvee_{y=\frac{-x_1x_8-x_3x_6+x_4x_5+x_2x_7}{x_2^2+x_4^2+x_6^2+x_8^2}} \Big(\widetilde{A}_1(x_1) \wedge \widetilde{D}_2(x_2) \wedge \widetilde{B}_1(x_3) \wedge \wedge \widetilde{C}_2(x_4) \wedge \widetilde{C}_1(x_5) \wedge \widetilde{B}_2(x_6) \wedge \widetilde{D}_1(x_7) \wedge \widetilde{A}_2(x_8)\Big).$$

4. Matrix representations of fuzzy quaternion numbers

Just as the quaternion numbers are represented as matrices (Moţ & Popa, 2014), so the fuzzy quaternion numbers have a matrix representations.

The first way in which we can represent the fuzzy quaternion numbers as a matrix is to use 2×2 fuzzy complex matrices and the representation given is one of a family of linearly related representations.

For this, we denote:

$$\begin{aligned} \mathbb{H}_{1F} &= \left\{ Q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \middle| z_1, z_2 \in \mathbb{C}_F \right\} = \\ &= \left\{ Q = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix} \middle| \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F \right\}. \end{aligned} \tag{4.1}$$

Operations of addition and multiplication in $\mathbb{H}_{1F}$ are made according to the rules of addition and multiplication of matrices.

For Q_1 and Q_2 in $\mathbb{H}_{1F}$ we have:

$$\begin{aligned} Q_1 + Q_2 &= \begin{pmatrix} \widetilde{A}_1 + \widetilde{D}_1 i & \widetilde{C}_1 + \widetilde{B}_1 i \\ -\widetilde{C}_1 + \widetilde{B}_1 i & \widetilde{A}_1 - \widetilde{D}_1 i \end{pmatrix} + \begin{pmatrix} \widetilde{A}_2 + \widetilde{D}_2 i & \widetilde{C}_2 + \widetilde{B}_2 i \\ -\widetilde{C}_2 + \widetilde{B}_2 i & \widetilde{A}_2 - \widetilde{D}_2 i \end{pmatrix} = \\ &= \begin{pmatrix} \widetilde{A}_1 + \widetilde{A}_2 + (\widetilde{D}_1 + \widetilde{D}_2) i & \widetilde{C}_1 + \widetilde{C}_2 + (\widetilde{B}_1 + \widetilde{B}_2) i \\ -(\widetilde{C}_1 + \widetilde{C}_2) + (\widetilde{B}_1 + \widetilde{B}_2) i & \widetilde{A}_1 + \widetilde{A}_2 - (\widetilde{D}_1 + \widetilde{D}_2) i \end{pmatrix}, \end{aligned}$$

where $\widetilde{A}_1 + \widetilde{A}_2$, $\widetilde{B}_1 + \widetilde{B}_2$, $\widetilde{C}_1 + \widetilde{C}_2$ and $\widetilde{D}_1 + \widetilde{D}_2$ were defined in previous section. The product of two matrices Q_1 and Q_2 also follows the usual definition for matrix multiplication.

It is easy to verify that the $\mathbb{H}_{1F}$ is closed under the operation of addition and multiplication. Moreover, any matrix of $\mathbb{H}_{1F}$ admits the opposed matrix, namely

$$-Q = \begin{pmatrix} -z_1 & -z_2 \\ \bar{z}_2 & -\bar{z}_1 \end{pmatrix} = \begin{pmatrix} -\widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i \\ \widetilde{C} - \widetilde{B}i & -\widetilde{A} + \widetilde{D}i \end{pmatrix},$$

which belongs to $\mathbb{H}_{1F}$ as well. For any non null matrix of $\mathbb{H}_{1F}$ we have that:

$$\begin{vmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{vmatrix} = \begin{vmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{vmatrix} = \widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2,$$

which is equal to zero, if and only if $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$. It result that any non null matrix of $\mathbb{H}_{1F}$ admits the inverse matrix, namely:

$$Q^{-1} = \frac{1}{\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2} \begin{pmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{pmatrix} = \frac{1}{\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2} \begin{pmatrix} \widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i \\ \widetilde{C} - \widetilde{B}i & \widetilde{A} + \widetilde{D}i \end{pmatrix}$$

Therefore $\mathbb{H}_{1F}$ has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{1F}$ where $\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1$ and $\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{array}{llll} \widetilde{A}_1 = [1,2,4] & \widetilde{B}_1 = [0,1,3] & \widetilde{C}_1 = [1,2,3] & \widetilde{D}_1 = [0,2,4] \\ \widetilde{A}_2 = [2,4,6] & \widetilde{B}_2 = [2,4,5] & \widetilde{C}_2 = [0,1,4] & \widetilde{D}_2 = [1,3,4] \end{array}$$

We have

$$\widetilde{Q}_1\widetilde{Q}_2 = \begin{pmatrix} [-41,-4,24] + [-9,21,55]i & [-18,5,46] + [-10,8,53]i \\ [-46,-5,18] + [-10,8,53]i & [-41,-4,24] + [-55,-21,9]i \end{pmatrix}$$

and

$$\widetilde{Q}_2\widetilde{Q}_1 = \begin{pmatrix} [-41,-4,24] + [-14,7,50]i & [-10,15,54] + [-13,16,50]i \\ [-54,-15,10] + [-13,16,50]i & [-41,-4,24] + [-50,-7,14]i \end{pmatrix}$$

Based on the previous results we obtain the following theorem:

Theorem 4.1. *$\mathbb{H}_{1F}$ has a noncommutative field structure.*

Theorem 4.2. *$\mathbb{H}_F$ and $\mathbb{H}_{1F}$ are isomorphic fields.*

Proof. We consider the mapping $\varphi : \mathbb{H}_F \to \mathbb{H}_{1F}$,

$$\varphi(\widetilde{q}) = \varphi(\widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k) = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix},$$

which is a bijective application and it mantains the operations:

(i) $\varphi(\widetilde{q}_1 + \widetilde{q}_2) = \varphi(\widetilde{q}_1) + \varphi(\widetilde{q}_2),\ \forall \widetilde{q}_1, \widetilde{q}_2 \in \mathbb{H}_F$

(ii) $\varphi(\widetilde{q}_1 \cdot \widetilde{q}_2) = \varphi(\widetilde{q}_1) \cdot \varphi(\widetilde{q}_2),\ \widetilde{q}_1, \widetilde{q}_2 \in \mathbb{H}_F$

Hence it is a isomorphism of fields. Thus $\mathbb{H}_F \simeq \mathbb{H}_{1F}$. □

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{1F}$ allows us to state that each fuzzy quaternion number of $\mathbb{H}_F$ admits a matrix representation under the form of the elements of $\mathbb{H}_{1F}$, namely $\begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix}$.

Proposition 4.1. *Any element of $\mathbb{H}_{1F}$ admits:*

$$Q = \widetilde{A}\boldsymbol{1} + \widetilde{B}I + \widetilde{C}J + \widetilde{D}k, \tag{4.2}$$

where $\boldsymbol{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ represent the matrix quaternion units.

Proof. It is verified by direct calculus:

$$Q = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{C} + \widetilde{B}i \\ -\widetilde{C} + \widetilde{B}i & \widetilde{A} - \widetilde{D}i \end{pmatrix} = \widetilde{A}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \widetilde{B}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \\ + \widetilde{C}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \widetilde{D}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \widetilde{A}\mathbf{1} + \widetilde{B}I + \widetilde{C}J + \widetilde{D}k.$$

□

Remark. The matrix quaternion units I, J, K can be written with the help of Pauli matrix $\sigma_x, \sigma_y, \sigma_z$, namely:

$$I = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_x, \; J = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_y, \; K = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_z.$$

Remark. If constraining any two of $\widetilde{B}$, $\widetilde{C}$ and $\widetilde{D}$ to zero it results a representation of fuzzy complex numbers:

i) If $\widetilde{B} = \widetilde{C} = \widetilde{0}$, then it results a diagonal fuzzy complex matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix} \widetilde{A} + \widetilde{D}i & \widetilde{0} \\ \widetilde{0} & \widetilde{A} - \widetilde{D}i \end{pmatrix}.$$

ii) If $\widetilde{B} = \widetilde{D} = \widetilde{0}$, then it results a fuzzy real matrix representation of fuzzy complex numbers:

$$Q = \begin{pmatrix} \widetilde{A} & \widetilde{C} \\ -\widetilde{C} & \widetilde{A} \end{pmatrix}.$$

Proposition 4.2. *The conjugate of a fuzzy quaternion corresponds to the Hermitian transpose (conjugate transpose) of the matrix*

$$Q^* = (\overline{Q})^T = \overline{Q^T} = \begin{pmatrix} \widetilde{A} - \widetilde{D}i & -\widetilde{C} - \widetilde{B}i \\ \widetilde{C} - \widetilde{B}i & \widetilde{A} + \widetilde{D}i \end{pmatrix},$$

where Q^T denotes the transpose of Q and $\overline{Q}$ denotes the matrix with complex conjugated entries.

The second way in which we can represent the fuzzy quaternion numbers as a matrix is to use 4×4 fuzzy real matrices and the representation given is one of a family of linearly related representations.

In order to obtain such a representation we consider:

$$\mathbb{H}_{2F} = \left\{ Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} \middle| \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D} \in \mathbb{R}_F \right\}.$$

Operations of addition and multiplication in $\mathbb{H}_{2F}$ is made according to the rules of addition and multiplication of matrices.

It is easy to verify that the set of matrices $\mathbb{H}_{2F}$ is closed under the operation of addition and multiplication of matrices. Moreover, any matrix of $\mathbb{H}_{2F}$, admits the opposed matrix and any non null matrix of $\mathbb{H}_{2F}$ admits its inverse. Indeed, as

$$\begin{vmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{vmatrix} = (\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2)^2,$$

is equal to zero, if and only if $\widetilde{A} = \widetilde{B} = \widetilde{C} = \widetilde{D} = 0$, it results that for any non null matrix of $\mathbb{H}_{2F}$, there exists the inverse matrix:

$$Q_1^{-1} = \frac{1}{(\widetilde{A}^2 + \widetilde{B}^2 + \widetilde{C}^2 + \widetilde{D}^2)^2} \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & -\widetilde{D} & -\widetilde{C} \\ -\widetilde{C} & -\widetilde{D} & \widetilde{A} & -\widetilde{B} \\ -\widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}.$$

Thus $\mathbb{H}_{2F}$ has a field structure.

For commutativity we give the following counterexample: let $Q_1, Q_2 \in \mathbb{H}_{2F}$ where $\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \widetilde{D}_1$ and $\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{D}_2$ are fuzzy triangular numbers defined by:

$$\begin{aligned} &\widetilde{A}_1 = [0,2,5], && \widetilde{B}_1 = [1,3,5], && \widetilde{C}_1 = [0,1,4], && \widetilde{D}_1 = [0,2,5] \\ &\widetilde{A}_2 = [1,2,3], && \widetilde{B}_2 = [0,3,4], && \widetilde{C}_2 = [2,4,5], && \widetilde{D}_2 = [0,2,4]. \end{aligned}$$

We have

$$\widetilde{Q}_1\widetilde{Q}_2 = \begin{pmatrix} [-60,-13,15] & [-51,-6,24] & [-57,-10,20] & [-60,-17,14] \\ [-24,6,51] & [-60,-13,15] & [-60,-17,14] & [-20,10,57] \\ [-20,10,57] & [-14,17,60] & [-60,-13,15] & [-51,-6,24] \\ [-14,17,60] & [-57,-10,20] & [-24,6,51] & [-60,-13,15] \end{pmatrix}$$

and

$$\widetilde{Q}_2\widetilde{Q}_1 = \begin{pmatrix} [-60,-13,15] & [-60,-18,15] & [-57,-10,20] & [-49,1,25] \\ [-15,18,60] & [-60,-13,15] & [-49,1,25] & [-20,10,57] \\ [-20,10,57] & [-25,-1,49] & [-60,-13,15] & [-60,-18,15] \\ [-25,-1,49] & [-57,-10,20] & [-15,18,60] & [-60,-13,15] \end{pmatrix}.$$

Based on the previous results we obtain the following theorem:

Theorem 4.3. *$\mathbb{H}_{2F}$ has a noncommutative field structure.*

Theorem 4.4. *$\mathbb{H}_F$ and $\mathbb{H}_{2F}$ are isomorphic fields.*

Proof. We consider the mapping $\psi : \mathbb{H}_F \to \mathbb{H}_{2F}$,

$$\psi(\widetilde{q}) = \varphi(\widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k) = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix}.$$

We note that ψ is bijective and preserves the operations - it follows immediately by direct calculation, thus it is a isomorphism of fields. Therefore $\mathbb{H}_F \simeq \mathbb{H}_{2F}$. □

Remark. The isomorphism $\mathbb{H}_F \simeq \mathbb{H}_{2F}$ allows us to stat that each fuzzy quaternion number of $\mathbb{H}_F$ admits a matrix representation under the form of the elements of $\mathbb{H}_{2F}$.

Remark. Any element of $\mathbb{H}_{2F}$ can be written

$$Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ \widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ \widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A}\begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B}\begin{pmatrix} \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C}\begin{pmatrix} \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. Similarly, a fuzzy quaternion number $\widetilde{q} = \widetilde{A} + \widetilde{B}i + \widetilde{C}j + \widetilde{D}k$ can be represented as

$$Q_2 = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & -\widetilde{D} & \widetilde{C} \\ -\widetilde{C} & \widetilde{D} & \widetilde{A} & -\widetilde{B} \\ -\widetilde{D} & -\widetilde{C} & \widetilde{B} & \widetilde{A} \end{pmatrix} = \widetilde{A}\begin{pmatrix} \widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \end{pmatrix} + \widetilde{B}\begin{pmatrix} \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & -\widetilde{1} \\ \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \end{pmatrix} +$$

$$+\widetilde{C}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & -\widetilde{1} & \widetilde{0} & \widetilde{0} \end{pmatrix} + \widetilde{D}\begin{pmatrix} \widetilde{0} & \widetilde{0} & \widetilde{0} & \widetilde{1} \\ \widetilde{0} & \widetilde{0} & -\widetilde{1} & \widetilde{0} \\ \widetilde{0} & \widetilde{1} & \widetilde{0} & \widetilde{0} \\ -\widetilde{1} & \widetilde{0} & \widetilde{0} & \widetilde{0} \end{pmatrix}.$$

Remark. The 48 possible matrix representation of fuzzy quaternion numbers are in fact the matrices whose transpose is its negation (skew-symmetric matrices, i.e. $-Q = Q^T$).

Proposition 4.3. *In this representations, the conjugate of a fuzzy quaternion number corresponds to the transpose of the matrix Q_n, $n = \overline{1,48}$*

$$\overline{q} = \widetilde{A} - \widetilde{B}i - \widetilde{C}j - \widetilde{D}k = (Q_n)^T.$$

For example, for the representation Q_1 of a fuzzy quaternion number, we have

$$\overline{q} = (Q_1)^T = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{C} & \widetilde{D} \\ -\widetilde{B} & \widetilde{A} & \widetilde{D} & -\widetilde{C} \\ -\widetilde{C} & -\widetilde{D} & \widetilde{A} & \widetilde{B} \\ -\widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}.$$

Or, for second representation Q_2,

$$\overline{q} = (Q_2)^T = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & -\widetilde{C} & -\widetilde{D} \\ \widetilde{B} & \widetilde{A} & \widetilde{D} & -\widetilde{C} \\ \widetilde{C} & -\widetilde{D} & \widetilde{A} & \widetilde{B} \\ \widetilde{D} & \widetilde{C} & -\widetilde{B} & \widetilde{A} \end{pmatrix}$$

Remark. If $\widetilde{C} = \widetilde{D} = \widetilde{0}$ then $\widetilde{q} = \widetilde{A} + \widetilde{B}i$ and it results the representation of fuzzy complex numbers as diagonal matrices with two 2×2 blocks.

For example, for the representation Q_1 when $\widetilde{C} = \widetilde{D} = \widetilde{0}$ it results

$$Q_1 = \begin{pmatrix} \widetilde{A} & -\widetilde{B} & \widetilde{0} & \widetilde{0} \\ \widetilde{B} & \widetilde{A} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{A} & -\widetilde{B} \\ \widetilde{0} & \widetilde{0} & \widetilde{B} & \widetilde{A} \end{pmatrix}$$

and for Q_2 we obtain

$$Q_2 = \begin{pmatrix} \widetilde{A} & \widetilde{B} & \widetilde{0} & \widetilde{0} \\ -\widetilde{B} & \widetilde{A} & \widetilde{0} & \widetilde{0} \\ \widetilde{0} & \widetilde{0} & \widetilde{A} & -\widetilde{B} \\ \widetilde{0} & \widetilde{0} & \widetilde{B} & \widetilde{A} \end{pmatrix}.$$

References

Chang, T.H. and T.C. Wang (2009). Using the fuzzy multi-criteria decision making approach for measuring the possibility of successful knowledge management. *Information Sciences* **179**, 355–370.

Das, R.K. and B. Mandal (2002). Fuzzy real line structure and metric space. *Indian J. Pure Appl. Math.* **33**(4), 565–571.

Dubois, D. and H. Prade (1980). *Fuzzy sets and systems: theory and aplications*. Academic Press, Inc.

Dzitac, I. (2015). The fuzzification of classical structures: A general view. *International Journal of Computers Communications & Control* **10**(6), 772–788.

Elomda, B.M. and H.A. Hefny (2013). An extension of fuzzy decision maps for for multi-criteria decision-making. *Egyptian Informatics Journal* **14**, 147–155.

Felbin, C. (1992). Finie dimensional fuzzy normed liniar space. *Fuzzy Sets and Systems* **48**, 239–248.

Fu, X. and Q. Shen (2011). Fuzzy complex numbers and their application for classifiers performance evaluation. *Pattern Recognition* **44**(7), 1403–1417.

Hanss, M. (2005). *Applied fuzzy arithmetic: an introduction with engineering applications*. Berlin Heidelberg: Springer-Verlag.

Janfada, M., H. Baghani and O. Baghani (2011). On felbin's-type fuzzy normed linear spaces and fuzzy bounded operators. *Iranian Journal of Fuzzy Systems* **8**(5), 117–130.

Kaleva, O. and S. Seikkala (1984). On fuzzy metric spaces. *Fuzzy Sets and Systems* **12**, 215–229.

Mizumoto, M. and J. Tanaka (1979). Some properties of fuzzy numbers. *Advances in Fuzzy Set theory and Applications (North-Holland, New York)* pp. 153–164.

Moţ, G. and L. Popa (2014). On quaternions. In: *Proc. of the Int. Symp. Research and Education in Innovation Era,5th Edition*. pp. 70–77.

Moura, R.P., F. Bergamaschi, R. Santiago and B. Bedregal (2014). Rotation of triangular fuzzy numbers via quaternion. In: *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*. pp. 2538–2543.

Moura, R.P., F.B. Bergamaschi, R.H.N. Santiago and B. Bedregal (2013). Fuzzy quaternion numbers. In: *FUZZ-IEEE*. pp. 1–6.

Nădăban, S., S. Dzitac and I. Dzitac (2016). Fuzzy topsis: A general view. *Procedia Computer Science* **91**, 823–831.

Sida, L., L. Popa and S. Nădăban (2016). On fuzzy quaternion numbers. In: *Proc. of the Int. Symp. Research and Education in Innovation Era,6th Edition*. pp. 116–119.

Xiao, J.Z. and X.H. Zhu (2002). On linearly topological structures and property of fuzzy normed linear space. *Fuzzy Sets and Systems* **125**(2), 153–161.

Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 72 – 77

Fuzzy M-Open Sets

Talal Al-Hawary[a,*]

[a] *Yarmouk University, Department of Mathematics, Irbid, Jordan.*

Abstract

In this paper, we introduce the relatively new notion of fuzzy $M-$open subset which is strictly weaker than fuzzy open. We prove that the collection of all fuzzy $M-$open subsets of a fuzzy space forms a fuzzy topology that is finer than the original one. Several characterizations and properties of this class are also given as well as connections to other well-known "fuzzy generalized open" subsets.

Keywords: Fuzzy M-open, Fuzzy countable set, Fuzzy anti locally countable space.
2010 MSC: 54C08, 54H40.

1

1. Introduction

Fuzzy topological spaces were first introduced by (Chakraborty & Ahsanullah, 1992; Chang, 1968). Let $(X, \mathfrak{T})$ be a fuzzy topological space (simply, Fts). If λ is a fuzzy set (simply, F-set), then the closure of λ, the interior of λ and the derived set of λ will be denoted by $Cl_{\mathfrak{T}}(\lambda)$, $Int_{\mathfrak{T}}(\lambda)$ and $d_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use $\overline{\lambda}$, $\overset{o}{\lambda}$ and $\acute{\lambda}'$ instead, respectively. A F-set λ is called *F-semi-open* (simply, *FSO*) (Mahmoud *et al.*, 2004) if there exists a fuzzy open (simply, F-open) set μ such that $\mu \leq \lambda \leq Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a FSO-set if and only if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda))$. A complement of a FSO-set is called *F-semi-closed* (simply, *FSC*). The fuzzy semi-interior of λ is the union of all fuzzy semi-open subsets contained in λ and is denoted by $sInt(\lambda)$. λ is called *fuzzy preopen* (simply, *FPO*) if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$.$\lambda$ is called fuzzy $\alpha-open$ if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and fuzzy $\beta-open$ if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda)))$. Finally, λ is called *fuzzy regular-open* (simply, *FRO*) if $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Complements of FRO-sets are called *fuzzy regular-closed* (simply, *FRC*). The collection of all FSO (resp., FPO, FRO, FRC, F$\alpha - open$ and F$\beta - open$) subsets of

*Corresponding author
Email address: talalhawary@yahoo.com (Talal Al-Hawary)

[1] This work has been done during the author's sabbatical leave at Jordan University of Science and Technology—Jordan.

X is denoted by $SO(X, \mathfrak{T})$ (resp., F$PO(X, \mathfrak{T})$, F$RO(X, \mathfrak{T})$, F$RC(X, \mathfrak{T})$, F$\alpha(X, \mathfrak{T})$ and F$\beta(X, \mathfrak{T})$). We remark that F$\alpha(X, \mathfrak{T})$ is a topological space and F$\alpha(X, \mathfrak{T}) = FSO(X, \mathfrak{T}) \wedge FPO(X, \mathfrak{T})$. A fuzzy space $(X, \mathfrak{T})$ is called *locally countable (P-space, anti locally countable,* respectively*)* if each $\lambda \in X$ has a countable neighborhood (countable intersections of fuzzy open subsets are fuzzy open, non-empty fuzzy open subsets are uncountable, respectively). For more on the preceding notions, the reader is referred to (Al-Hawary, 2017, 2008; Chakraborty & Ahsanullah, 1992; Chang, 1968; Chaudhuri & Das, 1993; Mahmoud *et al.*, 2004; Wong, 1974).

In this paper, we introduce the relatively new notions of FMO, which is weaker than the class of fuzzy open subsets. In section 2, we also show that the collection of all FMO subsets of a space $(X, \mathfrak{T})$ forms a fuzzy topology that is finer than $\mathfrak{T}$ and we investigate the connection of FMO notion to other classes of "fuzzy generalized open" subsets as well as several characterizations of FMO and fuzzy $M-$closed notions via the operations of interior and closure. In section 3, several interesting properties and constructions of FMO subsets are discussed in the case of anti locally countable spaces.

2. Fuzzy *M*-open set

We begin this section by introducing the notion of FMO and fuzzy $M-$closed subsets.

Definition 2.1. A fuzzy subset λ of a space $(X, \mathfrak{T})$ is called FMO (simply, FMO) if for every $\upsilon \leq \lambda$, there exists an open fuzzy subset $\omega \,\epsilon X$ such that $\upsilon \leq \omega$ and such that $\omega \backslash sInt(\lambda)$ is countable. The complement of a FMO subset is called fuzzy *$M-$closed* (simply, FMC).

Clearly every FO-set is FMO, but the converse needs not be true.

Example 2.1. Let $X = \{a, b\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}\}$. Set $\lambda = \chi_{\{b\}}$. Then λ is FMO but not FO.

Next, we show that the collection of all FMO subsets of a space $(X, \mathfrak{T})$ forms a fuzzy topology $\mathfrak{T}_M$ that contains $\mathfrak{T}$.

Theorem 2.1. *If $(X, \mathfrak{T})$ is a fuzzy space, then $(X, \mathfrak{T}_M)$ is a space such that $\mathfrak{T} \leq \mathfrak{T}_M$.*

Proof. We only need to show $(X, \mathfrak{T}_M)$ is a space. Clearly since 0 and 1 are FO-sets, they are FMO. If $\lambda, \psi \in \mathfrak{T}_M$ and $\upsilon \leq \lambda \wedge \psi$, then there exist FO-sets ω, υ in X both containing λ such that $\omega \backslash sInt(\lambda)$ and $\upsilon \backslash sInt(\psi)$ are countable. Now $\lambda \leq \omega \wedge \upsilon$ and for every $\delta \leq (\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) = (\omega \wedge \upsilon) \backslash (sInt(\lambda) \wedge sInt(\psi))$ either $\delta \leq \omega \backslash sInt(\lambda)$ or $\delta \leq \upsilon \backslash sInt(\psi)$. Thus $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) \leq \omega \backslash sInt(\lambda)$ or $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) \leq \omega \backslash sInt(\psi)$ and thus $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi)$ is countable. Therefore, $\lambda \wedge \psi \in \mathfrak{T}_M$. If $\{\lambda_\alpha : \alpha \in \Delta\}$ is a collection of FMO subsets of X, then for every $\lambda \leq \vee\{\lambda_\alpha : \alpha \in \nabla\}$, $\upsilon \leq \lambda_\beta$ for some $\beta \in \Delta$. Hence there exists a FO-set ω of X containing λ such that

$\omega \backslash sInt(\lambda)$ is countable. Now as $\omega \backslash sInt(\vee\{\lambda_\alpha : \alpha \in \nabla\}) \leq \omega \backslash \vee \{sInt(\lambda_\alpha) : \alpha \in \nabla\} \leq \omega \backslash sInt(\lambda)$, $\omega \backslash sInt(\vee\{\lambda_\alpha : \alpha \in \nabla\})$ is countable and hence $\vee\{\lambda_\alpha : \alpha \in \nabla\} \in \mathfrak{T}_M$. □

Corollary 2.1. *If $(X, \mathfrak{T})$ is a p-space, then $\mathfrak{T} = \mathfrak{T}_M$.*

Next we show that FMO notion is independent of both FPO and FSO notions.

Example 2.2. Consider $\mathbb{R}$ with the standard fuzzy topology. Then $\chi_{\mathbb{Q}}$ is FPO but not FMO. Also $\chi_{[0,1]}$ is FSO but not FMO.

Example 2.3. In Example 2.1, $\chi_{\{b\}}$ is FMO but neither FPO nor FO.

Next we characterize $\mathfrak{T}_M$ when X is a locally countable fuzzy space.

Theorem 2.2. *If $(X, \mathfrak{T})$ is a locally countable fuzzy space, then $\mathfrak{T}_M$ is the discrete fuzzy topology.*

Proof. Let λ be a fF-set in X and $\upsilon \leq \lambda$. Then there exists a countable neighborhood ω of λ and hence there exists a FO-set η containing λ such that $\eta \leq \omega$. Clearly $\eta \backslash sInt(\lambda) \leq \omega \backslash sInt(\lambda) \leq \omega$ and thus $\eta \backslash sInt(\lambda)$ is countable. Therefore λ is FMO and so $\mathfrak{T}_M$ is the discrete fuzzy topology. □

Corollary 2.2. *If $(X, \mathfrak{T})$ is a countable fuzzy space, then $\mathfrak{T}_M$ is the discrete fuzzy topology.*

The following result, in which a new characterization of FMO subsets is given, will be a basic tool throughout the rest of the paper.

Lemma 2.1. *A subset λ of a fuzzy space X is FMO if and only if for every $\upsilon \leq \lambda$, there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $\upsilon \leq \lambda$, then there exists a FO- subset ω containing λ such that $\omega \backslash sInt(\lambda)$ is countable. Let $\pi = \omega \backslash sInt(\lambda) = \omega \wedge (X \backslash sInt(\lambda))$. Then $\omega - \pi \leq sInt(\lambda)$. □

Conversely, let $\upsilon \leq \lambda$. Then there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$. Thus $\omega \backslash sInt(\lambda) = \pi$ is countable.

The next result follows easily from the definition and the fact that the intersection of fuzzy M-closed sets is again fuzzy M-closed.

Lemma 2.2. *A subset λ of a fuzzy space X is fuzzy M-closed if and only if $Cl_M(\lambda) = \lambda$.*

We next study restriction and deletion operations.

Theorem 2.3. *If λ is FMO subset of X, then $\mathfrak{T}_M|_\lambda \subseteq (\mathfrak{T}|_\lambda)_M$.*

Proof. Let $\rho \in \mathfrak{T}_M|_\lambda$. Then $\rho = \upsilon \wedge \lambda$ for some FMO subset υ. For every $\lambda \leq \rho$, there exist $\delta_\upsilon, \delta_\lambda \in \mathfrak{T}$ containing λ and countable sets γ_υ and γ_λ such that $\delta_\upsilon - \gamma_\upsilon \leq sInt(\upsilon)$ and $\delta_\lambda - \gamma_\lambda \leq sInt(\lambda)$. Therefore, $\upsilon \leq \lambda \wedge (\delta_\upsilon \wedge \delta_\lambda) \in \mathfrak{T}_\lambda$, $\gamma_\upsilon \vee \gamma_\lambda$ is countable and

$$\begin{aligned}
\lambda \wedge (\delta_\upsilon \wedge \delta_\lambda) - (\gamma_\upsilon \vee \gamma_\lambda) &\leq (\delta_\upsilon \wedge \delta_\lambda) \wedge (1 - \gamma_\upsilon) \wedge (1 - \gamma_\lambda) \\
&= (\delta_\upsilon - \gamma_\upsilon) \wedge (\delta_\lambda - \gamma_\lambda) \\
&\leq sInt(\upsilon) \wedge sInt(\lambda) \wedge \lambda \\
&= sInt(\upsilon \wedge \lambda) \wedge \lambda \\
&= sInt(\rho) \wedge \lambda \\
&\leq sInt_\lambda(\rho).
\end{aligned}$$

Therefore, $\rho \in (\mathfrak{T}|_\lambda)_M$. □

Corollary 2.3. *If λ is a FO subset of X, then $\mathfrak{T}_M|_\lambda \leq (\mathfrak{T}|_\lambda)_M$.*

In the next example, we show that if λ in the preceding Theorem is not FMO, then the result needs not be true.

Example 2.4. Consider $\mathbb{R}$ with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. Then λ is not FMO and so not FO. As $\chi_{(0,1)}$ is FMO, then $\theta = \chi_{(0,1)} \wedge \lambda \in \mathfrak{T}_M|_\lambda$ while if $\theta \in (\mathfrak{T}|_\lambda)_M$ then for every $\lambda \leq \theta$, there exists $\omega \in \mathfrak{T}|_\lambda$ and a countable $\delta \leq \lambda$ such that $\omega - \delta \leq sInt(\theta) = 0$. Thus $\omega \leq \delta$ and hence ω is countable which is a contradiction.

In the next example, we show that $(\mathfrak{T}|_\lambda)_M$ needs not be a subset of $\mathfrak{T}_M|_\lambda$.

Example 2.5. Consider $\mathbb{R}$ with the standard fuzzy topology, $\lambda = \chi_\mathbb{Q}$ and $\mu = \chi_{(0,2)}$. If $\mu \in \mathfrak{T}_M|_\lambda$, then $\mu = \delta \wedge \lambda$ for some $\delta \in \mathfrak{T}_M$ which is impossible as $\chi_{\sqrt{2}} \leq \delta - \lambda$. On the other hand to show $\mu \in (\mathfrak{T}|_\lambda)_M$, let $\upsilon \leq \mu$. If $\upsilon \leq \lambda$, pick $q_1, q_2 \leq \lambda$ such that $0 < q_1 < \upsilon < q_2 < 2$ and let $\omega = \chi_{(q_1,q_2)} \wedge \lambda$. Then $\lambda \leq \omega - 0 \leq \mu = sInt(\mu)$. Thus in both cases $\mu \in (\mathfrak{T}|_\lambda)_M$.

Theorem 2.4. *Let $(X, \mathfrak{T})$ be a fuzzy space and λ is FMC-set. Then $Cl_\mathfrak{T}(\lambda) \leq \gamma \vee \vartheta$ for some closed subset γ and a countable subset ϑ.*

Proof. Let λ be FMC-set. Then $1 - \lambda$ is FMO and hence for every $\lambda \leq 1 - \lambda$, there exists a FO-set ω containing λ and a countable set ϑ such that $\omega - \vartheta \leq sInt(1 - \lambda) \leq 1 - Cl_\mathfrak{T}(\lambda)$. Thus

$$Cl_\mathfrak{T}(\lambda) \leq 1 - (\omega - \vartheta) \leq 1 - (\omega \wedge (X - \vartheta)) \leq 1 \wedge ((1 - \omega) \vee \vartheta) \leq (1 - \omega) \vee \vartheta.$$

Letting $\gamma = 1 - \omega$. Then γ is closed such that $Cl_{\mathfrak{T}}(\lambda) \leq \gamma \vee \vartheta$.‘ □

3. Anti-locally countable fuzzy spaces

In this section, several interesting properties and constructions of FMO subsets are discussed in case of anti locally countable fuzzy spaces.

Theorem 3.1. *A fuzzy space* $(X, \mathfrak{T})$ *is anti locally countable if and only if* $(X, \mathfrak{T}_M)$ *is anti locally countable.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $\upsilon \leq \lambda$. Then by Lemma 2.1, there exists a FO- subset ω containing λ and a countable μ such that $\omega - \mu \leq sInt(\lambda)$. Hence $sInt(\lambda)$ is uncountable and so is λ. The converse follows from the fact that every FO-set is FMO. □

Corollary 3.1. *If* $(X, \mathfrak{T})$ *is anti locally countable fuzzy space and* λ *is FMO, then* $Cl_{\mathfrak{T}}(\lambda) = Cl_{\mathfrak{T}_M}(\lambda)$.

Proof. Clearly $Cl_{\mathfrak{T}_M}(\lambda) \leq Cl_{\mathfrak{T}}(\lambda)$. On the other hand, let $\lambda \leq Cl_{\mathfrak{T}}(\lambda)$ and μ be an FMO subset containing λ. Then by Lemma 2.1, there exists a FO- subset υ containing λ and a countable set η such that $\upsilon - \eta \leq sInt(\mu)$. Thus $(\upsilon - \eta) \wedge \lambda \leq sInt(\mu) \wedge \lambda$ and so $(\upsilon \wedge \lambda) - \eta \leq sInt(\mu) \wedge \lambda$. As $\lambda \in \upsilon$ and $\lambda \in Cl_{\mathfrak{T}}(\lambda)$, $\upsilon \wedge \lambda \neq 0$ and then as υ and λ are FMO, $\upsilon \wedge \lambda$ is FMO and as X is ani locally countable, $\upsilon \wedge \lambda$ is uncountable and so is $(\upsilon \wedge \lambda) - \eta$. Thus $\upsilon \wedge \lambda$ is uncountable as it contains the uncountable set $sInt(\mu) \wedge \lambda$. Therefore, $\mu \wedge \lambda \neq 0$ which means that $\lambda \in Cl_{\mathfrak{T}_M}(\lambda)$. □

By a similar argument, we can easily prove the following result:

Corollary 3.2. *If* $(X, \mathfrak{T})$ *is anti locally countable and* λ *is FMC, then* $Int_{\mathfrak{T}}(\lambda) = Int_{\mathfrak{T}_M}(\lambda)$.

Theorem 3.2. *Let* $(X, \mathfrak{T})$ *be an anti locally countable fuzzy space. Then* $F\alpha(X, \mathfrak{T}) \subseteq F\alpha(X, \mathfrak{T}_M)$.

Proof. If $\lambda \in F\alpha(X, \mathfrak{T})$, then $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.1, $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda))$ is FMC, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.2 again, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}_M}(\lambda)))$ which means $\lambda \in F\alpha(X, \mathfrak{T}_M)$. □

The converse of the preceding result needs not be true as shown next.

Example 3.1. Consider $\mathbb{R}$ with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. Then $\lambda \in F\alpha(\mathbb{R}, \mathfrak{T}_M)$ but $\lambda \notin F\alpha(\mathbb{R}, \mathfrak{T})$.

Similarly, one can show that in an anti locally countable fuzzy space, $F\beta(X, \mathfrak{T}_M) \leq F\beta(X, \mathfrak{T})$.

Theorem 3.3. *Let $(X, \mathfrak{T})$ be an anti locally countable fuzzy space. Then $d_{\mathfrak{T}}(\mu) = d_{\mathfrak{T}_M}(\mu)$ for every subset F-set μ.*

Proof. If $\lambda \leq d_{\mathfrak{T}}(\mu)$ and υ is any FMO subset containing λ, then there exists a FO- subset ω containing λ and a countable γ such that $\omega - \gamma \leq sInt(\upsilon) \leq \upsilon$. Thus $(\omega - \gamma) \wedge (\lambda - \{\lambda\}) \leq sInt(\upsilon) \wedge (\mu - \lambda) \leq \upsilon \wedge (\mu - \lambda)$ and as $\lambda \in d_{\mathfrak{T}}(\mu)$ and υ^o is open containing λ, we have $\upsilon^o \wedge (\mu - \lambda) \neq 0$ and so $\upsilon \wedge (\mu - \lambda) \neq 0$. Therefore $\lambda \in d_{\mathfrak{T}_M}(\mu)$. □

The converse is obvious as every FO subset is FMO.

Theorem 3.4. *Let $(X, \mathfrak{T})$ be an anti locally countable fuzzy space. Then $FRO(X, \mathfrak{T}) = FRO(X, \mathfrak{T}_M)$.*

Proof. If $\lambda \in FRO(X, \mathfrak{T})$, then $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ and by Corollary 3.1, $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(\lambda))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(\lambda)$ is FMC, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T}_M)$. Conversely, if $\lambda \in FRO(X, \mathfrak{T}_M)$, then $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$. Then as λ is FMO, by Corollary 3.1, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}}(\lambda))$ and as $Cl_{\mathfrak{T}}(\lambda)$ is FMC being a FC-set, then by $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T})$. □

The converse of the preceding result need not be true as shown next.

Example 3.2. Let $X = \{a, b, c, d, e\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}, \chi_{\{a,b,c,d\}}\}$. Then $(X, \mathfrak{T})$ is an anti locally countable fuzzy space such that $FRO(X, \mathfrak{T}) = \{0, 1\}$ while $FRO(X, \mathfrak{T}_M) = \mathfrak{T}$.

References

Al-Hawary, T. (2008). Fuzzy ω_0-open sets. *Bull. Korian Math. Soc.* **45**(4), 749–755.

Al-Hawary, T. (2017). Fuzzy l-closed sets, to appear in *MATEMATIKA*.

Chakraborty, M.K. and T.M.G. Ahsanullah (1992). Fuzzy topology on fuzzy sets and tolerance topology. *Fuzzy Sets and Systems* **45**(1), 103 – 108.

Chang, C.L (1968). Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications* **24**(1), 182 – 190.

Chaudhuri, A.K. and P. Das (1993). Some results on fuzzy topology on fuzzy sets. *Fuzzy Sets and Systems* **56**(3), 331 – 336.

Mahmoud, F.S., M.A. Fath Alla and S.M. Abd Ellah (2004). Fuzzy topology on fuzzy sets: fuzzy semicontinuity and fuzzy semiseparation axioms. *Applied Mathematics and Computation* **153**(1), 127 – 140.

Wong, C.K (1974). Fuzzy points and local properties of fuzzy topology. *Journal of Mathematical Analysis and Applications* **46**(2), 316 – 328.

Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 78 – 94

Decomposition of Continuity in a Fuzzy Sequential Topological Space

N. Tamang[a,*], S. De Sarkar[a]

[a] *Department of Mathematics, University of North Bengal, Dist. Darjeeling 734013, West Bengal, India*

Abstract

A new class of fuzzy sequential sets called fs-preopen sets is introduced and characterized. An fs-precontinuous mapping and strongly fs-precontinuous mapping are defined and studied. A necessary and sufficient condition under which an fs-preopen set is fs-open, has been established. Apart from that, different kinds of sets, namely, fs α-open set, fs δ-set, fs $\mathcal{A}$-set, locally fs-closed set, fs S-preopen set have been studied. The interrelationships between them are investigated. Using them and the associated continuities, various decompositions of fs-continuity have been established.

Keywords: Fuzzy sequential topological spaces, fs-preopen, fs α-open, fs δ-set, fs $\mathcal{A}$-set, locally fs-closed, fs S-preopen sets.
2010 MSC: 54A40, 03E72.

1. Introduction

In the last few decades, there has been interests in the study of generalized open sets and generalized continuity in topological spaces. Various authors studied different kinds of generalized open sets in topological spaces. In the fuzzy setting, fuzzy semi-open sets and fuzzy semicontinuity were introduced and studied by K. K. Azad (Azad, 1981), fuzzy pre-open sets (Mashhour *et al.*, 1982) by A. S. Mashhour, M. E. Abd El-Monsef and S. N. EI-Deeb. Again, fs-semiopen sets and fs-semicontinuity have been studied in (Tamang & Sarkar, 2016).

The purpose of this work is to study different generalized open sets and the associated continuous functions in a fuzzy sequential topological space.

Apart from the introduction in Section 1, Section 2 is devoted to the study of preopen sets and precontinuity in a fuzzy sequential topological space. An important result from this Section is a

*Corresponding author
Email addresses: nita_anee@yahoo.in (N. Tamang), suparnadesarkar@yahoo.co.in (S. De Sarkar)

necessary and sufficient condition for a preopen set to be open. Section 3 deals with the introduction and study of some more generalized open sets and their respective continuities. Finally the Section has been concluded with some decompositions of continuity.

Throughout the paper X will denote a non empty set and I the unit interval $[0, 1]$. Sequences of fuzzy sets in X called fuzzy sequential sets (fs-sets) will be denoted by the symbols $A_f(s)$, $B_f(s)$, $C_f(s)$, etc. An fs-set $X^l_f(s)$ is a sequence of fuzzy sets $\{X^n_f\}_n$ where $l \in I$ and $X^n_f(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,

A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties

(i) $X^r_f(s) \in \delta(s)$ for $r = 0$ and 1,

(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and

(iii) for any family $\{A_{fj}(s) \in \delta(s),\ j \in J\}$, $\bigvee_{j\in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS) (Singha *et al.*, 2014). The members of $\delta(s)$ are called open fuzzy sequential (fs-open) sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential (fs-closed) set in X. In an FSTS, the closure, interior, continuous functions, semiopen sets etc. are defined in the usual manner (See (Singha *et al.*, 2014), (Tamang & Sarkar, 2016), (Tamang *et al.*, 2016)).

2. FS-preopen sets and FS-precontinuity

Definition 2.1. (i) An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preopen set if $A_f(s) \leq (\overline{A_f(s)})^o$.
(ii) An fs-set $A_f(s)$ in an FSTS, is said to be an fs-preclosed set if its complement is fs-preopen or equivalently if $\overline{\overset{o}{A}_f(s)} \leq A_f(s)$.

If $A_f(s)$ is both fs-preopen and fs-preclosed, then it is called an fs-preclopen set.

Definition 2.2. An fs-set $A_f(s)$ is called fs-dense in an FSTS $(X, \delta(s))$, if $\overline{A_f(s)} = X^1_f(s)$.

Fundamental properties of fs-preopen (fs-preclosed) sets are:

- Every fs-open (fs-closed) set is fs-preopen (fs-preclosed).
- Arbitrary union (intersection) of fs-preopen (fs-preclosed) sets is fs-preopen (fs-preclosed).

Example 2.1 shows that an fs-preopen (fs-preclosed) set may not be fs-open (fs-closed), the intersection (union) of any two fs-preopen (fs-preclosed) sets need not be an fs-preopen (fs-preclosed) set. Unlike in a general topological space, the intersection of an fs-preopen set with an fs-open set may fail to be an fs-preopen set.

Example 2.1. Consider the fs-sets $A_f(s)$, $B_f(s)$ and $C_f(s)$ in $X = [0, 1]$, defined as follows:

$$\begin{aligned} A^1_f(x) &= 0, \text{ if } 0 \leq x \leq \frac{1}{2} \\ &= \frac{1}{4}, \text{ if } \frac{1}{2} < x \leq 1 \\ \text{and } A^n_f &= \overline{1} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} B_f^1(x) &= \frac{1}{2}, \text{ if } 0 \le x \le \frac{1}{2} \\ &= 0, \text{ if } \frac{1}{2} < x \le 1 \\ \text{and } B_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

$$\begin{aligned} C_f^1(x) &= \frac{3}{4}, \text{ if } 0 \le x \le \frac{1}{2} \\ &= 1, \text{ if } \frac{1}{2} < x \le 1 \\ \text{and } C_f^n &= \overline{0} \text{ for all } n \neq 1. \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Now,
(i) $A_f(s)$ and $C_f(s)$ are fs-preopen sets but their intersection is not fs-preopen.
(ii) $C_f(s)$ is fs-preopen but is not fs-open.

Theorem 2.1. *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preopen if and only if there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \le O_f(s) \le \overline{A_f(s)}$.*

Proof. Straightforward. □

Corollary 2.1. *Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is fs-preclosed if and only if there exists an fs-closed set $C_f(s)$ in X such that $\overset{o}{A}_f(s) \le C_f(s) \le A_f(s)$.*

Proof. Straightforward. □

Theorem 2.2. *An fs-set is fs-clopen (both fs-closed and fs-open) if and only if it is fs-closed and fs-preopen.*

Proof. Proof is omitted. □

Theorem 2.3. *In an FSTS, every fs-set is fs-preopen if and only if every fs-open set is fs-closed.*

Proof. Suppose every fs-set in an FSTS $(X, \delta(s))$, is fs-preopen and let $A_f(s)$ be an fs-open set. Then, $A_f^c(s) = \overline{A_f^c(s)}$ is fs-preopen and hence $\overline{A_f^c(s)} \le (\overline{\overline{A_f^c(s)}})^o = (\overline{A_f^c(s)})^o = (A_f^c(s))^o$. Thus, $A_f^c(s)$ is fs-open and hence $A_f(s)$ is fs-closed.

Conversely, suppose every fs-open set is fs-closed and let $A_f(s)$ be any fs-set. By the assumption, $\overline{A_f(s)} = (\overline{A_f(s)})^o$ and hence $A_f(s)$ is fs-preopen. □

Theorem 2.4. (a) *Closure of an fs-preopen set is fs-regular closed.*
(b) *Interior of an fs-preclosed set is fs-regular open.*

Proof. We prove only (a). Let $A_f(s)$ be an fs-preopen set in X. Since $(\overline{A_f(s)})^o \le \overline{A_f(s)}$, we have $\overline{(\overline{A_f(s)})^o} \le \overline{\overline{A_f(s)}} = \overline{A_f(s)}$. Now $A_f(s)$ being fs-preopen, $A_f(s) \le (\overline{A_f(s)})^o$ and hence $\overline{A_f(s)} \le \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ is fs-regular closed. □

The set of all fs-preopen sets in X, is denoted by $FSPO(X)$.

Theorem 2.5. *In an FSTS* $(X, \delta(s))$, (i) $\delta(s) \subseteq FSPO(X)$, (ii) *If* $V_f(s) \in FSPO(X)$ *and* $U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$, *then* $U_f(s) \in FSPO(X)$.

Proof. (i) Follows from definition.

(ii) Let $V_f(s) \in FSPO(X)$, that is, $V_f(s) \leq (\overline{V_f(s)})^o$. We have,

$$U_f(s) \leq V_f(s) \leq \overline{U_f(s)}$$

Therefore, $U_f(s) \leq V_f(s) \leq (\overline{V_f(s)})^o \leq (\overline{U_f(s)})^o$. Hence the result. □

Definition 2.3. An fs-set $A_f(s)$ in an FSTS, is called an fs-preneighbourhood of an fs-point $P_f(s) = (p^M_{fx}, r)$, if there exists an fs-preopen set $B_f(s)$ such that $P_f(s) \leq B_f(s) \leq A_f(s)$.

Theorem 2.6. *For an fs-set* $A_f(s)$ *in an FSTS* $(X, \delta(s))$, *the following are equivalent:*
(i) *$A_f(s)$ is fs-preopen.*
(ii) *There exists an fs-regular open set $B_f(s)$ containing $A_f(s)$ such that* $\overline{A_f(s)} = \overline{B_f(s)}$.
(iii) ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$.
(iv) *The semi-closure of $A_f(s)$ is fs-regular open.*
(v) *$A_f(s)$ is an fs-preneighbourhood of each of its fs-points.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be fs-preopen. This implies

$$\begin{aligned} & A_f(s) \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o} \leq \overline{A_f(s)} \\ \Rightarrow\ & \overline{(\overline{A_f(s)})^o} = \overline{A_f(s)} \\ \Rightarrow\ & \overline{A_f(s)} = \overline{B_f(s)} \end{aligned}$$

where $B_f(s) = (\overline{A_f(s)})^o$ is an fs-regular open set containing $A_f(s)$.

$(ii) \Rightarrow (iii)$ Let $\overline{A_f(s)} = \overline{B_f(s)}$, where $B_f(s)$ is an fs-regular open set containing $A_f(s)$. Then,

$$A_f(s) \leq B_f(s) = (\overline{B_f(s)})^o = (\overline{A_f(s)})^o$$

Also, $(\overline{A_f(s)})^o$ is fs-semiclosed. Let $C_f(s)$ be an fs-semiclosed set containing $A_f(s)$. Thus,

$$(\overline{A_f(s)})^o \leq (\overline{C_f(s)})^o \leq C_f(s).$$

Hence ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$.

$(iii) \Rightarrow (iv)$ Obvious.

$(iv) \Rightarrow (i)$ Suppose ${}_scl(A_f(s))$ is fs-regular open. Now,

$$\begin{aligned} & A_f(s) \leq {}_scl(A_f(s)) \\ \Rightarrow\ & (\overline{A_f(s)})^o \leq (\overline{{}_scl(A_f(s))})^o = {}_scl(A_f(s)) \leq (\overline{A_f(s)})^o \\ \Rightarrow\ & A_f(s) \leq {}_scl(A_f(s)) = (\overline{A_f(s)})^o \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

$(i) \Rightarrow (v)$ and $(v) \Rightarrow (i)$ are obvious. □

Corollary 2.2. *An fs-set is fs-regular open if and only if it is fs-semiclosed and fs-preopen.*

Proof. Proof is omitted. □

Theorem 2.7. *In an FSTS $(X, \delta(s))$, the following are equivalent:*
(i) $\overline{A_f(s)} \in \delta(s)$ *for all* $A_f(s) \in \delta(s)$.
(ii) *Every fs-regular closed set in X is fs-preopen.*
(iii) *Every fs-semiopen set in X is fs-preopen.*
(iv) *The closure of every fs-preopen set in X is fs-open.*
(v) *The closure of every fs-preopen set in X is fs-preopen.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be an fs-regular closed set, that is, $\overline{\overset{o}{A}_f(s)} = A_f(s)$. By (i), $A_f(s) \in \delta(s)$ and hence $A_f(s)$ is fs-preopen.

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-semiopen set, that is, $A_f(s) \leq \overline{\overset{o}{A}_f(s)}$. By (ii), $\overline{\overset{o}{A}_f(s)}$ is fs-preopen. Also, we have, $A_f(s) \leq \overline{\overset{o}{A}_f(s)} \leq \overline{A_f(s)}$. Thus, $A_f(s)$ is fs-preopen.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be an fs-preopen set, that is, $A_f(s) \leq (\overline{A_f(s)})^o$. This implies, $\overline{A_f(s)} \leq \overline{(\overline{A_f(s)})^o}$. Thus, $\overline{A_f(s)}$ being fs-semiopen, is fs-preopen and the result follows.

$(iv) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s) \in \delta(s)$. Then, $A_f(s)$ is fs-preopen and hence $\overline{A_f(s)}$ is fs-preopen. Therefore, $\overline{A_f(s)} \leq (\overline{A_f(s)})^o \leq \overline{A_f(s)}$ and hence $\overline{A_f(s)} \in \delta(s)$. □

Theorem 2.8. *In an FSTS $(X, \delta(s))$, the following are equivalent:*
(i) *Every non-zero fs-open set is fs-dense.*
(ii) *For every non-zero fs-preopen set $A_f(s)$, we have ${}_scl(A_f(s)) = X^1_f(s)$.*
(iii) *Every non-zero fs-preopen set is fs-dense.*

Proof. $(i) \Rightarrow (ii)$ Let $A_f(s)$ be a non-zero fs-preopen set. By Theorem 2.6 (iii), ${}_scl(A_f(s)) = (\overline{A_f(s)})^o$. Also, there exists an fs-open set $O_f(s)$ such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. By (i), $\overline{O_f(s)} = X^1_f(s)$. Therefore, $\overline{A_f(s)} = X^1_f(s)$ and hence ${}_scl(A_f(s)) = X^1_f(s)$.

$(ii) \Rightarrow (iii)$ Easy to prove.

$(iii) \Rightarrow (i)$ Since every fs-open set is fs-preopen, the proof is straightforward. □

Definition 2.4. Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. We define fs-preclosure ${}_pcl(A_f(s))$ and fs-preinterior ${}_pint(A_f(s))$ of $A_f(s)$ by

$$
\begin{aligned}
{}_pcl(A_f(s)) &= \wedge\{B_f(s); A_f(s) \leq B_f(s) \text{ and } B^c_f(s) \in FSPO(X)\} \\
{}_pint(A_f(s)) &= \vee\{C_f(s); C_f(s) \leq A_f(s) \text{ and } C_f(s) \in FSPO(X)\}
\end{aligned}
$$

Clearly, ${}_pcl(A_f(s))$ is the smallest fs-preclosed set containing $A_f(s)$ and ${}_pint(A_f(s))$ is the largest fs-preopen set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq {}_pcl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_pint(A_f(s)) \leq A_f(s)$.
(ii) $A_f(s)$ is fs-preopen if and only if $A_f(s) = {}_pint(A_f(s))$
(iii) $A_f(s)$ is fs-preclosed if and only if $A_f(s) = {}_pcl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_pint(A_f(s)) \leq {}_pint(B_f(s))$ and ${}_pcl(A_f(s)) \leq {}_pcl(B_f(s))$.

Definition 2.5. A mapping $g : (X, \delta(s)) \to (Y, \delta'(s))$ is said to be
(i) fs-precontinuous if $g^{-1}(B_f(s))$ is fs-preopen in X, for every $B_f(s) \in \delta'(s)$.
(ii) fs-preopen if $g(A_f(s))$ is fs-preopen in Y, for every $A_f(s) \in \delta(s)$.
(iii) fs-preclosed if $g(A_f(s))$ is fs-preclosed in Y, for every fs-closed set $A_f(s)$ in X.

It is easy to check that an fs-continuous (fs-open, fs-closed) function is fs-precontinuous (fs-preopen, fs-preclosed). That the converse may not be true, is shown by Example 2.2.

Example 2.2. Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\} \\ B_f(s) &= \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\} \\ C_f(s) &= \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Let $\delta'(s) = \{C_f(s), X_f^0(s), X_f^1(s)\}$ and define $g : (X, \delta(s)) \to (X, \delta'(s))$ by $g(x) = x$ for all $x \in X$. The function g is fs-precontinuous but not fs-continuous.

Again, the map $h : (X, \delta'(s)) \to (X, \delta(s))$ defined by $h(x) = x$ for all $x \in X$, is both fs-preopen and fs-preclosed but neither fs-open nor fs-closed.

Theorem 2.9. *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ be a map. Then the following conditions are equivalent:*
(i) *g is fs-precontinuous.*
(ii) *the inverse image of an fs-closed set in Y under g, is fs-preclosed in X.*
(iii) *For any fs-set $A_f(s)$ in X, $g({}_pcl(A_f(s))) \leq \overline{g(A_f(s))}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose g be an fs-precontinuous map and $B_f(s)$ be an fs-closed set in Y. Then,

$$\begin{aligned} & B_f^c(s) \text{ is fs-open in } Y \\ \Rightarrow\ & (g^{-1}(B_f(s)))^c = g^{-1}(B_f^c(s)) \text{ is fs-preopen in } X \\ \Rightarrow\ & g^{-1}(B_f(s)) \text{ is fs-preclosed in } X. \end{aligned}$$

$(ii) \Rightarrow (iii)$ Let $A_f(s)$ be an fs-set in X. Then, $g^{-1}(\overline{g(A_f(s))})$ is fs-preclosed in X and hence $g^{-1}(\overline{g(A_f(s))}) = {}_pcl(g^{-1}(\overline{g(A_f(s))}))$. Again,

$$\begin{aligned} & A_f(s) \leq g^{-1}(g(A_f(s))) \\ \Rightarrow\ & {}_pcl(A_f(s)) \leq {}_pcl(g^{-1}(g(A_f(s)))) \leq g^{-1}(\overline{g(A_f(s))}) \\ \Rightarrow\ & g({}_pcl(A_f(s))) \leq g(g^{-1}(\overline{g(A_f(s))})) \leq \overline{g(A_f(s))}. \end{aligned}$$

$(iii) \Rightarrow (i)$ Let $B_f(s)$ be an fs-open set in Y. Then for the fs-closed set $B_f^c(s)$, we have

$$g({}_pcl(g^{-1}(B_f^c(s)))) \leq \overline{g(g^{-1}(B_f^c(s)))} \leq \overline{B_f^c(s)} = B_f^c(s)$$

Thus, ${}_pcl(g^{-1}(B^c_f(s))) \leq g^{-1}(B^c_f(s))$. Therefore, ${}_pcl(g^{-1}(B^c_f(s))) = g^{-1}(B^c_f(s))$ and hence $(g^{-1}(B_f(s)))^c = g^{-1}(B^c_f(s))$ is fs-preclosed in X. □

In Theorem 2.8 (Tamang & Sarkar, 2016), it has been proved that the inverse image of an fs-semiopen set is fs-semiopen, under an fs-semicontinuous open map. The next Theorem shows that the result is true even if we take an fs-semicontinuous preopen map.

Theorem 2.10. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiopen set in Y under g, is fs-semiopen in X.*

Proof. Let $B_f(s)$ be an fs-semiopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & O_f(s) \leq B_f(s) \leq \overline{O_f(s)} \\ \Rightarrow \quad & g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq g^{-1}(\overline{O_f(s)}) \end{aligned}$$

We claim that $g^{-1}(\overline{O_f(s)}) \leq \overline{g^{-1}(O_f(s))}$. Let $P_f(s) \in g^{-1}(\overline{O_f(s)})$. This implies $g(P_f(s)) \in \overline{O_f(s)}$. Consider a weak open Q-nbd $U_f(s)$ of $P_f(s)$, then $\overline{g(U_f(s))}$ is a weak Q-nbd of $g(P_f(s))$. Therefore,

$$\begin{aligned} & \overline{g(U_f(s))} q_w O_f(s) \\ \Rightarrow \quad & W_f(s) q_w O_f(s) \text{ where } W_f(s) = \overline{g(U_f(s))} \\ \Rightarrow \quad & W^n_f(y) + O^n_f(y) > 1 \text{ for some } y \in Y \\ \Rightarrow \quad & O_f(s) \text{ is a weak open Q-nbd of the fs-point } (p^n_{fy}, W^n_f(y)) \\ \Rightarrow \quad & g(U_f(s)) q_w O_f(s) \\ \Rightarrow \quad & U_f(s) q_w g^{-1}(O_f(s)) \\ \Rightarrow \quad & P_f(s) \in \overline{g^{-1}(O_f(s))}. \end{aligned}$$

Thus $g^{-1}(O_f(s)) \leq g^{-1}(B_f(s)) \leq \overline{g^{-1}(O_f(s))}$. Since $g^{-1}(O_f(s))$ is fs-semiopen, $g^{-1}(B_f(s))$ is fs-semiopen. □

Corollary 2.3. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen mapping. Then the inverse image of every fs-semiclosed set in Y under g, is fs-semiclosed in X.*

Proof. The proof is omitted. □

Corollary 2.4. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-semicontinuous preopen map and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be fs-semicontinuous. Then hog is fs-semicontinuous.*

Proof. The proof is omitted. □

Theorem 2.11. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preopen set in Y under g, is fs-preopen in X.*

Proof. Let $B_f(s)$ be an fs-preopen set in Y. Then there exists an fs-open set $O_f(s)$ in Y such that

$$\begin{aligned} & B_f(s) \leq O_f(s) \leq \overline{B_f(s)} \\ \Rightarrow \quad & g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq g^{-1}(\overline{B_f(s)}). \end{aligned}$$

As in Theorem 2.10, we have $g^{-1}(\overline{B_f(s)}) \leq \overline{g^{-1}(B_f(s))}$. Thus $g^{-1}(B_f(s)) \leq g^{-1}(O_f(s)) \leq \overline{g^{-1}(B_f(s))}$, where $g^{-1}(O_f(s))$ is fs-preopen. Hence $g^{-1}(B_f(s))$ is fs-preopen. □

Corollary 2.5. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen mapping. Then the inverse image of every fs-preclosed set in Y under g, is fs-preclosed in X.*

Proof. The proof is omitted. □

Corollary 2.6. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-precontinuous preopen map and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be an fs-precontinuous map. Then hog is fs-precontinuous.*

Proof. The proof is omitted. □

Theorem 2.12. *Suppose $g : (X, \delta(s)) \to (Y, \eta(s))$ be an fs-continuous open map. Then the g-image of an fs-preopen set in X, is fs-preopen in Y.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. Then there exists an fs-open set $O_f(s)$ in X such that $A_f(s) \leq O_f(s) \leq \overline{A_f(s)}$. This implies $g(A_f(s)) \leq g(O_f(s)) \leq \overline{g(A_f(s))}$. Since $g(O_f(s))$ is fs-open in Y, $g(A_f(s))$ is fs-preopen. □

Corollary 2.7. *Pre-openness in an FSTS, is a topological property.*

Proof. Proof follows from Theorem 2.12. □

Theorem 2.13. *Let $g : (X, \delta(s)) \to (Y, \eta(s))$ and $h : (Y, \eta(s)) \to (Z, \delta'(s))$ be two mappings, such that hog is fs-preclosed. Then g is fs-preclosed if h is an injective fs-precontinuous preopen mapping.*

Proof. Let $A_f(s)$ be an fs-closed set in X. Then, $hog(A_f(s))$ is fs-preclosed in Z and hence $g(A_f(s)) = h^{-1}(hog(A_f(s)))$ is fs-preclosed in Y. □

Theorem 2.14. *If $g : (X, \delta(s)) \to (Y, \delta'(s))$ be fs-precontinuous and $h : (Y, \delta'(s)) \to (Z, \eta(s))$ be fs-continuous, then hog : $(X, \delta(s)) \to (Z, \eta(s))$ is fs-precontinuous.*

Proof. Omitted. □

Previously, we showed that the intersection of any two fs-preopen sets may not be fs-preopen and an fs-preopen set may not be fs-open. Now, we investigate and establish conditions, under which the intersection of any two fs-preopen sets is fs-preopen and conditions, under which an fs-preopen set is fs-open.

Theorem 2.15. *The intersection of any two fs-preopen sets is fs-preopen if the closure is preserved under finite intersection.*

Proof. Proof is simple and hence omitted. □

Theorem 2.16. *In an FSTS* $(X, \delta(s))$*, if every fs-set is either fs-open or fs-closed, then every fs-preopen set in X is fs-open.*

Proof. Let $A_f(s)$ be an fs-preopen set in X. If $A_f(s)$ is not fs-open, then it is fs-closed and hence $\overline{A_f(s)} = A_f(s)$. Therefore, $A_f(s) \leq (\overline{A_f(s)})^o = \overset{o}{A}_f(s)$ and hence the theorem. □

For a fuzzy sequential topological space $(X, \delta(s))$, $\delta^*(s)$ will denote the fuzzy sequential topology on X, obtained by taking $FSPO(X)$ as a subbase.

Definition 2.6. A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is called strongly fs-precontinuous if the inverse image of each fs-preopen set in Y is fs-open in X.

By the definition of a strong fs-precontinuous mapping, the following two results are obvious.

Theorem 2.17. (i) *A map* $g : (X, \delta(s)) \to (Y, \eta(s))$ *is strongly fs-precontinuous if and only if* $g : (X, \delta(s)) \to (Y, \eta^*(s))$ *is fs-continuous.*
(ii) *If* $g : (X, \delta(s)) \to (Y, \eta(s))$ *is strongly fs-precontinuous, then it is fs-continuous.*

Remark. Converse of (ii) of Theorem 2.17 may not be true, as is shown by the following Example.

Example 2.3. Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{1}, \overline{1}, ...\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{2}}, \overline{0}, \overline{0}, ...\right\}$$

$$C_f(s) = \left\{\overline{\frac{3}{8}}, \overline{1}, \overline{1}, ...\right\}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS. Consider the identity map $id : (X, \delta(s)) \to (X, \delta(s))$. Then, id is fs-continuous but not strongly fs-precontinuous, as the inverse image of fs-preopen set $C_f(s)$ is not fs-open.

We conclude the section with a necessary and sufficient condition for an fs-preopen set to be fs-open.

Theorem 2.18. *In an FSTS* $(Y, \eta(s))$*, the following are equivalent:*
(i) *Every fs-preopen set in Y is fs-open.*
(ii) *Every fs-continuous function* $g : (X, \delta(s)) \to (Y, \eta(s))$ *is strongly fs-precontinuous, where* $(X, \delta(s))$ *is any FSTS.*

Proof. $(i) \Rightarrow (ii)$ is straightforward.
$(ii) \Rightarrow (i)$ The identity map $g : (Y, \eta(s)) \to (Y, \eta(s))$ is fs-continuous and hence is strongly fs-precontinuous. Let $B_f(s)$ be an fs-preopen set in Y, then $B_f(s) = g^{-1}(B_f(s))$ is fs-open in Y. □

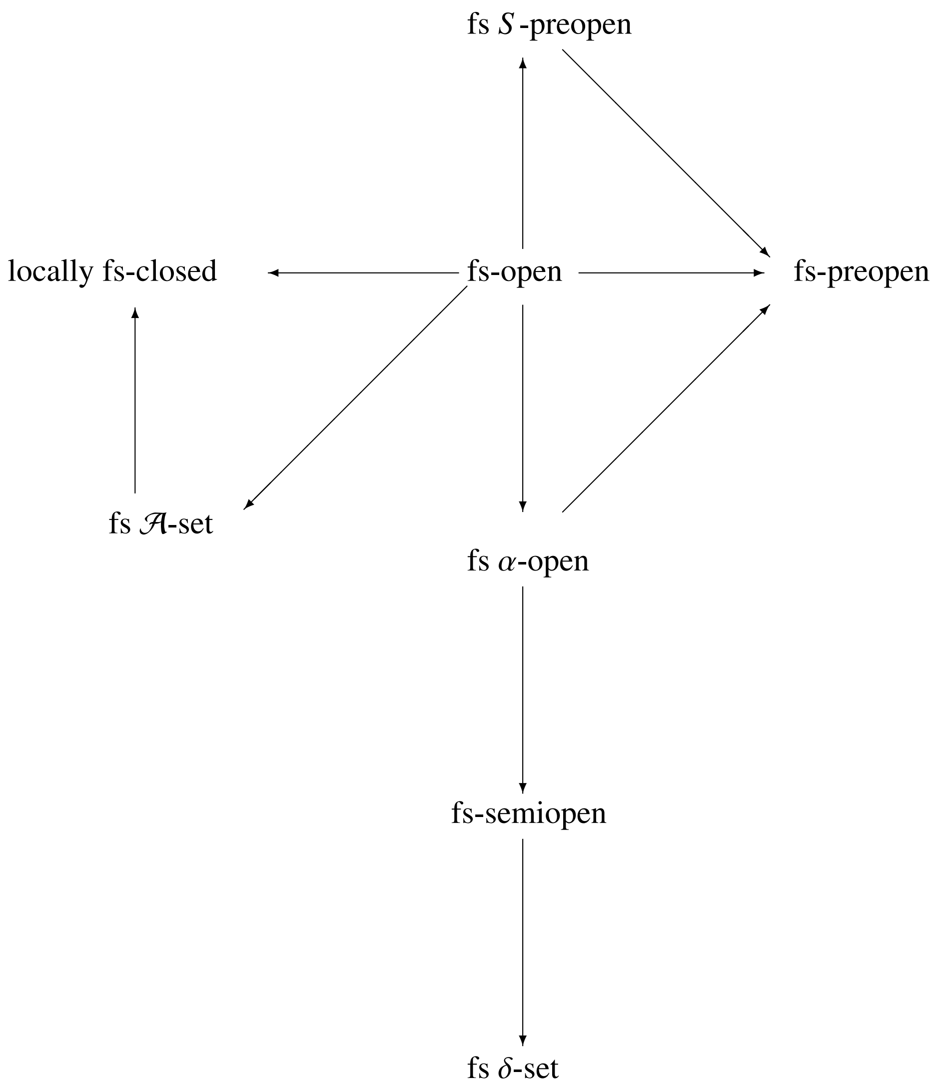
3. Decomposition of Continuity

In (Tamang & Sarkar, 2016) and in the last section, nearly open sets like fs-semiopen, fs-preopen and fs-regular open sets in a fuzzy sequential topological space have been studied. Here, we study some more of such sets and the respective continuities and the section is concluded establishing a few decompositions of fs-continuity. In this section, for our convenience, we denote the closure and interior by *cl* and *int* respectively.

Definition 3.1. Let $(X, \delta(s))$ be an FSTS. An fs-set $A_f(s)$ is called
(i) fs α-open if $A_f(s) \leq int\ cl\ intA_f(s)$;
(ii) locally fs-closed if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-closed;
(iii) an fs $\mathcal{A}$-set if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed;
(iv) an fs δ-set if $int\ clA_f(s) \leq cl\ intA_f(s)$;
(v) fs S-preopen if $A_f(s)$ is fs-preopen and $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $intV_f(s)$ is fs-regular open.

We denote the collection of all fs α-open sets, fs-semiopen sets, fs-preopen sets, fs $\mathcal{A}$-sets, fs S-preopen sets, locally fs-closed sets and fs δ-sets in an FSTS $(X, \delta(s))$, by $\alpha(X)$, $FSSO(X)$, $FSPO(X)$, $\mathcal{A}(X)$, $FSSPO(X)$, $FSLC(X)$ and $\delta(X)$ respectively.

The relationships among different fs-sets defined above, are given by the following diagram:



The implications in the above diagram are not reversible. To show this, here we give examples. In (Tamang & Sarkar, 2016) and in Section 2 respectively, it is already shown that an fs-semiopen and an fs-preopen set may not be fs-open.

Example 3.1. Example to show that an fs α-open set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{1}, \overline{1}, ... \right\} \\ B_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{0}, \overline{0}, ... \right\} \\ C_f(s) &= \left\{ \overline{\frac{3}{8}}, \overline{1}, \overline{1}, ... \right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), B_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $C_f(s)$ is fs α-open but is not fs-open.

Example 3.2. Example to show that a locally fs-closed set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, \overline{\frac{1}{4}}, \overline{\frac{3}{4}}, ... \right\} \\ B_f(s) &= \left\{ \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, ... \right\} \end{aligned}$$

Let $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$. Then $(X, \delta(s))$ is an FSTS, where $B_f(s)$ is locally fs-closed but not fs-open.

Example 3.3. Example to show that an fs $\mathcal{A}$-set may not be fs-open.

Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ in a set X, defined as follows:

$$\begin{aligned} A_f(s) &= \left\{ \overline{1}, \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, ... \right\} \\ B_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{0}, \overline{\frac{1}{2}} \overline{0}, ... \right\} \\ C_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, \overline{\frac{1}{2}}, ... \right\} \\ D_f(s) &= \left\{ \overline{\frac{1}{2}}, \overline{1}, \overline{\frac{1}{2}}, \overline{1}, ... \right\} \end{aligned}$$

Consider the fuzzy sequential topological space $(X, \delta(s))$, where $\delta(s) = \{A_f(s), B_f(s), X_f^0(s), X_f^1(s)\}$. Here, $C_f(s) = A_f(s) \wedge D_f(s)$, where $A_f(s)$ is fs-open and $D_f(s)$ is fs-regular closed. Hence $C_f(s)$ is an fs $\mathcal{A}$-set but is not fs-open.

Example 3.4. Example to show that a locally fs-closed set may not be an fs $\mathcal{A}$-set.
Consider the FSTS $(X, \delta(s))$, given in Example 3.2. Here, $B_f(s)$ is a locally fs-closed set but not an fs $\mathcal{A}$-set.

Example 3.5. Example to show that an fs-semiopen set may not be an fs $\mathcal{A}$-set.
Consider the FSTS $(X, \delta(s))$, given in Example 3.1.
The fs-set $C_f(s)$ is an fs-semiopen set but not an fs $\mathcal{A}$-set.

Example 3.6. Example to show that an fs-semiopen set may not be fs α-open.
In the FSTS, given in Example 3.3, the fs-set $C_f(s)$ is fs-semiopen but not fs α-open.

Example 3.7. Example to show that an fs-preopen set may not be fs α-open.
Consider the fs-sets $A_f(s)$, $B_f(s)$ in a set X, defined as follows:

$$A_f(s) = \left\{\overline{\frac{1}{4}}, \overline{\frac{1}{4}}, \overline{\frac{1}{4}}, ...\right\}$$

$$B_f(s) = \left\{\overline{\frac{1}{5}}, \overline{\frac{1}{5}}, \overline{\frac{1}{5}}, ...\right\}$$

Then $\delta(s) = \{A_f(s), X_f^0(s), X_f^1(s)\}$ is a fuzzy sequential topology on X. In this FSTS, $B_f(s)$ is fs-preopen but not fs α-open.

Example 3.8. Example to show that an fs δ-set may not be fs-semiopen.
In the FSTS, given in Example 3.2, the fs-set $B_f(s)$ is an fs δ-set but is not fs-semiopen.

Example 3.9. Example to show that an fs-preopen set may not be an fs S-preopen set.
Consider the FSTS, given in Example 3.1, the fs-set $C_f(s)$ is fs-preopen but not fs S-preopen.

Example 3.10. Example to show that an fs S-preopen set may not be an fs-open set.
Consider the FSTS, given in Example 3.7, the fs-set $B_f(s)$ is fs S-preopen but not fs-open.

Definition 3.2. A mapping $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ is called
(i) fs α-continuous if $g^{-1}(B_f(s))$ is fs α-open in X, for every fs-open set $B_f(s)$ in Y.
(ii) fs lc-continuous if $g^{-1}(B_f(s))$ is locally fs-closed in X, for every fs-open set $B_f(s)$ in Y.
(iii) fs $\mathcal{A}$-continuous if $g^{-1}(B_f(s))$ is an fs $\mathcal{A}$-set in X, for every fs-open set $B_f(s)$ in Y.
(iv) fs δ-continuous if $g^{-1}(B_f(s))$ is an fs δ-set in X, for every fs-open set $B_f(s)$ in Y.
(v) fs S-precontinuous if $g^{-1}(B_f(s))$ is fs S-preopen in X, for every fs-open set $B_f(s)$ in Y.

Theorem 3.1. *An fs-set in an FSTS, is fs α-open if and only if it is fs-semiopen and fs-preopen.*

Proof. Let $A_f(s)$ be an fs α-open set, that is, $A_f(s) \leq int\ cl\ intA_f(s)$. Clearly, $A_f(s)$ is fs-semiopen. Also,

$$\begin{aligned} intA_f(s) \leq clA_f(s) &\Rightarrow cl\ intA_f(s) \leq clA_f(s) \\ &\Rightarrow int\ cl\ intA_f(s) \leq int\ clA_f(s) \\ &\Rightarrow A_f(s) \leq int\ clA_f(s) \end{aligned}$$

Thus, $A_f(s)$ is fs-preopen.

Conversely, suppose $A_f(s)$ be fs-semiopen and fs-preopen, that is, $A_f(s) \leq cl\ intA_f(s)$, $A_f(s) \leq int\ clA_f(s)$. Then,

$$\begin{aligned} & int\ clA_f(s) \leq clA_f(s) \leq cl\ intA_f(s) \\ \Rightarrow\ & A_f(s) \leq int\ cl\ intA_f(s) \end{aligned}$$

Hence, $A_f(s)$ is fs α-open. □

Corollary 3.1. *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is fs α-continuous if and only it is fs-semicontinuous and fs-precontinuous.*

Definition 3.3. Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then αfs-closure ${}_\alpha clA_f(s)$ and αfs-interior ${}_\alpha intA_f(s)$ of $A_f(s)$ are defined by

$$\begin{aligned} {}_\alpha clA_f(s) &= \wedge\{V_f(s);\ A_f(s) \leq V_f(s) \text{ and } V_f^c(s) \in \alpha(X)\} \\ {}_\alpha intA_f(s) &= \vee\{U_f(s);\ U_f(s) \leq A_f(s) \text{ and } U_f(s) \in \alpha(X)\} \end{aligned}$$

Complement of an fs α-open set is called an fs α-closed set. Hence, it is clear that ${}_\alpha cl(A_f(s))$ is the smallest fs α-closed set containing $A_f(s)$ and ${}_\alpha int(A_f(s))$ is the largest fs α-open set contained in $A_f(s)$. Further,

(i) $A_f(s) \leq {}_\alpha cl(A_f(s)) \leq \overline{A_f(s)}$ and $\overset{o}{A}_f(s) \leq {}_\alpha int(A_f(s)) \leq A_f(s)$.
(ii) $A_f(s)$ is fs α-open if and only if $A_f(s) = {}_\alpha int(A_f(s))$
(iii) $A_f(s)$ is fs α-closed if and only if $A_f(s) = {}_\alpha cl(A_f(s))$
(iv) $A_f(s) \leq B_f(s) \Rightarrow {}_\alpha int(A_f(s)) \leq {}_\alpha int(B_f(s))$ and ${}_\alpha cl(A_f(s)) \leq {}_\alpha cl(B_f(s))$.

Theorem 3.2. *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then,*
(i) ${}_\alpha intA_f(s) = A_f(s) \wedge int\ cl\ intA_f(s)$.
(ii) if $A_f(s)$ is both fs-preopen and fs-preclosed, then $A_f(s) = int\ clA_f(s) \wedge A_f(s)$ and thus $A_f(s)$ is fs S-preopen;
(iii) if $A_f(s) = U_f(s) \wedge V_f(s)$, where $U_f(s)$ is fs-open and $intV_f(s)$ is fs-regular open, then ${}_\alpha intA_f(s) = intA_f(s)$;
(iv) if $A_f(s)$ is an fs δ-set, then ${}_\alpha intA_f(s) = {}_p intA_f(s)$.

Proof. (i) Easy to prove.

(ii) Given $A_f(s) \leq int\ clA_f(s)$ and $cl\ intA_f(s) \leq A_f(s)$. Then,

$$A_f(s) = int\ clA_f(s) \wedge A_f(s).$$

Since $intA_f(s) = int\ cl\ intA_f(s)$, hence $A_f(s)$ is fs S-preopen.

(iii) We have $int\ cl\ intA_f(s) \leq int\ cl\ intV_f(s) = intV_f(s)$. Therefore,

$$\begin{aligned} {}_\alpha intA_f(s) &= A_f(s) \wedge int\ cl\ intA_f(s) \\ &\leq A_f(s) \wedge intV_f(s) \\ &= intA_f(s) \end{aligned}$$

Also, $intA_f(s) \leq {}_{\alpha}intA_f(s)$. Hence $intA_f(s) = {}_{\alpha}intA_f(s)$.

(*iv*) Given $int\ clA_f(s) \leq cl\ intA_f(s)$. Since ${}_{\alpha}intA_f(s)$ is an fs-preopen set contained in $A_f(s)$, we have

$$ {}_{\alpha}intA_f(s) \quad \leq \quad {}_{p}intA_f(s) $$

Now,

$$ {}_{p}intA_f(s) \leq int\ clA_f(s) \leq int\ cl\ intA_f(s) $$

Thus, ${}_{p}intA_f(s) \leq A_f(s) \wedge int\ cl\ intA_f(s) = {}_{\alpha}intA_f(s)$. Hence the result. □

Lemma 3.1. *An fs-set $A_f(s)$ is locally fs-closed if and only if $A_f(s) = U_f(s) \wedge cl(A_f(s))$, where $U_f(s)$ is an fs-open set.*

Proof. Omitted. □

Theorem 3.3. *Let $A_f(s)$ be an fs-set in an FSTS $(X, \delta(s))$. Then $A_f(s)$ is an fs $\mathcal{A}$-set if it is fs-semiopen and locally fs-closed.*

Proof. Suppose $A_f(s)$ be fs-semiopen and locally fs-closed. Then, $A_f(s) \leq cl\ intA_f(s)$ and $A_f(s) = U_f(s) \wedge clA_f(s)$, where $U_f(s)$ is fs-open. Since $clA_f(s) = cl\ intA_f(s)$ is fs-regular closed, the result follows. □

Corollary 3.2. *A mapping $g : (X, \delta(s)) \to (Y, \eta(s))$ is fs $\mathcal{A}$-continuous if it is fs-semicontinuous and fs lc-continuous.*

Remark. Unlike in a general topological space, the converse of Theorem 3.3 may not be true and it has been shown by the next Example.

Example 3.11. Let $X = \{x, y\}$. Consider the fs-sets $A_f(s)$, $B_f(s)$, $C_f(s)$, $D_f(s)$ and $E_f(s)$ in X, where

$A^1_f = \overline{0.3}$, $A^n_f(x) = 1$ and $A^n_f(y) = 0$ for all $n \neq 1$;
$B^1_f(x) = 0.4$, $B^1_f(y) = 0.7$, $B^n_f(x) = 0$ and $B^n_f(y) = 1$ for all $n \neq 1$;
$C^1_f = \overline{0.7}$ and $C^n_f = \overline{1}$ for all $n \neq 1$;
$D^1_f(x) = 0.6$, $D^1_f(y) = 0.3$, $D^n_f(x) = 1$ and $D^n_f(y) = 0$ for all $n \neq 1$;
$E^1_f(x) = 0.4$, $E^1_f(y) = 0.3$ and $E^n_f = \overline{0}$ for all $n \neq 1$.

Let $\delta(s) = \{A_f(s), B_f(s), C_f(s), A_f(s) \wedge B_f(s), A_f(s) \vee B_f(s), X^0_f(s), X^1_f(s)\}$. In the FSTS $(X, \delta(s))$, $D_f(s)$ being an fs-regular closed set, the fs-set $E_f(s) = B_f(s) \wedge D_f(s)$ is an fs $\mathcal{A}$-set but not fs-semiopen.

Theorem 3.4. *Let $(X, \delta(s))$ be an FSTS and $A_f(s)$ be an fs-set in X. Then the following statements are equivalent:*
(i) $A_f(s)$ is an fs-open set.
(ii) $A_f(s)$ is fs α-open and locally fs-closed.
(iii) $A_f(s)$ is fs-preopen and locally fs-closed.
(iv) $A_f(s)$ is fs-preopen and an fs $\mathcal{A}$-set.
(v) $A_f(s)$ is fs S-preopen and an fs δ-set.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (iv)$ Let $A_f(s)$ be fs-preopen and locally fs-closed. Then,

$$A_f(s) \leq int\, clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. $clA_f(s)$ being fs-regular closed, the result follows.

$(iv) \Rightarrow (i)$ Let $A_f(s)$ be an fs-preopen and an fs $\mathcal{A}$-set. Then,

$$A_f(s) \leq int\, clA_f(s) \quad \text{and} \quad A_f(s) = U_f(s) \wedge clA_f(s),$$

where $U_f(s)$ is fs-open. Since $intA_f(s) = U_f(s) \wedge int\, clA_f(s)$, $A_f(s)$ is fs-open.

$(i) \Rightarrow (v)$ Obvious.

$(v) \Rightarrow (i)$ Let $A_f(s)$ be an fs S-preopen and an fs δ-set. Using Theorem 3.2, (iii) and (iv),

$$intA_f(s) = {}_{\alpha}intA_f(s) = {}_{p}intA_f(s) = A_f(s).$$

Hence, $A_f(s)$ is fs-open. □

By Theorems 3.1, 3.3 and 3.4, we have the following relationships among the different classes of fs-sets of an FSTS $(X, \delta(s))$:
(i) $\alpha(X) = FSPO(X) \cap FSSO(X)$.
(ii) $\mathcal{A}(X) \supseteq FSSO(X) \cap FSLC(X)$.
(iii) $\delta(s) = \alpha(X) \cap FSLC(X)$.
(iv) $\delta(s) = FSPO(X) \cap FSLC(X)$.
(v) $\delta(s) = FSPO(X) \cap \mathcal{A}(X)$.
(vi) $\delta(s) = FSSPO(X) \cap \delta(X)$.

Theorem 3.5. *In an FSTS $(X, \delta(s))$, the following are equivalent:*
(i) $clA_f(s) \in \delta(s)$ for every $A_f(s) \in \delta(s)$.
(v) $\mathcal{A}(X) = \delta(s)$.

Proof. $(i) \Rightarrow (ii)$ It is obvious that $\delta(s) \subseteq \mathcal{A}(X)$. For the reverse inclusion, let $A_f(s) \in \mathcal{A}(X)$, then

$$A_f(s) = U_f(s) \wedge V_f(s),$$

where $U_f(s)$ is fs-open and $V_f(s)$ is fs-regular closed. By (i), $V_f(s) \in \delta(s)$ and hence $A_f(s) \in \delta(s)$.

$(ii) \Rightarrow (i)$ Suppose $\mathcal{A}(X) = \delta(s)$. Let $A_f(s) \in \delta(s)$, then $clA_f(s)$ is fs-regular closed and hence belongs to $\mathcal{A}(X) = \delta(s)$. □

We conclude the chapter by stating the following decompositions of fs-continuity:

Theorem 3.6. *Let $g : (X, \delta(s)) \rightarrow (Y, \eta(s))$ be a map. Then g is fs-continuous if and only if*
(i) *g is fs α-continuous and fs lc-continuous.*
(ii) *g is fs-precontinuous and fs lc-continuous.*
(iii) *g is fs-precontinuous and fs $\mathcal{A}$-continuous.*
(iv) *g is fs S-continuous and fs δ-continuous.*

References

Azad, K. K. (1981). On fuzzy semicontinuity, fuzzy amost continuity and fuzzy weakly continuity. *J. Math. Anal. Appl.* **82**, 14–32.

Mashhour, A. S., M. E. Abd El-Monsef and S. N. EI-Deeb (1982). On precontinuous and weak precontinuous mappings. *Proc. Math. Phys. Soc. Egypt* **53**, 47–53.

Singha, M., N. Tamang and S. De Sarkar (2014). Fuzzy sequential topological spaces. *International Journal of Computer and Mathematical Sciences* **3**(4), 2347–8527.

Tamang, N. and S. De Sarkar (2016). Some nearly open sets in a fuzzy sequential topological space. *International Journal of Mathematical Archive* **7**(2), 120–128.

Tamang, N., M. Singha and S. De Sarkar (2016). On the notions of continuity and compactness in fuzzy sequential topological spaces. *International Journal of Engineering and Advanced Research Technology* **2**(10), 9–15.