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Structural Machines as a Mathematical Model of Biological and Chemical Computers

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Abstract

To rigorously describe and study the living morphological computers, we develop a formal model of algorithms and computations called a structural machine providing theoretical tools for exploration of possibilities of biological computations and extension of their applications. We study properties of structural machines demonstrating how they can model the most popular in computer science model of computation — Turing machine. We also prove that structural machines can solve some problems more efficiently than Turing machines. We also show how structural machines model a programmable amorphous biological computer called a Physarum machine, as well as an inductive Turing machine.

Keywords: model of computation, algorithm, structural machine, Turing machine, Physarum machine, bio-inspired computing, computing automaton, slime mould, unconventional computing, computational efficiency.
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1. Introduction

To better understand and further develop information processing, researchers created different mathematical models of algorithms and computation building foundations of theoretical computer science. After development and proliferation of digital computers, Turing machine became the most popular of those models impersonating the paradigm of algorithmic computational devices according to the Church-Turing Thesis. In spite of this, creation of new mathematical models has continued. There were three reasons for this situation in theoretical computer science. First, many researchers tried to build a model more powerful than Turing machine. Regardless of many attempts, the first mathematical model of algorithms and computation, for which it was proved

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that it was really more powerful than Turing machine, was inductive Turing machine constructed in (Burgin, 1983). Note that although Turing machines with an oracle are more powerful than conventional Turing machines, a Turing machine with an oracle is not a constructive theoretical device, while inductive Turing machines are virtually as constructive as conventional Turing machines. Second, by its construction, Turing machine is a very simple edifice well suited for theoretical research but reflecting only some essential features of digital computers while ignoring other features. At the same time, exactly these properties of a Turing machine made it very inefficient as a computing device. Therefore, researchers thrived to build more efficient than Turing machines mathematical models, representing more key features of ever-developing computers. Examples of such models are Kolmogorov algorithms (Kolmogorov, 1953), Random Access Machines (Shepherdson & Sturgis, 1963), Random Access Machines with the Stored Program (Elgot & Robinson, 1964) and some others. The third incentive for the development of new models has been necessity to address various kinds of computations that the conventional Turing machine was not able to properly model. This intent brought forth many novel models such as:

- cellular automata (Von Neumann *et al.*, 1966), Turing machines with many heads and tapes, vector machines and systolic arrays (Kung & Leiserson, 1978) for modeling and exploration of parallel computations;
- Petri nets (Petri, 1962) and grid automata (Burgin, 2003; Burgin & Eberbach, 2009*a*) for modeling and exploration of concurrent computations;
- formal grammars (Chomsky, 1956) for modeling and exploration of linguistic transformations; and some others.

Discovery of unconventional types of computation and research in building unconventional computers such as biological, chemical, or non-linear physical computers (Adamatzky, 2001; Adamatzky *et al.*, 2005; Calude & Dinneen, 2015; Adamatzky, 2016) demand innovative mathematical models of computation that reflect peculiarity of unconventional computations (Stepney *et al.*, 2005; Cooper, 2013; Kendon *et al.*, 2015) and original models of algorithms that represent control in such computations. This is the exact goal of the development and exploration of structural machines in this paper. The paper is organized as follows. In the second section, we describe structural machines and study their properties. In the third section, we study relations between structural machines and other popular computational models. In particular, we show how structural machines model Turing machines (Theorem 1) and inductive Turing machines (Theorem 3) demonstrating (Theorem 2) that structural machines are more efficient than Turing machines. In the fourth section, we briefly describe Physarum machines, which perform biological computations, and show how structural machines model Physarum machines. In the fifth section, we discuss the obtained results and suggest future directions for research.

2. Structural machines

There are structural machines of different orders. Here we study structural machines of the first order, which work with first-order structures, and structural machines of the second order, which work with first-order and second-order structures.

Definition 2.1. From (Burgin, 2012). A first-order structure is a triad of the form $\mathbf{A} = (A, r, \mathcal{R})$ In this expression, we have:

- the set A, which is called the substance of the structure $\mathbf{A}$ and consists of elements of the structure $\mathbf{A}$, which are called structure elements of the structure $\mathbf{A}$
- the set $\mathcal{R}$, which is called the arrangement of the structure $\mathbf{A}$ and consists of relations in the structure $\mathbf{A}$, which are called structure relations of the structure $\mathbf{A}$
- the incidence relation r, which connects groups of elements from A with relations from $\mathcal{R}$.

For instance, if $\mathbf{R}$ is an n-ary relation from $\mathcal{R}$ and $a_1, a_2, a_3, \ldots$, an are elements from A, then the expression $r(R; a_1, a_2, a_3, \ldots, a_n)$ means that the elements $a_1, a_2, a_3, \ldots, a_n$ belong to the relation $\mathbf{R}$ with the name R, i.e., r connects the elements $a_1, a_2, a_3, \ldots, a_n$ with the relation $\mathbf{R}$.

Describing structures, it is significant to distinguish relations and their names because when a structural machine functions, relations, as a rule, are changing, while their names can remain the same. For instance, when a structure $\mathbf{A}$ on a set A has a ternary relation $\mathbf{R}$ with the name R and in the process of computation, the machine additionally connects three elements from A by a link assigning it to the relation $\mathbf{R}$. As a result, $\mathbf{R}$ becomes larger but preserves the same name R.

However, it is also possible, that a structural machine changes names of the structure relations, for example by splitting one relation into two new relations of the same arity.

Lists, queues, words, texts, graphs, directed graphs, labeled graphs, mathematical and chemical formulas, tapes of Turing machines and Kolmogorov complexes are particular cases of structures that have only unary and binary relations. Note that labels, names, types and properties are unary relations.

In the case when the set $\mathcal{R}$ of relations consists of one binary and several unary relations, then the first order structure is a labeled (named) graph. When $\mathcal{R}$ contains only binary and unary relations, then the first order structure is a labeled (named) multigraph.

Example 2.1. Let us consider the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ where $A = \{a, b, c, d, e, f, g, h\}$ and $\mathcal{R}$ consists of one binary relation P and one ternary relation Q, the graphical representation of which is given in Figure 1.

In the case of a Physarum machine, we can interpret elements of the relation Q in Figure 1 as two clusters of nutrients, colonized by a single slime mould. There are higher degrees of connections between growth zones and branches of the protoplasmic network inside clusters but there are only few links bringing these two clusters together.

Another possibility is when a first order structure $A = (A, r, \mathcal{R})$, in which the set $\mathcal{R}$ consists of a single binary relation R can faithfully represent the structure of a living slime mould established by blobs of slime mould and active zones. Namely, elements from the set A represent blobs of slime mould and active zones, while elements from the relation R represent connecting tubes.

However, a slime mould often has a more sophisticated structure. The network of blobs, active zones and protoplasmic tubes is not uniform but forms clusters. These clusters are also connected by thick protoplasmic tubes, which represent the incidence relation that connects groups of elements from A. To model a slime mould with clusters, we need second-order structures.

Definition 2.2. From (Burgin, 2012). A second-order structure is a triad of the form

$$\mathbf{A} = (A, r, \mathcal{R})$$

Here

- the set A, which is called the substance of the structure $\mathbf{A}$ and consists of elements of the structure $\mathbf{A}$, which are called structure elements of the structure $\mathbf{A}$
- the set $\mathcal{R}$, which is called the arrangement of the structure $\mathbf{A}$ and consists of relations in the structure $\mathbf{A}$, which are called structure relations of the structure $\mathbf{A}$
- r is the incidence relation that connects groups of elements from A with relations from $\mathcal{R}$
- $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$
- $\mathcal{R}_1$ is the set of relations in the set A
- $\mathcal{R}_2$ is the set of relations in the set $\mathcal{R}_1$, i.e., elements from $\mathcal{R}_2$ are relations between relations from $\mathcal{R}_1$

Second-order structures are used to represent data processed by structural machines of the second order.

Relations from the set $\mathcal{R}$ determine the intrinsic structure of the structure $\mathbf{A} = (A, r, \mathcal{R})$. However, efficient operation with, utilization and modeling of first-order and higher-order structures demands additional (extrinsic) structures (Burgin, 2012). One of these extrinsic structures is pretopology (ech, 1966) determined by neighborhoods in the set A.

Definition 2.3. If $\mathbf{R}$ is an n–ary relation from $\mathcal{R}$, then:

1. the substantial R–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RS}\, a = \{a\} \cup \{d; \exists i, k \exists a_2, \ldots, a_n \in A((1 \le k \le n) \wedge (a_1, a_2, \ldots, a_i = a, \ldots, a_k = d, \ldots, a_n) \in \mathbf{R})\}$$

2. the link R–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RL} a = \{(a_1, a_2, \ldots, a_n) \in \mathbf{R}; a_1, a_2, \ldots, a_n \in O_R a\}$$

3. the full R–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RF} a = O_{RS}\, a \cup O_{RL} a$$

Informally, the substantial R-neighborhood of a structure element a consists of all structure elements connected to a by the relation $\mathbf{R}$. The link R–neighborhood of a structure element a consists of all elements from the relation $\mathbf{R}$ that contain a. The full R–neighborhood of a structure element a is the union of the substantial R–neighborhood and link R–neighborhood of a.

Examples:

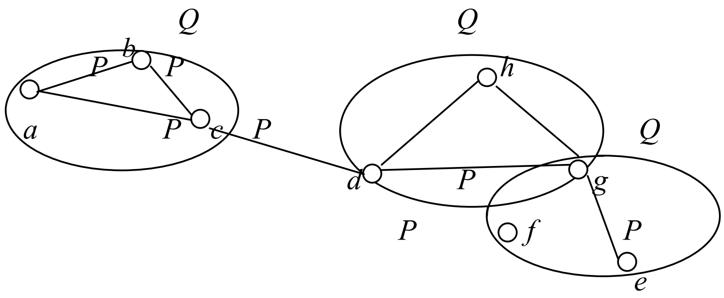


Figure 1. The graphical representation of a first order structure.

1. Taking the structure presented in Fig. 1, we see that $\{a, b, c, f\}$ is the substantial P–neighborhood of the structure element a determined by the relation P from the structure in Fig. 1.
2. Taking the structure presented in Fig. 1, we see that $\{a, b, c\}$ is the substantial Q–neighborhood of the structure element a determined by the relation Q from the structure in Fig. 1.
3. Taking the structure presented in Fig. 1, we see that $\{e, g\}$ is the substantial P–neighborhood of the structure element e determined by the relation P from the structure in Fig. 1.
4. Taking the structure presented in Fig. 1, we see that $\{f, g, e\}$ is the substantial Q–neighborhood of the structure element f determined by the relation Q from the structure in Fig. 1.
5. Taking the structure presented in Fig. 1, we see that $\{(a, b), (a, c)\}$ is the link P–neighborhood of the structure element a determined by the relation Q from the structure in Fig. 1.
6. Taking the structure presented in Fig. 1, we see that $\{(a, b, c)\}$ is the link Q–neighborhood of the structure element a and of the structure element b where both neighborhoods are determined by the relation Q from the structure in Fig. 1.

Definition 3 implies the following result.

Lemma 2.1. *Each structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ has the uniquely defined substantial R—neighborhood.*

Relations between relations from $\mathcal{R}$ imply relations between neighborhoods.

Proposition 2.1. *For any structure element a, the inclusion $R \subseteq Q$ of two relations $R, Q \in \mathcal{R}$ implies inclusions of neighborhoods*

$O_{RS}a \subseteq O_{QS}a$, $O_{RL}a \subseteq O_{QL}a$ *and* $O_{RF}a \subseteq O_{QF}a$.

Neighborhoods of elements allow us to build neighborhoods of sets of elements.

Definition 2.4. If R is an n–ary relation from $\mathcal{R}$ and $Z \subseteq A$, then:
(α) the substantial R–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RS}Z = \cup_{a \in Z} O_{RS}\, a$$

(β) the link R–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{RL}Z = \cup_{a \in Z} O_{RL} a$$

(γ) the full R–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{RF}Z = O_{RS}Z \cup O_{RL}Z$$

Lemma 1 implies the following result.

Lemma 2.2. *The substantial (link or full) R–neighborhood of any set Z of structure elements is uniquely defined.*

Proposition 1 implies the following result.

Proposition 2.2. *For any set Z of structure elements, the inclusion $R \subseteq Q$ of two relations $R, Q \in \mathcal{R}$ implies inclusions of neighborhoods $O_{RS}Z \subseteq O_{QS}Z, O_{RL}Z \subseteq O_{QL}Z$ and $O_{RF}Z \subseteq O_{QF}Z$.*

Substantial neighborhoods of sets of structure elements determine pretopology in the set A of all structure elements. We remind that a pretopological space is defined as a set X with a preclosure operator (ech closure operator) cl. Let 2^X be the power set of X. A preclosure operator on a set X is a mapping $cl : 2^X \to 2^X$ that satisfies the following axioms (ech, 1966):

- $cl(\emptyset) = \emptyset$
- $Z \subseteq cl(Z)$ for any $Z \subseteq X$
- $cl(Z \cup Y) \subseteq cl(Z) \cup cl(Y)$ for any $Z, Y \subseteq X$
- $Y \subseteq Z$ implies $cl(Y) \subseteq cl(Z)$ for any $Z, Y \subseteq X$

Properties of substantial R–neighborhoods allow us to prove the following result.

Proposition 2.3. *Substantial R–neighborhoods define a pretopology in the set A.*

Proof. Let us define the closure $cl(Z)$ of a set $Z \subseteq A$ equal to its substantial R–neighborhood O_{RSX} and check the axioms of pretopological spaces.

Axioms 1 and 2 are true by definition as $cl(\emptyset) = \emptyset$ and $Z \subseteq cl(Z)$ for any $Z \subseteq A$. In addition,

$cl(Z \cup Y) = O_{RS}(Z \cup Y) = \cup_{a \in Z \cup Y} O_{RS}a = (\cup_{a \in Z} O_{RS}a) \cup (\cup_{a \in Y} O_{RS}a) = O_{RS}Z \cup O_{RS}Y = cl(Z) \cup cl(Y)$

This gives us Axiom 3. Axiom 4 is implied by Proposition 2. □

Definition 2.5. • The substantial $\mathcal{R}$–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{S}}a = \cup_{\mathbf{R} \in \mathcal{R}} O_{RS}a$$

- The link $\mathcal{R}$–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{\mathcal{R}\mathbf{L}}a = \cup_{\mathbf{R}\in\mathcal{R}} O_{RS}\, a$$

- The full $\mathcal{R}$–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{F}a} = \cup_{\mathbf{R}\in\mathcal{R}} O_{RF}\, a$$

Definition 5 implies the following result.

Lemma 2.3. *The substantial (link or full) $\mathcal{R}$–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is uniquely defined.*

Properties of union and Definitions 4 and 5 imply the following result.

Lemma 2.4. $O_{\mathcal{R}\mathbf{F}}a = O_{\mathbf{RS}}a \cup O_{\mathbf{RL}}a$.

Proposition 3 and Definition 4 imply the following result.

Corollary 2.1. *Substantial $\mathcal{R}$–neighborhoods define a pretopology in the set A.*

Neighborhoods of elements allow us to build neighborhoods of sets of elements.

Definition 2.6. If $Z \subseteq A$, then:
α : the substantial $\mathcal{R}$–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathcal{R}\mathbf{S}}Z = \cup_{a\in Z} O_{\mathcal{R}\mathbf{S}a}$$

β: the link $\mathcal{R}$–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is a set of the form

$$O_{\mathcal{R}\mathbf{L}}Z = \cup_{a\in Z} O_{\mathcal{R}\mathbf{L}}a$$

γ: the full $\mathbf{R}$–neighborhood of the set Z in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set

$$O_{\mathbf{RF}}Z = O_{\mathbf{RS}}Z \cup O_{\mathbf{RL}}Z$$

Lemma 3 implies the following result.

Lemma 2.5. *The substantial (link or full) $\mathcal{R}$–neighborhood of any set Z of structure elements is uniquely defined.*

Let us study properties of neighborhoods in structures.

Definition 2.7. If R is an n-ary relation from $\mathcal{R}$, then the substantial R–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is symmetric if for any elements $a_1, a_2, a_3, \dots, a_n$ from A and any permutation $i_1, i_2, i_3, \dots, i_n$ of the numbers $1, 2, 3, \dots, n$, we have

$$(a_1, a_2, a_3, \dots, a_n) \in R$$

if and only if

$$(a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_n}) \in R$$

Lemma 2.6. *If the substantial R–neighborhood $O_{RS}a$ of a structure element a is symmetric, a structure element b belongs to the R–neighborhood $O_{RS}a$ if and only if a belongs to the substantial R–neighborhood $O_{RS}b$ of b.*

Definition 2.8. If **R** is an n-ary relation from $\mathcal{R}$, then the R–neighborhood of a structure element a in a structure $\mathbf{A} = (A, r, \mathcal{R})$ is transitive if for any elements $a_1, a_2, a_3, \ldots, a_n, d_1, d_2, d_3, \ldots, d_n$ from A and any numbers i and j from the set $\{1, 2, 3, \ldots, n\}$, we have

If $(a_1, a_2, a_3, \ldots, a_n) \in \mathbf{R}$ and $(d_1, \ldots, a_i, \ldots, d_n) \in \mathbf{R}$, then there are elements $b_1, b_2, b_3, \ldots, b_n$ from A such that $(b_1, \ldots, a_i, \ldots, d_j, \ldots, b_n) \in \mathbf{R}$

Proposition 2.4. *If a relation $\mathbf{R} \in \mathcal{R}$ is symmetric and transitive, then substantial R–neighborhoods of two structure elements either coincide or do not intersect.*

Corollary 2.2. *If all relations in $\mathcal{R}$ are symmetric and transitive, then substantial $\mathcal{R}$–neighborhoods of two structure elements either coincide or do not intersect.*

Corollary 2.3. *If $\mathcal{R}$ consists of one symmetric binary relation R, i.e.,* **A** *is a graph and R is also transitive, then any substantial $\mathcal{R}$–neighborhood (R–neighborhood) is a complete graph*

Proposition 2.5. *If a relation $\mathbf{R} \in \mathcal{R}$ is symmetric and transitive, then the system of substantial R–neighborhoods forms a base of a topology in A.*

Corollary 2.4. *If all relations in $\mathcal{R}$ are symmetric and transitive, then the system of substantial $\mathcal{R}$–neighborhoods forms a base of a topology in A.*

Definition 2.9. The type $T(A)$ of a first-order structure $\mathbf{A} = (A, r, \mathcal{R})$ is the set $\{(R, \alpha(R)); R \in \mathcal{R}\}$ of pairs $(R, \alpha(R))$; where $\alpha(R))$ is the arity of the relation **R** with the name R.

f

We assume that two first-order structures $\mathbf{A} = (A, r, \mathcal{R})$ and $\mathbf{B} = (B, p, \mathcal{P})$ have the same type if there is a one-to-one mapping $f : T(A) \to T(B)$ such that if $f(R, \alpha(R)) = (P, \alpha(P)), then \alpha(R) = \alpha(P)$.

For instance, all binary relations have the same type.

A structural machine M works with structures of a given type and has three components:

- The control device C_M regulates the state of the machine M
- The processor P_M performs transformation of the processed structures and its actions (operations) depend on the state of the machine M and the state of the processed structures
- The functional space Sp_M consists of three components:
  - The input space In_M , which contains the input structure.
  - The output space Out_M , which contains the output structure.
  - The processing space PS_M , in which the input structure(s) is transformed into the output structure(s).

We assume that all structures — the input structure, the output structure and the processed structures — have the same type.

Computation of a structural machine M determines the trajectory of computation, which is a tree in general case and a sequence in the deterministic case and a single processor unit.

There are two forms functional spaces Sp_M and USp_M :

- Sp_M is the set of all structures that can be processed by the structural machine M and is called a categorical functional space
- USp_M is a structure for which all structures that can be processed by the structural machine M are substructures and is called a universal functional space

There are three basic types of control devices:

- A central control device controls all processors of the structural machine
- A cluster control device controls a cluster of processors in the structural machine
- An individual control device controls a single processor in the structural machine

There are two basic types of processors:

- A localized processor is a single abstract device
- A distributed processor, which is also called a total processor, consists of a system of unit processors or processor units

In turn, there are three basic types of distributed processors:

- A homogeneous distributed processor consists of a system of identical unit processors, i.e., all these unit processors are copies of one processor
- An almost homogeneous distributed processor consists of a system in which almost all unit processors are identical
- A heterogeneous distributed processor consists of a system of different unit processors

As a result, we have three structural types of processors.

There are also three basic classes of distributed processors:

- In a synchronous distributed processor, all unit processors perform each operation at the same time
- In an almost synchronous distributed processor, almost all unit processors perform each operation at the same time
- In an asynchronous distributed processor, unit processors function independently

A cellular automaton is a synchronous distributed processor, while a Petri net is an asynchronous distributed processor.

It is natural to suppose that each unit processor performs only local operations. In a general case, each unit processor moves from one structure element to another, performing operations in their neighborhoods (Fig. 2). This makes it possible to consider a localized processor as a special type of a distributed processor with one unit processor.

A standard example of a localized processor is the head of a Turing machine with one head or a finite automaton. One head of TM correspond to one growth zone of the slime mould.

A standard example of a homogeneous distributed processor is the system of all heads of a Turing machine with several heads. It is also possible to perceive a Physarum machine as multiprocessor structural machine.

Examples of heterogeneous distributed processors are processing devices in evolutionary automata such as evolutionary finite automata, evolutionary Turing machines or evolutionary inductive Turing machines (Burgin & Eberbach, 2009*b*).

Note that not all heterogeneous distributed processors are the same and to discern their properties it is possible to use measures of homogeneity constructed in (Burgin & Bratalskii, 1986).

There are three types of localized processors:

- A processor localized to one structure element (e.g., a node)
- A processor localized to an R–neighborhood of one structure element (e.g., a node) where $\mathbf{R}$ is a relation from $\mathcal{R}$
- A processor localized to an $\mathcal{R}$–neighborhood of one structure element (e.g., a node)

In what follows, localization of a processor is formalized by the concept of the processor topos.

There are three sorts of distributed processors:

- A constant distributed processor has a fixed number of localized unit processors.
- A variable distributed processor can change the number of localized unit processors.
- A growing distributed processor can increase the number of localized unit processors.

Growing distributed processors are special kinds of variable distributed processors.

There are three types of variable (growing) distributed processors:

- In a bounded variable (growing) distributed processor, the quantity of localized unit processors is always between two numbers, e.g., between 1 and 10, (is bounded by some number).
- An unbounded variable distributed processor can use any finite number of localized unit processors in its functioning.
- An infinite distributed processor can use any (even infinite) number of localized unit processors.

Cellular automata give examples of structural machines with infinite distributed processors. One-dimensional cellular automata work with such structures as words. Two-dimensional cellular automata work with such structures as two-dimensional arrays.

Now let us consider characteristics of unit processors in structural machines. Each unit processor p of a structural machine M has its topos, observation zone and operation zone.

Definition 2.10. The topos T_p of a unit processor p is the part of the structure occupied by this processor. When we take into account time of processing, the topos T_p of the processor p is denoted by $T_p(t)$.

This definition allows defining topoi for total processors of structural machines.

Definition 2.11. The topos T_A of a total (distributed) processor A is the union of the topoi of all its unit processors. When we take into account time of processing, the topos T_A of the processor A is denoted by $T_A(t)$.

Note that in the general case, the topos of a processor can be infinite although the constructive conditions on algorithms prohibit infinite topoi. Cellular automata give an example of a processor with an infinite topos.

Axiom T. Topoi of different unit processors do not intersect.

Localization of unit processors implies restrictions on their topoi. Namely, the topos of a unit processor localized to one structure element consists of this structure element, the topos of a unit processor localized to an R–neighborhood of one structure element is a part of this neighborhood, and the topos of a unit processor localized to an $\mathcal{R}$–neighborhood of one structure element is a part of that neighborhood.

Definition 2.12. The observation zone Ob_p of a unit processor p is the part of the structure Sp_M observed by this processor from its topos. When we take into account time of processing, the observation zone Ob_p of the processor p is denoted by observation zone $Ob_p(t)$.

As in the case of topoi, we define observation zones for total processors.

Definition 2.13. The observation zone Ob_A of a total (distributed) processor A is the union of the observation zones of all its unit processors. When we take into account time of processing, the observation zone Ob_A of the processor A is denoted by observation zone $Ob_A(t)$.

It is natural to suppose that observation zones of processors impact their functioning.

Axiom Z. Operations performed by unit processors depend only on their observation zone.

Definition 2.14. The operation zone Op_p of a unit processor p is the part of the structure Sp_M that can be changed by this processor from its topos. When we take into account time of processing, the operation zone Op_p of the processor p is denoted by observation zone $Op_p(t)$.

For instance, the head of a Turing machine with one linear tape is unit processor. The topos of the head is one cell in the tape. Its observation zone is the same cell and the symbol written in it. Its operation zone is the symbol written in the cell occupied by the head.

Usually, these parts of the functional space PS_M satisfy the following conditions:

$$T_p \subseteq Op_p \subseteq Ob_p$$

and

$$T_p(t) \subseteq Op_p(t) \subseteq Ob_p(t)$$

Informally, it means that the topos of a processor is inside its operation zone, while it is possible to perform operations only inside the observation zone.

These conditions are true, for example, for Turing machines, but they are not satisfied for pushdown automata (Hopcroft *et al.*, 2001).

Often we have $Op_p = Ob_p$ and T_p consists of a single node (element from A).

As in the case of observation zones, we define operation zones for total processors.

Definition 2.15. The operation zone Op_A of a total (distributed) processor A is the union of the operation zones of all its unit processors.. When we take into account time of processing, the operation zone Op_A of the processor A is denoted by operation zone $Op_A(t)$.

Functioning of processors in structural machines includes not only structure transformations but also transitions from one topos to another.

Definition 2.16. The transition zone Tr_p of a unit processor p consists of all topoi where p can move in one step from its present topos.

For instance, the transition zone Tr_h of the head h of a Turing machine with one linear tape consists of three adjacent cells with h is situated in the middle cell.

In some cases, it is useful to assume that the transition zone of a unit processor is included in its observation zone.

Definition 2.17. A processor unit p is called: ¢ topologically uniform if all its topoi are isomorphic ¢ operationally uniform if all its operation zones are isomorphic ¢ transitionally uniform if all its transition zones are isomorphic ¢ observationally uniform if all its observation zones are isomorphic ¢ topologically standardized if all its topoi have the same type, e.g., are R–neighborhoods ¢ operationally standardized if all its operation zones have the same type, e.g., are R–neighborhoods ¢ transitionally standardized if all its transition zones have the same type, e.g., are R–neighborhoods ¢ observationally standardized if all its observation zones have the same type, e.g., are R–neighborhoods

For instance, processor unit p is topologically uniform if all its topoi consist of a single node.

Lemma 2.7. *Any topologically (operationally, transitionally or observationally) uniform processor unit p is topologically (correspondingly, operationally, transitionally or observationally) standardized.*

Usually, processors of abstract and physical automata (machines) are topologically operationally, transitionally and observationally uniform. At the same time, processors in chemical and biological automata (machines) can be non-uniform. An example of a non-uniform processor is a processor that can read from or write to up to five cells in the memory. Thus, when this processor writes to one cell, its topos consists of one cell, while when this processor writes to three cell, its topos consists of these three cells.

Topoi, observation zones and operation zones of unit processors allow us to define topoi, observation zones and operation zones of distributed processors.

There are different types of processor units. A processor unit can be:

- Controlled (by the central control device of the structural machine).
- Autonomous, when it has its own control device.
- Cooperative, when it has its own control device but the functioning of this processor unit depends on the states both of its own control device and of the central control device of the structural machine.

For instance, in a many-head Turing machine T, all heads are controlled processor units. The control device of T controls them. At the same time, all finite automata in a cellular automaton are autonomous processor units.

We remind that a finite state machine also called a finite state automaton is an abstract system that can be in a finite number of different finite states and functioning of which is described as changes of its states.

Proposition 2.6. *A structural machine M is a finite state machine if and only if:*

- *Its structural space Sp_M is finite, i.e., in the case of universal structural space, it is a finite structure, or in the case of categorical structural space, it consists of a finite number of finite structures.*
- *The number of unit processors is finite and each of them can be in a finite number of different finite states.*

For instance, a finite automaton is a finite state machine, while a Turing machine is not a finite state machine.

Definition 2.18. A temporally finite state machine is an abstract system that can be in a finite number of different finite states at any moment of time and functioning of which is described as changes of its states.

Proposition 2.7. *A structural machine M is a temporally finite state machine if and only if:*

- *At any moment of time, its structural space Sp_M is finite, i.e., in the case of universal structural space, it is a finite structure, or in the case of categorical structural space, it consists of a finite number of finite structures.*
- *At any moment of time, the number of unit processors is finite and each of them can be in a finite number of different finite states.*

- *Any operation of each unit processor involves only a finite number of structure elements and relations*

For instance, a Turing machine is a temporally finite state machine, while finite dimensional and general machines of (Blum *et al.*, 1989) are not temporally finite state machines.

Definition 2.19. An operation of a processor is local or more exactly, unilocal if it is performed with one structural element (e.g., node) and some (all) of its relations (a pointed operation), e.g., deleting a structural element (e.g., a node) and all its binary connections (links or edges), adding a link to a structural element or changing a label of a structural element.

For instance, the head h of a Turing machine performs only local operations, while the head of a pushdown automaton can perform nonlocal operations. Processors of automata that perform operations of unrestricted formal grammars are mostly nonlocal.

Definition 2.20. An operation of a processor P is $\mathcal{R}$–local if it is performed with elements (e.g., nodes) from the $\mathcal{R}$–neighborhood of a definite element (e.g., node) and with some (all) of their relations (a singularly local operation).

An operation of a processor P is topologically $\mathcal{R}$–local if it is performed with elements (e.g., nodes) from the $\mathcal{R}$–neighborhood of a topos of P (e.g., node) and with some (all) of their relations (a topologically local operation).

Proposition 2.8. *If (1) a structural machine M operates with structures in which $\mathcal{R}$ contains only one binary relation, (2) a topos of a topologically uniform processor P of M is one structural element, and (3) an operation O of P is topologically local, then O is singularly local.*

Definition 2.21. a) An operation of a processor is $\mathcal{R}$–local or totally local if it is performed with elements (e.g., nodes) from the $\mathcal{R}$–neighborhood of a definite element (e.g., node) and with some (all) of their relations. b) An operation of a processor P is topologically $\mathcal{R}$–local if it is performed with elements (e.g., nodes) from the $\mathcal{R}$–neighborhood of a topos of P (e.g., node) and with some (all) of their relations (a wholly local operation).

Proposition 2.9. *If a topos of a topologically uniform processor P is one structural element and an operation O of P is wholly local, then O is totally local.*

Definitions imply the following result.

Proposition 2.10. *If R belongs to $\mathcal{R}$, then any R–local operation is $\mathcal{R}$–local.*

Let us consider operations performed by processors of structural machines.

The first group of operations consists of the transition operations:

1. Moving the processor from one topos, e.g., a structure element, to another topos. This operation is local when both elements belong to some relation from $\mathcal{R}$.
2. Changing the operation zone of the processor
3. Changing the observation zone of the processor

The second group of operations consists of the substantial transforming operations:

1. Adding a structure element, e.g., a node.
2. Deleting (removing) a structure element, e.g., a node, from a neighborhood of the element where the processor is situated and all relations that include this element.
3. Deleting (removing) a link from a relation R that connects some structure elements with the element where the processor is situated.
4. Adding a link to a relation $\mathbf{R}$ that connects some structure elements with the element where the processor is situated.
5. Deleting (removing) a relation $\mathbf{R}$ from $\mathcal{R}$.
6. Adding a new relation to R.

The third group of operations consists of the symbolic transforming operations:

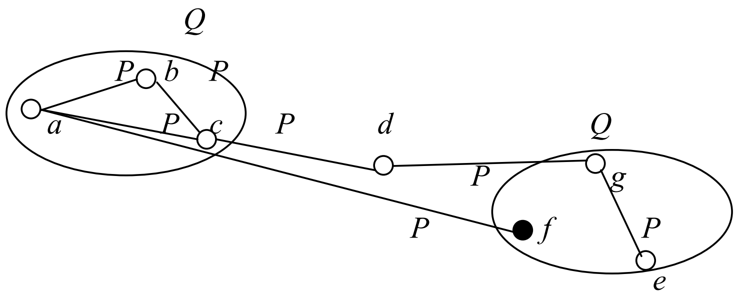
1. Renaming a node
2. Naming a node
3. Denaming a node, i.e., deleting the name of a node
4. Renaming a link
5. Naming a link
6. Denaming a link, i.e., deleting the name of a link

Example 2.2. Operation of deleting the element f from the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ in Fig. 2 where (A) shows the structure before operation and (B) shows the structure after operation. Note that such operations often change relations in the processed structure. Besides, the processor (processor unit) moves from the place (position) f to the place (position) g (see Fig. 2). This operation is performed according to the instruction $(q, f, f) \to (q, g, \sim f)$, in which q is the state of the processor (processor unit), f is the place (position) of the processor (processor unit) before the operation, g is the place (position) of the processor (processor unit) after the operation and $\sim f$ means elimination of f.

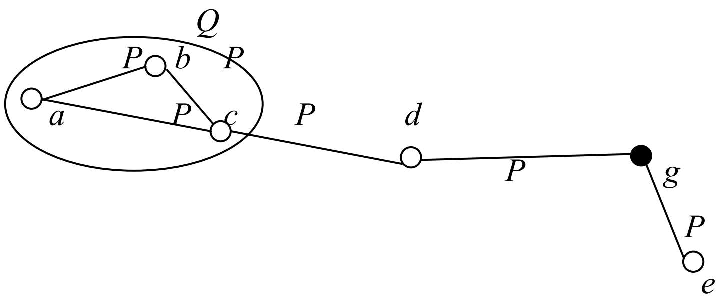
Example 2.3. Operation of adding the relation P for elements g and f in the first order structure $\mathbf{A} = (A, r, \mathcal{R})$ where (A) shows the structure before operation and (B) shows the structure after operation (see Fig. 3). Besides, the processor (processor unit) moves from the place (position) g to the place (position) e. This operation is performed according to the instruction $(p, g, f) \to (p, e, P(g, f))$, in which p is the state of the processor (processor unit), g is the place (position) of the processor (processor unit) before the operation, e is the place (position) of the processor (processor unit) after the operation, f is an observed element and means addition of the pair (g, f) to the relation P.

Structural machines can simulate Turing machines, Kolmogorov algorithms (machines), storage modification machines and cellular automata. We prove some of these results in the next section.

Structural machines also can simulate processes generated by logical calculi (Schumann & Adamatzky, 2011), λ-calculus and formal grammars being able to perform operations used in

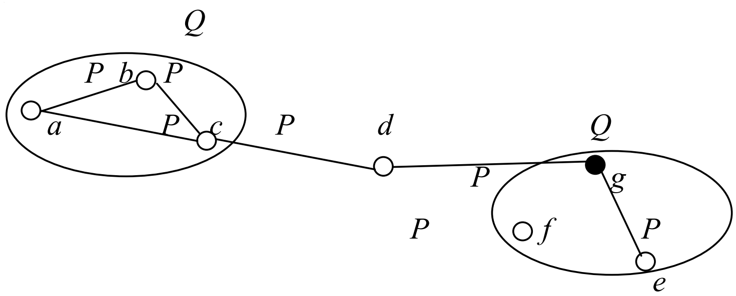


(a)

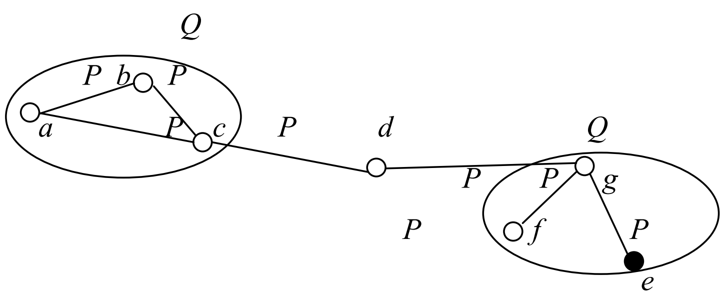


(b)

Figure 2. The graphical representation of an operation on first order structures.



(a)



(b)

Figure 3. The graphical representation of an operation on first order structures.

various databases. Structural machines can also compute partial recursive functions and limit partial recursive functions.

It is possible to build structural machines that can work not only with discrete but also with continuous data because structures can be continuous and there are no restrictions on relations in processed structures. This possibility turns artificial neural networks, Shannons differential analyzer (Shannon, 1941), a finite dimensional and general machines of (Blum *et al.*, 1989) and Type 2 Turing machines (Weihrauch, 2000) into special cases of structural machines.

This shows that it is practical to discern discrete structural machines, which work with discrete structures, have discrete systems of states and operations, and continuous structural machines. In continuous structural machines one two or all three of the following components can be continuous, i.e., continuous processed structures, continuous system of states and/or continuous operations.

Being able to construct various forms of structural machines and their flexibility show that it is natural to use structural machines for a theoretical study of natural computations performed by biological, chemical and physical systems. For instance, in Section 4, we show how to use structural machines as abstract automata for simulation of such biological automata as Physarum machines (Adamatzky, 2007) based on slime mold computations.

3. Structural machines, Turing machines and inductive Turing machines

In this section, we study relations between structural machines and other popular computational models.

Theorem 3.1. *A structural machine with a centralized processor can simulate any Turing machine with the linear time complexity.*

It is sufficient to simulate a Turing machine with one linear tape and one head (cf., for example, (Burgin, 2005; Hopcroft *et al.*, 2001)).

When we want to model a Turing machine by a structural machine, it is easy to build a linear structure of elements with sequential elements connected by a binary relation and this structure will represent the linear tape of the Turing machine. However, the question is how the structural machine will discern left from right simulating the moves of the Turing machine. Here we give three solutions to this problem.

Let us consider a Turing machine $T = (A, Q, q_0, F, R)$. This formula shows that the Turing machine T is determined by the alphabet $\Sigma = \{a_1, a_2, \ldots, a_m\}$, the set of states $Q = \{q_0, q_1, q_2, \ldots, q_n\}$, the set of the final states $F = \{p_1, p_2, \ldots, p_u\} \subseteq Q$, the start state q_0 and the system of rules R, each of which has the form

$$q_h a_i \to a_j q_k$$

$$q_h \Lambda \to a_j q_k$$

$$q_h a_i \to \Lambda q_k$$

$$q_h a_i \to q_k L$$

$$q_h a_i \to q_k c R$$

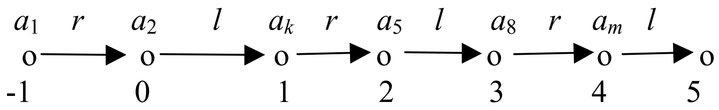


Figure 4. The graphical representation of processed first-order structures.

where $a_i, a_j \in A$, $q_h, q_k \in Q$, R means that the head of T moves to the right, L means that the head of T moves to the left and Λ is the empty symbol, which denotes empty cell of the memory.

Here we give a linear solution to the problem of simulation, i.e., we use only one-dimensional first-order structures in the simulation.

Let us describe the structural machine $M_T = (C, P, \mathcal{R})$ with the control device C, processor P and processing space $\mathcal{R}$, which coincides with the input space and output space, such that M_T simulates the Turing machine T.

In it, C is a finite automaton with the states $Q^o = \{q_0^{odd}, q_1^{odd}, q_2^{odd}, \ldots, q_n^{odd}, q_0^{even}, q_1^{even}, q_2^{even}, \ldots, q_n^{even}\}$, the final states $F^o = \{p_1^{odd}, p_2^{odd}, \ldots, p_u^{odd}, p_1^{even}, p_2^{even}, \ldots, p_u^{even}\} \subseteq Q^o$, and the start state q_0^{odd}.

The processor P is similar to the head of a Turing machine but it observes not only the content of a cell but also its connections, and P can change not only the content of the observed cell but also its connections.

$\mathcal{R} = \{R, L, \Sigma\}$ where: R is a binary relation elements of which are denoted (named) by r; L is a binary relation elements of which are denoted (named) by l; Σ contains unary relations a where $a \in \Sigma$.

Relations r and l are also called connections or links. The unary relations a from Σ are used for naming the structure elements by symbols from Σ.

The machine M_T processes first-order structures that have the form $A = (N, z, \mathcal{R}_a)$ where $N = \{-h, -h+1, \ldots, -1, 0, 1, 2, 3, \ldots, k; -h < k+1\}$, the processing space $\mathcal{R}_a$ contains two binary relations $R_a \subseteq \mathbf{R}$ and $L_a \subseteq L$ in which links r connect elements $2t$ and $2t+1$ for all integer numbers t with $-h \leq 2t \leq k - -1$, while links l connect elements $2t - -1$ and $2t$ for all integer numbers t with $-h + 1 \leq 2t \leq k$; and m unary relations $a_1, a_2, \ldots, a_m$ and z is the assignment of relations from $\mathcal{R}_a$ (names from the alphabet Σ) to elements from N. Note that the number $-h$ can be positive, e.g., when $h = -3$.

An arbitrary input structure has the form $A_0 = (N_0 = \{1, 2, 3, \ldots, m\}, z_0, \mathcal{R}_0)$ where $\mathcal{R}_0 =$ contains two binary relations $\mathbf{R}_0 \subseteq R$ and $L_0 \subseteq L$ in which links r connect elements $2t$ and $2t+1$ for all t with $1 \leq 2t \leq x - -1$, while links l connect elements $2t - -1$ and $2t$ for all t with $1 \leq 2t \leq x$; and m unary relations $a_1, a_2, \ldots, a_m$ and z_0 is the assignment of relations from $\mathcal{R}_0$ (names from the alphabet Σ) to the elements from N_0 , of the relation r to the pairs of elements $2t$ and $2t+1$ for all t with $1 \leq 2t \leq x - -1$ and of the relation l to the pairs of elements $2t - -1$ and $2t$ for all t with $1 \leq 2t \leq x$.

Expressions in the rules for the machine M_T have the following meanings:
$a_i(r, l)$ in the left part of a rule means that the processor observes three relations a_i, r and l at the named element where the processor is situated.
$a_i(l)$ in the left part of a rule means that the processor observes only two relations a_i and l at the named element where the processor is situated.

$a_i(r)$ in the left part of a rule means that the processor observes only two relations a_i and r at the named element where the processor is situated.
$o(r, l)$ in the left part of a rule means that the processor observes two relations (connections) r and l at the element where the processor is situated.
$o(l)$ in the left part of a rule means that the processor observes only one relation (connection) l at the element where the processor is situated.
$o(r)$ in the left part of a rule means that the processor observes only one relation (connection) r at the element where the processor is situated.

The symbol o means a structure element to which no unary relation a_i is assigned, that is, an empty or not named structure element.

The symbol a_i in a rule means a structure element to which the unary relation (name) a_i is assigned.

We build rules of the in the machine M_T following way:
Renaming rules
Two rules $q_h^{odd}a_i(r, l) \to a_jq_k^{odd}$ and $q_h^{even}a_i(r, l) \to a_jq_k^{even}$ are assigned to the rule $q_ha_i \to a_jq_k$
Two rules $q_h^{odd}o(r, l) \to a_jq_k^{odd}$ and $q_h^{even}o(r, l) \to a_jq_k^{even}$ are assigned to the rule $q_h\Lambda \to a_jq_k$
Two rules $q_h^{odd}a_j(r, l) \to oq_k^{odd}$ and $q_h^{even}a_j(r, l) \to oq_k^{even}$ are assigned to the rule $q_ha_j \to \Lambda q_k$

Transition rules
Two rules $q_h^{odd}a_i(r, l) \to q_k^{even}r$ and $q_h^{even}ai(r, l) \to q_k^{odd}l$ are assigned to the rule $q_ha_i \to q_kL$
Two rules $q_h^{odd}a_i(r, l) \to q_k^{even}l$ and $q_h^{even}a_i(r, l) \to qkoddr$ are assigned to the rule $q_ha_i \to q_kR$
Two rules $q_h^{odd}(r, l) \to q_k^{even}r$ and $q_h^{even}o(r, l) \to a_jq_koddl$ are assigned to the rule $q_h\Lambda \to q_kL$
Two rules $q_h^{odd}(r, l) \to q_k^{even}l$ and $q_h^{even}o(r, l) \to qkoddr$ are assigned to the rule $q_h\Lambda \to q_kR$
Construction rules
$q_h^{odd}aj(l) \to q_h^{odd}a_j(l)[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],
$q_h^{odd}aj(r) \to q_h^{odd}a_j(r)[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],
$q_h^{even}aj(l) \to q_h^{even}[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],
$q_h^{even}aj(r) \to q_h^{even}[o]$ [creation of a new structural element near the named (full) end-element where the processor is situated],
$q_h^{odd}aj(l) \to q_h^{odd}a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],
$q_h^{odd}aj(r) \to q_h^{odd}a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],
$q_h^{even}aj(l) \to q_h^{even}a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],
$q_h^{even}aj(l) \to q_h^{even}a_j(r, l)$ [connecting a new structural element to the nearby (full) named end-element where the processor is situated],
$q_h^{odd}o(l) \to q_h^{odd}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],

$q_h^{odd}o(r) \rightarrow q_h^{odd}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],
$q_h^{even}o(l) \rightarrow q_h^{even}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],
$q_h^{even}o(r) \rightarrow q_h^{even}[o]$ [creation of a new structural element near the empty (not named) end-element where the processor is situated],
$q_h^{odd}o(l) \rightarrow q_h^{odd}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],
$q_h^{even}o(r) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],
$q_h^{even}o(l) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated],
$q_h^{even}o(l) \rightarrow q_h^{even}o(r, l)$ [connecting a new structural element to the nearby empty (not named) end-element where the processor is situated].

Construction operations add new structural elements and lacking relations, allowing processor to perform prescribed movements in all cases.

The structural machine M_T works following these rules. When it comes either to a state q_h^{even} or q_h^{odd} with q_h from F, then the machine M_T stops and the created structure with filled structure elements is the result of computation of the structural machine M_T.

When the processor of the structural machine M_T is at an odd structure element, then the state of the machine is some q_h^{odd}, and when the processor is at an even structure element, then the state of the machine is some q_h^{even}. When the head of the Turing machine T comes to the end of the tape, the processor of the structural machine M_T creates an empty structure element and the adequate connection to this element. The goal is to allow the processor of M_T to go (by this connection) to the empty structure element, simulating in such a way the move of the head of T to the empty cell.

As a result of these operations, the structural machine M_T simulates the Turing machine T, while the number of performed operations is $O(n)$ when T makes n operations.

Theorem is proved.

Being able to simulate Turing machines, structural machines can be more efficient than Turing machines. To show this, we take some alphabet Σ and consider the following algorithmic problem.

The Word Symmetry Problem. Given an arbitrary word w in the alphabet Σ, find if $w = uu^\star$ for some word u where $u^\star$ is the inverse of u.

A word $w = uu^\star$ is called a palindrome because written backwards it coincides with itself.

Theorem 3.2. *A structural machine with a centralized processor can solve the Word Symmetry Problem with the linear time complexity, i.e., with time $T(n) = O(n)$.*

Let us describe the structural machine $M = (C, P, \mathcal{R})$ with the control device C, processor P and processing space $\mathcal{R}$, which coincides with the input space and output space, such that M solves the Word Symmetry Problem with the linear time complexity.

In it, the control device C is a finite automaton with the states $Q = \{q_0, (q_0, a), (q_1, a), (q_2, a), q_f; a \in \Sigma\}$, the final states $F = \{q_f\} \subseteq Q$, and the start state q_0 .

The processor P is similar to the head of a Turing machine but it observes not only the content of a cell but also its connections, and P can change not only the content of the observed cell but

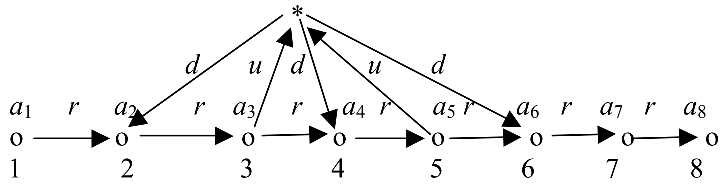


Figure 5. The graphical representation of processed first order structures.

also its connections.

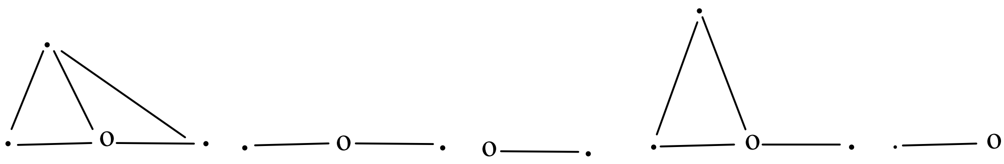
$\mathcal{R} = \{R, L, U, D, £\}$ where:
R is a binary relation elements of which are denoted (named) by r;
U is a binary relation elements of which are denoted (named) by u;
DB is a binary relation elements of which are denoted (named) by db;
DE is a binary relation elements of which are denoted (named) by de;
L is a binary relation elements of which are denoted (named) by l;
Σ contains unary relations a where $a \in \Sigma$.
Relations r, u, d and l are also called connections or links. The unary relations a from Σ are used for naming the structure elements by symbols from Σ.

The machine M processes first-order structures that have the form $A = (N \cup \{^\star\}; z, \mathcal{R}a)$ where $N = \{1, 2, 3, \ldots, k\}$, $\mathcal{R}a = \{$ our binary relations $D_a \subseteq D$, $U_a \subseteq U$, $R_a \subseteq R$ and $L_a \subseteq L$ in which links r and l can connect elements t and $t + 1$ for all $t = 1, 2, 3, \ldots, k - 1$, while links u and d can connect elements t with the element *; and m unary relations $a_1, a_2, \ldots, a_m \in \Sigma$ $\}$ and z is the assignment of relations from $\mathcal{R}a$ (names from the alphabet Σ) to elements from N.

An arbitrary input structure has the form $A_0 = (N_0 = \{1, 2, 3, \ldots, h\}, z_0, \mathbf{R}_0)$ where $R_0 = \{$ a binary relation $\mathbf{R}_0 \subseteq \mathbf{R}$ and m unary relations $a_1, a_2, \ldots, a_m \in \Sigma\}$ and links r from R_0 connect elements t and $t + 1$ for all $t = 1, 2, 3, \ldots, h - 1$.

The processor P of the machine M performs the following operation:
mv$[c(a, b)]$ denotes the transition of P from its topos (the cell where it is situated) with the name a to the cell with the name b by the link with the name c, which connects these two cells.
ch$[q \to p]$ denotes changing the state of M from q to the name p.
rn$[c(a, b) \to k(a, b)]$ denotes renaming of the link with the name c by giving it the new name k.
rn$[a \to b]$ denotes renaming of the cell where P is situated by changing its name a to the name b.
rn$[c(a, e); e \to b]$ denotes renaming of the cell that has the name e and is connected by a link with the name c to the cell where P is situated by changing its name e to the name b.
bd$[^\star]$ denotes building a new cell.
nm$[^\star \to b]$ denotes naming a new cell.
bd$[c(a, ^\star)]$ denotes building a link with the name c from the cell where P is situated with the name a to a new cell.
bd$[c(a, b)]$ denotes building a link with the name c from the cell where P is situated with the name a to a cell with the name b.

Note that all these operations are local and $\mathcal{R}$–local. There are five types of neighborhoods



(a) (b) (c) (d) (e)

Figure 6. Types of neighborhoods of elements in processed first-order structures.

(Fig. 6) and all operations are performed only with parts (structure elements and links) of these neighborhoods.

Now we describe the rules of the machine M. In the rules, the left side describes the observation/control zone, while the right side shows the performed operation. Symbols a_j denote arbitrary elements from the alphabet Σ. We build the rules of the machine M in following way:

$[a_1; q_0] \to \mathbf{rn}[a \to N]$; $\mathbf{ch}[q_0 \to q_f]$

$[a_1; r(a_1, a_2); q_0] \to \mathbf{ch}[q_0 \to (q_0, a_1)]$; $mathbf fmv[r(a_1, a_2)]$

$[a_2; r(a_1, a_2); (q_0, a_1)] \to \mathbf{rn}[a_2 \to N]$; $\mathbf{ch}[(q_0, a_1) \to q_f]$ for all $a_2 \neq a_1$ $[a_1; r(a_1, a_2); (q_0, a_1)] \to \mathbf{rn}[a_2 \to T]$; $\mathbf{ch}[(q_0, a_1) \to q_f]$

$[a_2; r(a_1, a_2), r(a_2, a_3); (q_0, a_1)] \to \mathbf{bd}[\star]$; $\mathbf{nm}[\star \to \alpha]$; $\mathbf{bd}[db(a_2, \alpha)]$; $\mathbf{ch}[(q_0, a_1) \to (q_1, a_1)]$; $\mathbf{rn}[r(a_1, a_2) \to l(a_1, a_2)]$; $\mathbf{mv}[r(a_2, a_3)]$

$[a_3; r(a_2, a_3), r(a_3, a_4), db(a_2, \alpha); (q_1, a_1)] \to \mathbf{bd}[u(a_3, \alpha)]$; $\mathbf{mv}[r(a_3, a_4)]$; $\mathbf{ch}[(q_1, a_1) \to (q_0, a_1)]$

$[a_4; r(a_3, a_4), r(a_4, a_5), u(a_3, \alpha); (q_0, a_1)] \to \mathbf{bd}[u(a_4, \alpha)]$; $\mathbf{mv}[r(a_4, a_5)]$; $\mathbf{ch}[(q_0, a_1) \to (q_1, a_1)]$

$[a_k; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \to \mathbf{rn}[a_k \to N]$; $\mathbf{ch}[(q_0, a_1) \to q_f]$ for all $a_k \neq a_1$

$[a_k; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_1, a_1)] \to \mathbf{rn}[a_k \to N]$; $\mathbf{ch}[(q_1, a_1) \to q_f]$

$[a_k = a_1; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \to \mathbf{ch}[(q_0, a_1) \to q_1]$; $\mathbf{mv}[r(a_{k-1}, a_k)]$ (the case $a_k = a_1$)

$[a_{k-1}; r(a_{k-1}, a_k), u(a_{k-1}, \alpha); (q_0, a_1)] \to \mathbf{ch}[(q_0, a_1) \to (q_2, a_{k-1})]$; $\mathbf{rn}[r(a_{k-1}, a_k) \to l(a_{k-1}, a_k)]$; $\mathbf{rn}[r(a_{k-1}, a_{k-2}) \to l(a_{k-1}, a_{k-2})]$; $\mathbf{rn}[u(a_{k-2}, \alpha) \to de(a_{k-2}, \alpha)]$; $\mathbf{mv}[u(a_{k-1}, \alpha)]$

$[\alpha; de(a_{k-2}, \alpha), db(a_2, \alpha); (q_2, a_{k-1})] \to \mathbf{mv}[db(a_2, \alpha)]$

$[a_2; r(a_2, a_3), db(a_2, \alpha), l(a_1, a_2); (q_2, a_{k-1})] \to \mathbf{rn}[a_k \to N]$; $\mathbf{ch}[q_0 \to q_f]$ for all $a_{k-1} \neq a_2$

$[a_2; r(a_2, a_3), db(a_2, \alpha), l(a_1, a_2); (q_2, a_2)] \to \mathbf{rn}[db(a_2, \alpha) \to u(a_2, \alpha)]$; $\mathbf{mv}[r(a_2, a_3)]$ (the case $a_{k-1} = a_2$)

$[a_3; r(a_2, a_3), r(a_3, a_4), u(a_3, \alpha); (q_2, a_2)] \to \mathbf{ch}[(q_2, a_2) \to (q_2, a_3)]$; $\mathbf{rn}[r(a_2, a_3) \to l(a_2, a_3)]$; $\mathbf{rn}[r(a_3, a_4) \to l(a_3, a_4)]$; $\mathbf{rn}[u(a_4, \alpha) \to db(a_4, \alpha)]$; $\mathbf{mv}[u(a_3, \alpha)]$

$[\alpha; de(a_{k-2}, \alpha), db(a_4, \alpha); (q_2, a_3)] \to \mathbf{mv}[de(a_{k-2}, \alpha)]$ $[a_{k-2}; r(a_{k-2}, a_{k-3}), de(a_{k-2}, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \to \mathbf{rn}[a_{k-2} \to N]$; $\mathbf{ch}[(q_3, a_3) \to q_f]$ for all $a_{k-2} \neq a_3$

$[a_{k-2} = a_3; r(a_{k-2}, a_3), de(a_2, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \to \mathbf{rn}[de(a_2, \alpha) \to u(a_2, \alpha)]$; $\mathbf{mv}[r(a_{k-2}, a_{k-3})]$ (the case $a_{k-2} = a_3$)

$[a_{k-2}; r(a_{k-2}, a_{k-3}), de(a_{k-2}, \alpha), l(a_{k-1}, a_{k-2}); (q_2, a_3)] \to \mathbf{rn}[a_{k-2} \to T]$; $\mathbf{ch}[(q_3, a_3) \to q_f]$

The result of computations is defined in the following way:

- When the machine M comes to the final state and the name of the processor topos is N, then the result is negative, i.e., the input word is not symmetric.

- When the machine M comes to the final state and the name of the processor topos is T, then the result is positive, i.e., the input word is symmetric.

Using these rules the machine M checks if the input word is symmetric or not. In the process of computation, the processor P comes to each structure element (cell) not more than two times and in each cell, the processor P performs not more than five operations. Assuming, as it is done in the theory of algorithms and computation, that each operation takes one unit of time, we see that the machine M can solve the Word Symmetry Problem with the time complexity $T(n) = 10n$. Theorem is proved. At the same time, by Barzdins theorem, any deterministic Turing machine can solve the Word Symmetry Problem only with time complexity $O(n^2)$ (cf. (Trahtenbrot, 1974)). It means that computational complexity of structural machines is essentially less than computational complexity of Turing machines for some problems.

Another problem with time complexity $O(n^2)$ for Turing machines and with time complexity $O(n)$ for structural machines is inversion of a given word.

Theorem 3.3. *A structural machine with a centralized processor can simulate any simple inductive Turing machine.*

Proof. It is demonstrated how structural machine with a centralized processor can simulate any Turing machine with one tape. In the theory of algorithms and computation, it is proved that a Turing machine with one tape can simulate a Turing machine with any number of tapes. At the same time, a simple inductive Turing machine has exactly the same structure and operations (instructions) as a Turing machine with three tapes. Consequently, working in the inductive mode, a structural machine with a centralized processor can simulate any simple inductive Turing machine. □

It is also possible to simulate membrane computations (Păun & Rozenberg, 2002) with structural machines, while some classes of structural machines work as neural Turing machines (Graves *et al.*, 2014). However, these results are presented in another work of the authors.

4. Modeling slime mold computations and Physarum machines by structural machines

Physarum polycephalum belongs to the species of order *Physarales*, subclass *Myxogastromycetidae*, class *Myxomycetes*, division *Myxostelida*. It is commonly known as a true, acellular or multi-headed slime mould, see introduction in (Stephenson *et al.*, 1994). Plasmodium is a 'vegetative' phase, a single cell with a myriad of diploid nuclei. The plasmodium is visible to the naked eye. The plasmodium looks like an amorphous yellowish mass with networks of protoplasmic tubes.

The plasmodium behaves and moves as a giant amoeba forming a network of biochemical oscillators (Matsumoto *et al.*, 1986; Nakagaki *et al.*, 2000). The plasmodium's behavior is determined by external stimuli and excitation waves travelling and interacting inside the plasmodium. The plasmodium can be considered as a reaction-diffusion (Adamatzky, 2007) encapsulated in an elastic growing membrane.

When plasmodium is placed on an appropriate substrate, the plasmodium propagates, searches for sources of nutrients and follows gradients of chemo-attractants, humidity and illumination

Figure 7. Physarum forms a network of protoplasmic tubes on virtually any surface.

Figure 8. A spanning tree approximated by live Physarum. Planar data points are represented by oat flakes (dark blobs), edges of the tree by protoplasmic tubes; few duplicated tubes are considered as a single edge.

forming veins of protoplasm, or protoplasmic tubes (Fig. 7). The veins can branch, and eventually the plasmodium spans the sources of nutrients with a dynamic proximity graph, resembling, but not perfectly matching graphs from the family of k-skeletons (Kirkpatrick *et al.*, 1985).

Due to its unique features and relative ease of experimentation with, the plasmodium has become a test biological substrate for implementation of various computational tasks. The problems solved by the plasmodium include maze-solving, spanning trees and proximity graphs (Fig. 8, calculation of efficient networks, construction of logical gates, sub-division of spatial configurations of data points, and robot control (see overview in (Adamatzky, 2016)). A computation in the plasmodium is implemented by interacting biochemical and excitation waves, redistribution of electrical charges on plasmodiums membrane and spatiotemporal dynamics of mechanical waves. Plasmodium of P. polycephalum performs complex computation by three general mechanisms: morphological adaptation of its body plan and transport network, wave propagation of information through its protoplasmic transport network, and competition and entrainment of oscillations in partial bodies relatively small fragments of plasmodium connected via protoplasmic tubes. All three mechanisms are closely associated with one another (for example, morphological adaptation is dependent on local oscillatory activity and protoplasmic flux). In (Adamatzky, 2007), it is demonstrated how to simulate Kolmogorov algorithms (Kolmogorov, 1953; Kolmogorov & Uspensky, 1958) with living slime mould in experimental laboratory conditions.

To model a Physarum machine by a structural machine, we have to interpret components of a slime mould as components of a structural machine and behaviour of the slime mould as computations of the structural machine.

A Physarum machine PM is realized by a many-headed slime mould, which is a single cell with a myriad of diploid nuclei. It is possible to treat this cell as a primitive object SM with a set of inner states. Examples of such states are “to be alive or “not to be alive. In a context of physical measurements it would be more correct to use.

In this context, we represent the object SM by the control device C_M of the structural machine M, which models the Physarum machine. The control device C_M can be assigned to be an active growing zone.

A many-headed slime mould has several active growth zones exploring concurrently the physical space around the slime mould (Fig. 9). Thus, it is natural to treat a Physarum machine as a structural machine with a distributed processor P and to interpret each active growth zone as the operation zone of a unit processor p.

As it was already demonstrated, a first-order structure $\mathbf{A} = (A, r, \mathcal{R})$, in which the set $\mathcal{R}$ consists of a single binary relation R naturally represents the structure of a living slime mould established by blobs of slime mould and active zones where structural elements (e.g., nodes) from the set A represent blobs of slime mould and active zones, while elements from the relation R (e.g., edges) represent connecting tubes A Physarum machine has two types of nodes: stationary nodes presented by sources of nutrient (oat flakes), and dynamic nodes, which are sites where two or more protoplasmic tubes originate (Adamatzky, 2007).

However, a slime mould often has a more sophisticated structure. Despite being a single cell, the slime mould can colonize substantial areas, up to hundreds of centimeters. The network of blobs, active zones and protoplasmic tubes is not uniform but forms clusters. These clusters are also connected by thick protoplasmic tubes, which represent the incidence relation that connects groups of elements from A. Therefore, we use second-order structures to model a slime mould with clusters. Thus, taking a second-order structure $\mathbf{A} = (A, r, \mathcal{R})$, in which the set $\mathcal{R}$ consists of a binary relation R, a system of binary relations $C_1, C_2, C_3, \ldots, C_n$, and a binary relation Q, we represent the structure of a living slime with clusters in the following way:

- elements from the set A represent blobs of slime mould and active zones,

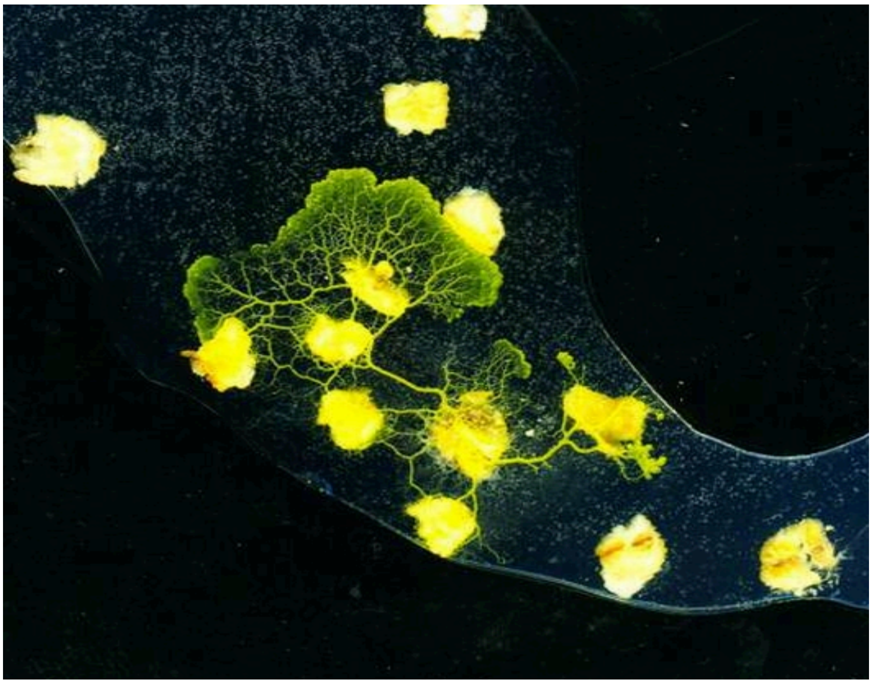


Figure 9. Physarum propagates in several directions simultaneously attracted by sources of nutrients.

- elements from the relation R represent tubes connecting blobs of the slime mould and active zones,
- each relation C_i represents one cluster of the slime mould, namely, if the cluster with the number i consists of blobs and active zones $a_1, a_2, a_3, \dots, a_m$, then $C_i = (a_1, a_2, a_3, \dots, a_m) \subseteq A^m$
- elements from the relation Q represent tubes connecting clusters.

This allows us to consider the sensorial space of the slime mould as the input space In_M of the machine M because the slime mould sees the world as a configuration of gradient fields.

The output space Out, which contains the output structure. The output space is the morphology of the slime mould, i.e., the configuration of growth zones, blobs occupying nutrients, and network protoplasmic tubes connecting them, is moulded by the output space Out_M of the machine M.

In a similar way, the cyto-skeletal network inside the slime mould body forms the processing space of the Physarum machine and is naturally modeled by the processing space PS_M of the structural machine M.

In slime mould, oscillatory patterns control the behaviors of the cell. In structural machines, oscillatory patterns are represented by the names of the nodes (structural elements) and links between these elements.

5. Conclusion

We determined and studied structural machines demonstrating that they can simulate Turing machines (Theorem 1) and inductive Turing machines (Theorem 3). We also proved that structural machines are more efficient than Turing machines (Theorem 2). In addition, we explained how structural machines describe and model behavior biological computers such as Physarum machine, which is based on slime mould *P. polycephalum*. This shows the big computational potential of slime mould as a biological computer. Further work can go in two directions: implementation of practical algorithms on structural machines and development of structural machines models for ultra-cellular computing based on cytoskeleton. The development of practical algorithms is necessary to allow the structural machines to 'enter the real world' and not just remain one of the many formal accomplishments of theoretical computer science. Physarum machines can solve dozens of problems from computational geometry, graph optimization and control. They also can be used as organic electronic elements (Erokhin *et al.*, 2012; Gale *et al.*, 2015; Adamatzky, 2014; Whiting *et al.*, 2015). The structural machine might form a platform for developing Physarum programming languages, compilers and interface between human operators and the slime mould (Schumann & Adamatzky, 2011; Schumann & Pancerz, 2014, 2015). The development of structural machine models of ultra-cellular computing is necessary because the behavior of the slime mould, as of most other cells, is governed by actin and tubuline networks inside the cells. Here we mention actin because it is a dominating cytoskeleton protein in P. polycephalum. Actin is a filament-forming protein forming a communication and information processing cytoskeletal network of eukaryotic cells. Actin filaments play a key role in developing synaptic structure, memory and learning of animals and humans. This is why it is important to develop abstractions of the information processing on the actin filaments. While designing experimental laboratory prototypes of computing devices from living slime mould *P. polycephalum* (Adamatzky, 2016), we found that actin networks might play a key role in distributed sensing, decentralized information processing and parallel decision making in a living cell (Adamatzky & Mayne, 2015; Mayne *et al.*, 2015). The actin-automata exhibit a wide a range of mobile and stationary patterns, which were later used to design computational models of quantum (Siccardi & Adamatzky, 2016) and Boolean (Siccardi *et al.*, 2016) gates implementable on actin fibre, as well as realization of universal computation with cyclic tag systems (Martínez *et al.*, 2017). The previously proposed model of an actin filament in a form of a finite-state machine, or automaton network, (Adamatzky & Mayne, 2015) constitutes a very special case of studied in this paper structural machines, which provide much more powerful tools for exploration of possibilities of biologically based computation. Detailed formalization of the information processing capabilities of the actin networks, including their polymerization and growths, and interaction with other intra-cellular proteins would immensely advance nano-computing and theoretical computer science making an imperative impact on development of future and emergent computing architectures. Structural machines provide means for simulation of membrane computations (Păun, 2000; Păun & Rozenberg, 2002), while some classes of structural machines work as neural networks or neural Turing machines (Graves *et al.*, 2014). Exposition of these results is given in another work of the authors. It is necessary to remark that due to their flexibility, definite classes of structural machines allow much better modeling of quantum computations than conventional models. Now there are various theoretical models of quantum computation: quantum

Turing machines (Deutsch, 1985) and quantum circuits (Feynman, 1986) correspond to discrete computing, while modular functors describe topological quantum computation (Freedman *et al.*, 2003), to mention but a few. It is interesting that while some theoretical models are recursive algorithms, which are not more powerful than Turing machines (Deutsch, 1985), other theoretical models are more powerful than Turing machines (Kieu, 2003). Structural machines provide a theoretical framework for unification of different models of quantum computation. This opportunity brings us to the open problem of modeling of quantum computation by structural machines.

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New Subclasses of Analytic and Bi-Univalent Functions Involving a New Integral Operator Defined by Polylogarithm Function

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Abstract

In the present investigation, we introduce two new subclasses of the function class σ of bi-univalent functions in the open unit disc. Also we find coefficient estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class and several related classes are also considered and connections to earlier known results are made.

Keywords: Analytic functions, univalent functions, bi-univalent functions, coefficient bounds.
2010 MSC: 30C45.

1. Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Further, the class of all functions in A which are univalent in U is denoted by the symbol S.
The Koebe one-quarter theorem (Duren, 1983) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z\,, \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w\,, \quad \left(|w| < r_0(f)\;,\; r_0(f) \geq \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, see (Srivastava *et al.*, 2010). The concept of bi-univalent function class was firstly studied by Lewin (Lewin, 1967) and obtained that the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie (Brannan & Clunie, 1980) conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu (Netanyahu, 1969) showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha (Brannan & Taha, 1986) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0 < \alpha \leq 1)$ respectively. The classes $\delta_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\delta^{\star}(\alpha)$ and $K(\alpha)$, were also introduced similarly. For each of the function classes $\delta_{\Sigma}^{\star}(\alpha)$ and $K_{\Sigma}(\alpha)$, non-sharp estimates on the initial coefficients were found by them. In recent years, bounds for various subclasses of bi-univalent functions were investigated by many authors ((Frasin & Aouf, 2011), (Srivastava *et al.*, 2010), (Xu *et al.*, 2012*b*)). For each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \backslash \{1, 2\}$, the problem of determining coefficient estimate is still an open problem. In the year 2010, the following subclasses of the bi-univalent function class Σ was introduced by Srivastava et al. (Srivastava *et al.*, 2010) and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ was obtained.

Definition 1.1. (Srivastava *et al.*, 2010) A function $f(z)$ given by the TaylorMaclaurin series expansion (1.1) is said to be in the class $\mathcal{H}_{\sigma}^{\alpha}$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|\arg\left(f'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(g'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

where the function g is given by

$$f^{-1}(w) = g(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

Theorem 1.1. *(Srivastava* et al.*, 2010) Let the function $f(z)$ given by (1.1) be in the class* $\mathrm{H}_{\Sigma}^{\alpha}(0 < \alpha \leq 1)$. *Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}} \qquad \textit{and} \qquad |a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

Definition 1.2. (Srivastava *et al.*, 2010) A function $f(z)$ given by (1.1) is said to be in the class $H_{\Sigma}^{\beta}(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|Re\left(f'(z)\right)\right| > \beta \quad (0 \leq \beta < 1,\ z \in U)$$

and

$$\left|Re\left(g'(w)\right)\right| > \beta \quad (0 \leq \beta < 1,\ w \in U)$$

where the function g is given by $f^{-1}(w) = g(w)$.

Theorem 1.2. *(Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class $\mathrm{H}_{\Sigma}^{\beta}$ $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \qquad \textit{and} \qquad |a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}.$$

Here, in our present sequel to some of the aforecited works (especially [15]), the following subclass of the analytic function class A is introduced. Also, by using the method of (Srivastava *et al.*, 2010), (Frasin & Aouf, 2011), (Xu *et al.*, 2012*b*) and (Xu *et al.*, 2012*a*) different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Porwal and Darus and get more accurate estimates than that given in (Porwal & Darus, 2013). For the functions $f \in A$ given by (1.1) and $g \in A$, $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their Hadamard product or convolution (Duren, 1983) is defined by the power series

$$(f * g)(z) = z + \sum_{k=2} a_k b_k z^k.$$

For $f(z) \in A$, Al-Shaqsi (AL-Shaqsi, 2014) defined the following integral operator:

$$\begin{aligned} L_c^{\delta} f(z) &= (1+c)^{\delta} \Phi_{\delta}(c; z) * f(z) \\ &= -\frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} \log\left(\frac{1}{t}\right)^{\delta-1} f(zt)\, dt \end{aligned} \tag{1.2}$$

$$(c > 0, \delta > 1, z \in U)$$

where Γ stands for the usual gamma function, $\Phi_{\delta}(c; z)$ is the well known generalization of the Riemann-zeta and polylogarithm functions, or the δth polylogarithm function, given by

$$\Phi_{\delta}(c; z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^{\delta}}$$

where any term without $k+c = 0$ (see (Lerch, 1887) and (Bateman, 1953)(sections 1.10 and 1.12)). Also, $\Phi_{-1}(0; z) = \frac{z}{(1-z)^2}$ is Koebe function. One can find more details about polylogarithms in theory of univalent functions in the study of Ponnusamy and Sabapathy (Ponnusamy & Sabapathy, 1996).

We also state that the operator $\mathcal{L}_c^{\delta} f(z)$ given by the relation (1.2) can be expressed by the series expansions as follows:

$$\mathcal{L}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^{\delta} a_k z^k.$$

First of all, we present the following lemma to prove our main result

Lemma 1.1. *(Pommerenke & Jensen, 1975) If $h \in \mathbb{P}$ then $| c_k | \leq 2$ for each k, where $\mathbb{P}$ is the family of all functions h analytic in E for which $Re(h(z)) > 0$, then*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots .$$

2. Coefficient Estimates for the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$

Definition 2.1. The class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$ of the functions $f(z)$ determined by the equality (1.1) consists of those functions $f(z)$ that satisfy the following conditions:
$f \in \sigma$,

$$Re\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z}\right) > \beta \tag{2.1}$$

where $0 \leq \beta < 1, \lambda \geq 1, c > 0, Re\delta > 1, z \in U$ and

$$Re\left(\frac{(1-\lambda)\mathcal{L}_c^{\delta} g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w}\right) > \beta. \tag{2.2}$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function introduced and studied by Al-Shaqsi and the function g is given by $g(w) = f^{-1}(w)$.

Remark. If we let $c = 0$ and $\delta = -n$, for $n \in \mathbb{N} \cup \{0\}$, then we obtain

$$\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c) = H_{\Sigma}(n, \beta, \lambda)$$

studied by Porwal and Darus (Porwal & Darus, 2013). This class contains the function $f \in \Sigma$ satisfying

$$Re\left(\frac{(1-\lambda)\mathcal{D}^n f(z) + \lambda \mathcal{D}^{n+1} f(z)}{z}\right) > \beta$$

and

$$Re\left(\frac{(1-\lambda)\mathcal{D}^n g(w) + \lambda \mathcal{D}^{n+1} g(w)}{w}\right) > \beta.$$

where $\mathcal{D}^n$ stands for Salagean derivative introduced by Sâlâgean (Salagean, 1983).

The class $\mathcal{B}_{\sigma}^{-n}(\beta, \lambda, 0)$ includes many earlier classes, which are mentioned below:

1. If we let $n = 0$, then we have

$$\mathcal{B}_{\sigma}^{-n}(\beta, \lambda, 0) = H_{\Sigma}^{\lambda}(\beta)$$

studied by Frasin and Aouf (Frasin & Aouf, 2011). This class contains the functions $f \in \Sigma$ satisfying

$$Re\left(\frac{(1-\lambda)f(z)}{z} + \lambda f'(z)\right) > \beta$$

and

$$Re\left(\frac{(1-\lambda)g(w)}{w} + \lambda g'(w)\right) > \beta.$$

2. If we let $n = 0$ and $\lambda = 1$, then we have

$$\mathcal{B}_{\sigma}^{-n}(\beta, 1, 0) = H_{\Sigma}(\beta)$$

studied by Srivastava et al.(Srivastava *et al.*, 2010). This class contains the functions $f \in \Sigma$ satisfying

$$Re\left(f'(z)\right) > \beta$$

and

$$Re\left(g'(w)\right) > \beta.$$

The next theorem gives the estimate on coefficient of the function in the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$ given in Definition 2.1.

Theorem 2.1. *Let the function $f(z)$ given by equation (1.1) be in the class $\mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}} \tag{2.3}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(1 + \frac{\lambda}{1+c}\right)^2 \left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1 + \frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}} \tag{2.4}$$

where $0 \leq \beta < 1$ and $\lambda \geq 1$.

Proof. Let $f \in \mathcal{B}_{\sigma}^{\delta}(\beta, \lambda, c), \lambda \geq 1$ and $0 \leq \beta < 1$. Using argument inequalities in (2.1) and (2.2), we can state their forms as follows:

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta} f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z} = \beta + (1-\beta)p(z) \quad (z \in U) \tag{2.5}$$

and

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta} g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w} = \beta + (1-\beta)q(w) \quad (w \in U) \tag{2.6}$$

where $p(z)$ and $q(w)$ given by the equalities

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.7}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \tag{2.8}$$

satisfy the inequalities $Re(p(z)) > 0$ and $Re(q(w)) > 0$ respectively. Equating coefficients (2.5) and (2.6) yields

$$\left(1 + \frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta} a_2 = (1-\beta)p_1, \tag{2.9}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} a_3 = (1-\beta)p_2, \tag{2.10}$$

and

$$-\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta} a_2 = (1-\beta)q_1 \tag{2.11}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} (2a_2^2 - a_3) = (1-\beta)q_2. \tag{2.12}$$

From (2.9) and (2.11) , we have

$$p_1 = -q_1 \tag{2.13}$$

and

$$2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta} a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{2.14}$$

Also, adding (2.10) to (2.12) , we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} a_2^2 = (1-\beta)(p_2+q_2). \tag{2.15}$$

Applying Lemma 1.1 for equality (2.15), we have

$$|a_2|^2 \le \frac{(1-\beta)(|p_2|+|q_2|)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}} \le \frac{2(1-\beta)}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}$$

This gives the bound on $|a_2|$ as asserted in (2.3).

Next, to find the bound on $|a_3|$, by subtracting (2.12) from (2.10), we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta} (a_3 - a_2^2) = (1-\beta)(p_2-q_2) \tag{2.16}$$

which, upon substitution of value of a_2^2 from (2.14) yields

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{(1-\beta)(p_2-q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}.$$

Applying the lemma 1 for the coefficients p_1, q_1, p_2 and q_2, we readily get

$$|a_3| \le \frac{4(1-\beta)^2}{\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2(1-\beta)}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}.$$

□

Remark. Choosing $c = 0$ in Theorem 2.1, we have the following corollaries:

1. If we let $\delta = -n, (n \in \mathbb{N} \cup \{0\})$, then we obtain the following:

Corollary 2.1. *(Porwal & Darus, 2013) Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma(n,\beta,\lambda)$,*

$0 \le \beta < 1, \lambda \ge 1, n \in \mathbb{N}_0$. *Then,*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(1-\lambda)\,3^n + \lambda 3^{n+1}}}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{\left[(1-\lambda)\,2^n + \lambda 2^{n+1}\right]^2} + \frac{2(1-\beta)}{(1-\lambda)\,3^n + \lambda 3^{n+1}}.$$

2. Especially, choosing $n = 0$ in Corollary 2.1, we have the following result:

Corollary 2.2. *(Frasin & Aouf, 2011) Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma^\lambda(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{1+2\lambda}}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(1+2\lambda)}.$$

Remark. The estimates for $|a_2|$ and $|a_3|$ of Corollary 2.2 and Corollary 2.3 show that Theorem 2.1 coincides with the the estimates obtained by Frasin and Aouf (Frasin & Aouf, 2011).

3. If we choose $n = 0$ and $\lambda = 1$, then we obtain the following corollary:

Corollary 2.3. *(Srivastava et al., 2010) Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3}}$$

and

$$|a_3| \le \frac{(1-\beta)(5-3\beta)}{3}.$$

3. Coefficient Estimates for the class $\mathcal{H}_\sigma^\delta(\alpha, \lambda, c)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\sigma^\delta(\alpha, \lambda, c)$ if the following conditions are satisfied:

$$f \in \sigma, \quad \left|\arg\left(\frac{(1-\lambda)\mathcal{L}_c^\delta f(z) + \lambda \mathcal{L}_c^{\delta-1} f(z)}{z}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ \lambda \ge 1,\ z \in U) \tag{3.1}$$

and

$$\left|\arg\left(\frac{(1-\lambda)\mathcal{L}_c^\delta g(w) + \lambda \mathcal{L}_c^{\delta-1} g(w)}{w}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ \lambda \ge 1,\ w \in U) \tag{3.2}$$

where $\mathcal{L}_c^{\delta-1}$ stands for polylogarithm function and the function (by Al-Shaqsi) $g(w) = f^{-1}(w)$.

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{H}_{\sigma}^{\delta}(\alpha,\lambda,c)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\alpha-(\alpha-1)\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}}} \tag{3.3}$$

and

$$|a_3| \le \frac{4\alpha^2}{\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}} + \frac{2\alpha}{\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}}, where 0 \le \beta < 1 and \lambda \ge 1. \tag{3.4}$$

Proof. Let $f \in \mathcal{H}_{\sigma}^{\delta}(\alpha,\lambda,c)$, $\lambda \ge 1$ and $0 < \alpha \le 1$. We can write the argument inequalities in (3.1) and (3.2) as follows:

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta}f(z)+\lambda\mathcal{L}_c^{\delta-1}f(z)}{z} = [p(z)]^{\alpha}, \quad z \in U \tag{3.5}$$

$$\frac{(1-\lambda)\mathcal{L}_c^{\delta}g(w)+\lambda\mathcal{L}_c^{\delta-1}g(w)}{w} = [q(w)]^{\alpha}, \quad w \in U \tag{3.6}$$

where $p(z)$ and $q(w)$ are given by (2.7) and (2.8) and satisfy the inequalities $Re(p(z)) > 0$ and $Re(q(w)) > 0$ respectively. Now, equating the coefficients of (3.5) and (3.6), we have

$$\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta}a_2 = \alpha p_1, \tag{3.7}$$

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{3.8}$$

$$-\left(1+\frac{\lambda}{1+c}\right)\left(\frac{1+c}{2+c}\right)^{\delta}a_2 = \alpha q_1, \tag{3.9}$$

and

$$\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}(2a_2^2-a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2, \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2\left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}a_2^2 = \alpha^2(p_1^2+q_1^2) \tag{3.12}$$

Also from (3.8) and (3.10), we obtain

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2). \tag{3.13}$$

By using the relation (3.12) in (3.13), we find that

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}a_2^2 = \alpha(p_2+q_2) + \left(1+\frac{\lambda}{1+c}\right)^{2}\left(\frac{1+c}{2+c}\right)^{2\delta}a_2^2.$$

Thus we get

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\alpha - (\alpha-1)\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.14}$$

Then, applying Lemma1.1 for the aforementioned equality, we get desired estimate on $|a_2|$ as asserted in (3.3). Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}\left(a_3-a_2^2\right) = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{3.15}$$

Also from (3.11) , (3.12) and (3.15) we find that

$$a_3 = \frac{\alpha(p_2-q_2)}{2\left(1+\frac{2\lambda}{1+c}\right)\left(\frac{1+c}{3+c}\right)^{\delta}} + \frac{\alpha^2(p_1^2+q_1^2)}{2\left(1+\frac{\lambda}{1+c}\right)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}. \tag{3.16}$$

By applying the Lemma1 for the equality (3.16), we obtain desired estimate and this complats the proof of the theorem. □

Remark. If we let $c = 0$ in Theorem 3.1 and

1. $\delta = -n$, we obtain the following corollary:

Corollary 3.1. *(Porwal & Darus, 2013) Let the function $f(z)$ given by (1.1) be in the class $B_{\Sigma}(n,\alpha,\lambda), 0<\alpha\leq 1, \lambda\geq 1, n\in \mathbb{N}_0$. Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{4^n(\lambda+1)^2+\alpha\left(2.3^n(1+2\lambda)-4^n(\lambda+1)^2\right)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2} + \frac{2\alpha}{(1-\lambda)3^n+\lambda 3^{n+1}}.$$

2. Choosing $\delta = 0$, we obtain the following corollary:

Corollary 3.2. *(Frasin & Aouf, 2011). Let the function $f(z)$ given by (1.1) be in the class $B_{\Sigma}(\lambda,\beta), 0<\alpha\leq 1, \lambda\geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2+\alpha\left(1+2\lambda-\lambda^2\right)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}.$$

3. Also, if we choose $\lambda = 1$, we have the following corollary:

Corollary 3.3. *(Srivastava et al., 2010). Let the function $f(z)$ given by (1.1) be in the class $H_{\Sigma}^{\alpha}, 0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{2+\alpha}}$$

and

$$|a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

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Statistical Convergence and C^*-operator Algebras

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Abstract

In this article, we define in terms of Berezin symbols, reproducing kernels and statistical radial convergence the notion of generalized Engliš algebra, which is a C^*-operator algebra on the Hardy space $H^2(\mathbb{D})$, and study its some properties.

Keywords: Berezin symbol, Engliš algebra, statistical convergence, Toeplitz operators, Hardy space.
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1. Introduction

This article is mainly motivated with the papers by Engliš (Engliš, 1995), Karaev (Karaev, 2002, 2004, 2010) and Pehlivan and Karaev (Pehlivan & Karaev, 2004), where the authors systematically applied the Berezin symbols method in summability theory; and conversely, the summability methods are used in investigation of some important problems for C^*-operator algebras on the Hardy space and also on the Bergman space. In particular, Pehlivan and Karaev investigated in (Pehlivan & Karaev, 2004) compactness of the weak limit of the sequence of compact operators on a Hilbert space by using the notion of so-called statistical convergence.

In this article, we use statistical radial limits for the study of some special C^*-operator algebras on the classical Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. These results generalize some results in the paper (Engliš, 1995).

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right)\right|^2 dt < +\infty.$$

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Alternately, H^2 consists of all functions in $L^2(\mathbb{T})$ whose negative Fourier coefficients vanish; here $\mathbb{T} = \partial\mathbb{D}$ is the unit circle in $\mathbb{C}$. The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 will be denoted P_+, and $P_- : I - P_+$. For $\varphi \in L^\infty(\mathbb{T})$ the Toeplitz operator T_φ and Hankel operator H_φ with symbol φ are defined by $T_\varphi f = P_+\varphi f$ and $H_\varphi f = P_-\varphi f$ and are bounded linear operators from H^2 into H^2 $\left(\text{i.e., } T_\varphi \in \mathcal{B}\left(H^2\right)\right)$ and $L^2(\mathbb{T}) \ominus H^2$, respectively. For $\lambda \in \mathbb{D}$, the reproducing kernel (Szegö kernel) of H^2 is the function $k_\lambda \in H^2$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for every $f \in H^2$. The normalized reproducing kernel $\widehat{k_\lambda}$ is the function $\frac{k_\lambda}{\|k_\lambda\|}$. It is well-known (and can be easily shown) that $k_\lambda(z) := \frac{1}{1-\overline{\lambda}z}$. The algebra of all bounded linear operators on the Hilbert space H is denoted by $\mathcal{B}(H)$. For $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space over some set Ω with the reproducing kernel $k_{\mathcal{H},\lambda} \in \mathcal{H}$, its Berezin symbol $\widetilde{T}$ is the complex-valued function on Ω defined by

$$\widetilde{T}(\lambda) = \left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle \ \ (\lambda \in \Omega).$$

It is well-known that $\widetilde{T_\varphi}(\lambda) = \widetilde{\varphi}(\lambda)$, where $\widetilde{\varphi}$ denotes the harmonic extension of φ into the unit disc $\mathbb{D}$ (see Engliš (Engliš, 1995), Zhu (Zhu, 1990) and Karaev (Karaev, 2002)).

Engliš determined in (Engliš, 1995) in terms of nontangential and radial convergences some C^*-operator algebras on the Hardy space $H^2(\mathbb{D})$, which is defined by the boundary behavior of $\left\|T\widehat{k}_\lambda\right\|$, $\left\|T^*\widehat{k}_\lambda\right\|$ and $\left|\widetilde{T}(\lambda)\right|$. Let $\mathcal{T}$ be the C^*-algebra generated by $\left\{T_\varphi : \varphi \in L^\infty(\mathbb{T})\right\}$. The following celebrated result due to Douglas (Douglas, 1972) (see also in (Nikolski, 1986)).

Theorem D (Douglas). *There is a C^*-homomorphism*

$$\sigma : \mathcal{T} \to L^\infty(\mathbb{T})$$

of $\mathcal{T}$ onto $L^\infty(\mathbb{T})$ which satisfies $\sigma\left(T_\varphi\right) = \varphi \ (\forall\varphi \in L^\infty(\mathbb{T}))$. The kernel of σ coincides with the commutator ideal of $\mathcal{T}$, i.e., the ideal in $\mathcal{T}$ generated by all commutators

$$[R, S] := RS - SR \ \ (R, S \in \mathcal{T}).$$

σ is sometimes called the symbol map.

Note that the major goal of the Engliš's paper (Engliš, 1995) is to develop an alternative approach for proving results akin to the Douglas theorem. The symbol of an operator $T \in \mathcal{T}$ is then obtained in (Engliš, 1995) as the nontangential boundary value of a certain function on $\mathbb{D}$ associated with T (called the Berezin symbol (transform), $\widetilde{T}$, of T to be defined above). Following by (Engliš, 1995), remark that Engliš's method also works for some operator algebras larger than the Toeplitz algebra, thereby yielding a number of interesting generalizations of the classical Toeplitz symbol calculus. This method is also applicable to the Bergman space, where the resulting symbol calculus is related to the one obtained by Berger and Coburn in (Berger & Coburn, 1986, 1987), Gürdal and Şöhret in (Gurdal & Sohret, 2011) and Zhu in (Zhu, 1987).

In the present article, we replace nontangential and radial limits by so-called statistical nontangential and statistical radial limits (which are weaker than the usual one) and define generalizations of some Engliš's algebras. It turns out that the same results are true for these generalized Engliš's algebras. Before giving our results, let us recall the definition of the statistical convergence of real or complex numbers sequence.

If K is a subset of the positive integers $\mathbb{N}$, the K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see ((Niven & Zuckerman, 1980), Chapter 11) is given by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$. A sequence $(x_k : k = 1, 2, ...)$ of (real or complex) numbers is said to be statistically convergent to some number L if for each $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case we write $st - \lim_k x_k = L$. In what follows statistical convergence studied in many further papers (see, for instance, (Braha *et al.*, 2014), (Mursaleen *et al.*, 2014)).

The following notion is due to Fridy (Fridy, 1985). A sequence (x_k) is said to be statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \varepsilon\}| = 0.$$

We recall that (see (Fridy, 1985)) for two sequences $x = (x_k)$ and $y = (y_k)$ the notion "$x_k = y_k$ for almost all k" means that $\delta(\{k : x_k \neq y_k\}) = 0$. Fridy proved the following main result of this theory (Fridy, 1985).

Theorem F (Fridy). *The following statements are equivalent:*
(1) (x_k) is a statistically convergent sequence;
(2) (x_k) is a statistically Cauchy sequence;
(3) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ for almost all k.

The following result is immediate from Theorem F.

Corollary 1.1. *If (x_k) is a sequence such that st-$\lim_k x_k = L$, then (x_k) has a subsequence (y_k) such that $\lim_k y_k = L$ (in the usual sense).*

2. On some properties of Berezin Symbols

In this section, we prove some results concerning to Berezin symbols.

2.1. *Approach Regions*

For $0 < \alpha < 1$, following by Rudin (Rudin, 1974), pp. 240-241, let us define Ω_α to be the union of the disc $D(0;\alpha) := \{z \in \mathbb{C} : |z| < \alpha\}$ and the line segments from $z = 1$ to points of $D(0;\alpha)$. In other words, Ω_α is the smallest convex open set that contains $D(0;\alpha)$ and has the point 1 in its boundary. Near $z = 1$, Ω_α is an angle, bisected by the radius of $\mathbb{D}$ that terminates at 1, of opening 2θ, where $\alpha = \sin\theta$. Curves that approach 1 within Ω_α cannot be tangent to $\mathbb{T}$. Therefore Ω_α is called a nontangential approach region, with vertex 1. The regions Ω_α expand when α increases. The union is $\mathbb{D}$, their intersection is the radius $[0, 1)$. Rotated copies of Ω_α, with vertex at e^{it}, will be denoted by $e^{it}\Omega_\alpha$.

2.2. *Statistical Nontangential and Radial Limits.*

A function F, defined in $\mathbb{D}$, is said to have statistical nontangential limit λ at $e^{i\theta} \in \mathbb{T}$ if, for each $\alpha < 1$,

$$st - \lim_{j\to\infty} F\left(z_j\right) = \lambda$$

for every sequence $\left\{z_j\right\}$ that statistically converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$. If F is a function on $\mathbb{D}$, and f a function on $\mathbb{T}$, we say that F tends to f statistically radially, written

$$st - \lim F\left(z_n\right) = f\left(e^{it}\right)$$

whenever $(z_n) \to e^{i\theta}$ statistically radially (i.e., if st-$\lim_{n\to\infty} |z_n|\, e^{i\theta} = e^{i\theta}$) for almost all $\theta \in [0, 2\pi)$.

2.3. *Some Properties of Berezin Symbols of Operators.*

For any two functions $\varphi, \psi \in L^\infty(\mathbb{T})$, let us denote $\left[T_\varphi, T_\psi\right) := T_{\varphi\psi} - T_\varphi T_\psi$, which is called their semicommutator.

Theorem 2.1. *For $\varphi \in L^\infty(\mathbb{T})$, $\left\|H_\varphi \widehat{k}_\lambda\right\| \to 0$ statistically radially. Consequently, for any $\varphi, \psi \in L^\infty(\mathbb{T})$, $\left[T_\varphi, T_\psi\right)^{\sim}(\lambda) \to 0$ statistically radially.*

Proof. Following the method of the paper (Engliš, 1995), let $y := P_-\varphi$. By considering that $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^2}} \in H^\infty$, we may write

$$H_\varphi \widehat{k}_\lambda = P_-\left(\varphi \widehat{k}_\lambda\right) = P_-\left(\left(P_+\varphi\right)\widehat{k}_\lambda\right) + P_-(y\widehat{k}_\lambda) = P_-(y\widehat{k}_\lambda) = H_y \widehat{k}_\lambda,$$

and hence

$$\left\|H_\varphi \widehat{k}_\lambda\right\|^2 = \left\|y\widehat{k}_\lambda\right\|^2 - \left\|T_y \widehat{k}_\lambda\right\|^2 = \left\langle |y|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \left\|T_y \widehat{k}_\lambda\right\|^2 = \widetilde{|y|^2}(\lambda) - \left\|T_y \widehat{k}_\lambda\right\|^2.$$

Then, for any $\beta \in \mathbb{D}$, we have

$$\left\langle T_y k_\lambda, k_\beta\right\rangle = \left\langle k_\lambda, \overline{y} k_\beta\right\rangle = \widetilde{y}(\lambda)\left\langle k_\lambda, k_\beta\right\rangle,$$

since $\overline{y}$ is the boundary value of an analytic function. It follows that $T_\varphi \widehat{k}_\lambda = \widetilde{y}(\lambda)\widehat{k}_\lambda$ and

$$\left\|H_\varphi \widehat{k}_\lambda\right\|^2 = \widetilde{|y|^2}(\lambda) - |\widetilde{y}(\lambda)|^2.$$

By Fatou's theorem, both $\widetilde{|y|^2}$ and $|\widetilde{y}|^2$ tend radially to $|y|^2$, which implies that they tend statistically radially to $|y|^2$, and so their difference statistically tends to zero and we are done.

Observe that, $\left[T_\varphi, T_\psi\right) = H_{\overline{\varphi}}^* H_\psi$, and therefore we obtain that

$$\begin{aligned} \left|\left[T_\varphi, T_\psi\right)^{\sim}(\lambda)\right| &= \left|\left\langle H_{\overline{\varphi}}^* H_\psi \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| = \left|\left\langle H_\psi \widehat{k}_\lambda, H_{\overline{\varphi}} \widehat{k}_\lambda\right\rangle\right| \leq \\ &\leq \left\|H_\psi \widehat{k}_\lambda\right\| \left\|H_{\overline{\varphi}} \widehat{k}_\lambda\right\| \to 0 \end{aligned}$$

statistically radially, and the second assertion follows. The theorem is proved. □

Let us call the closed ideal in $\mathcal{T}$, generated by all semicommutators $\left[T_\varphi, T_\psi\right), \varphi, \psi \in L^\infty(\mathbb{T})$, the semicommutator ideal. The following strengthens Theorem 2.1.

Theorem 2.2. *If $T \in \mathcal{B}\left(H^2\right)$ belongs to the semicommutator ideal of $\mathcal{T}$, then $\widetilde{T} \to 0$ statistically radially.*

Proof. Since the linear combinations of Toeplitz operators of the form

$$T_{\varphi 1} T_{\varphi 2} \dots T_{\varphi n} \left[T_a, T_b\right) T_{\psi 1} T_{\psi 2} \dots T_{\psi m} \tag{1}$$

form a dense subset of the semicommutator ideal (and therefore it is a statistically dense subset of the ideal in question), it suffices to prove the assertion when T is of the form (1). So, we obtain (see also (Engliš, 1995))

$$\begin{aligned} T_c \left[T_a, T_b\right) &= T_c T_{ab} - T_c T_a T_b \\ &= \left(T_c T_{ab} - T_{cab}\right) + \left(T_{cab} - T_{ca} T_b\right) + \left(T_{ca} T_b - T_c T_a T_b\right) \\ &= -\left[T_c, T_{ab}\right) + \left[T_{ca}, T_b\right) + \left[T_c, T_a\right) T_b. \end{aligned}$$

It follows that we may even assume T to be of the form

$$T = \left[T_\varphi, T_\psi\right) A = H^*_{\overline{\varphi}} H_\psi A, \ A \in \mathcal{T}, \ \varphi, \psi \in L^\infty(\mathbb{T}).$$

Now by using that $\left[T_\varphi, T_\psi\right) = H^*_{\overline{\varphi}} H_\psi$, we have

$$\begin{aligned} \left|\widetilde{T}(\lambda)\right| &= \left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| = \left|\left\langle H^*_{\overline{\varphi}} H_\psi A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\ &= \left|\left\langle H_\psi A\widehat{k}_\lambda, H_{\overline{\varphi}}\widehat{k}_\lambda\right\rangle\right| \\ &\leq \left\|H_\psi A\right\| \left\|H_{\overline{\varphi}}\widehat{k}_\lambda\right\| \text{ (Cauchy-Schwarz inequality)} \end{aligned}$$

and by considering that $\left\|H_{\overline{\varphi}}\widehat{k}_\lambda\right\| \to 0$ statistically radially (see Theorem 2.1), we obtain that $\widetilde{T} \to 0$ statistically radially, as desired. □

Now we prove more general theorem which improves Engliš's result (see [4, Theorem 2]) and implies the theorem of Douglas (Theorem D) as an easy corollary.

Theorem 2.3. *For any T in $\mathcal{T}$, $\widetilde{T} \to \varphi$ statistically radially for some function $\varphi \in L^\infty(\mathbb{T})$. The mapping*

$$\sigma : \mathcal{T} \to L^\infty(\mathbb{T}), \ T \to \varphi$$

is a C^*-algebra morphism, its kernel is precisely the commutator ideal of $\mathcal{T}$, and $\sigma\left(T_\psi\right) = \psi$ for any Toeplitz operator T_ψ. Thus, σ coincides with the symbol map from Theorem D.

Proof. Let $\mathcal{J}$ be the semicommutator ideal in $\mathcal{T}$. As in (Berger & Coburn, 1987), by repeated applications of the identity $AT_aT_bB - AT_{ab}B = -A\left[T_a, T_b\right)B$, we have that $T_{\varphi 1} T_{\varphi 2} \dots T_{\varphi n} - T_{\varphi 1} T_{\varphi 2 \dots \varphi n} \in \mathcal{J}$ for any $\varphi_1, \varphi_2, \dots, \varphi_n \in L^\infty(\mathbb{T})$. By considering that linear combinations of operators of the form

$$T = T_\varphi + S, \ \varphi \in L^\infty(\mathbb{T}), S \in \mathcal{J}, \tag{2}$$

form a statistically dense subset of $\mathcal{T}$. According to the fact that $\widetilde{T_\varphi} = \widetilde{\varphi}$ and Theorem 2.2, $\widetilde{T} \to \varphi$ statistically radially, and thus we have $\|\varphi\|_\infty \leq \left\|\widetilde{T}\right\|_\infty \leq \|T\|$. It follows that the mapping

$$\sigma : T_\varphi + S \longmapsto \varphi$$

is well defined and extends continuously to the whole of $\mathcal{T}$, so that, in particular, every operator in $\mathcal{T}$ is of the form (2), because a statistical limit of Toeplitz operators is again a Toeplitz operator. Indeed, if

$$st - \lim_n \left\|T_{\varphi n} - X\right\| = 0$$

for some $X \in \mathcal{B}\left(H^2\right)$, then by Corollary 1, the sequence $\left(\left\|T_{\varphi_n} - X\right\|\right)_{n\geq 1}$ has a subsequence $\left(\left\|T_{\varphi_{n_k}} - X\right\|\right)_k$ such that $\lim_k \left\|T_{\varphi_{n_k}} - X\right\| = 0$ in the usual sense, which easily implies that X is a Toeplitz operator, and so $\mathcal{J}$ is statistically closed.

This mapping is clearly linear, preserves conjugation and, owing to Theorem 2.2, is also multiplicative. It is clear that its kernel is precisely the semicommutator ideal $\mathcal{J}$ of $\mathcal{T}$. Now it remains only to show that $\mathcal{J}$ coincides with the commutator ideal $\mathcal{G}$. Since $\left[T_\varphi, T_\psi\right] = \left[T_\psi, T_\varphi\right) - \left[T_\varphi, T_\psi\right)$, the inclusion $\mathcal{G} \subset \mathcal{J}$ is trivial. The reverse inclusion $\mathcal{J} \subset \mathcal{G}$ is proved in (Engliš, 1995), and therefore we omit it. The theorem is proved. □

Corollary 2.1. *If $A \in \mathcal{T}$, then $[A^*, A]^\sim \to 0$ statistically radially.*

3. Generalized Engliš algebras and extending the Toeplitz calculus

In the present section , we introduce the concept of generalized Engliš algebra of operators, study some properties and exhibit a family of generalized Engliš C^*- algebras containing $\mathcal{T}$ for which analogs of Theorem 2.3 still hold.

For this reason, let us define the following generalized Engliš algebra:

$$\mathcal{E} := \left\{H \in \mathcal{B}\left(H^2\right) : \left\|H\widehat{k_\lambda}\right\| \text{ and } \left\|H^*\widehat{k_\lambda}\right\| \to 0 \text{ statistically radially}\right\}.$$

In other words, we demand that

$$st - \lim_{r \nearrow 1} \left\|H\widehat{k}_{re^{i\theta}}\right\| = 0$$

for all $\theta \in [0, 2\pi) \setminus E$, where E is a set (depending on H) of zero Lebesgue measure; similarly for H^*. Note that in case of usual radial limits, this algebra is the usual Engliš algebra (Engliš, 1995).

In the following theorem we give some important properties of algebra $\mathcal{E}$.

Theorem 3.1. *We have:*
(a) *$\mathcal{E}$ is a C^*−algebra ;*
(b) *If $T_\varphi \in \mathcal{E}$, then $\varphi = 0$;*
(c) *For $\varphi, \psi \in L^\infty(\mathbb{T})$, $[T_\varphi, T_\psi) \in \mathcal{E}$;*
(d) *$\mathcal{E}$ is an "ideal with respect to Toeplitz operators", that is, $H \in \mathcal{E}$ and $\varphi \in L^\infty(\mathbb{T})$ implies that $HT_\varphi, T_\varphi H \in \mathcal{E}$.*

Proof. (a) Everything is trivial, except may be for the implication $A, B \in \mathcal{E} \Rightarrow AB \in \mathcal{E}$. But, $0 \leq \left\| AB\widehat{k_\lambda} \right\| \leq \|A\| \left\| B\widehat{k_\lambda} \right\| \longrightarrow 0$ statistically radially, and similarly for B^*A^*.
(b) Indeed, $\left\| T_\varphi \widehat{k_\lambda} \right\| \longrightarrow 0$ statistically radially implies that

$$\left| \widetilde{T}_\varphi(\lambda) \right| = \left| \left\langle T_\varphi \widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle \right| \leq \left\| T_\varphi \widehat{k_\lambda} \right\| \longrightarrow 0$$

statistically radially, and hence $\widetilde{T}_\varphi(\lambda) \longrightarrow 0$ statistically radially. But, we know that $\widetilde{T}_\varphi \longrightarrow \varphi$ statistically radially, so $\varphi = 0$.
(c) The proof is immediate from Theorem 2.1 and the equality $[T_\varphi, T_\psi) = H^*_{\overline{\varphi}} H_\psi$.
(d) Indeed, we have that

$$0 \leq \left\| T_\varphi H \widehat{k_\lambda} \right\| \leq \left\| T_\varphi \right\| \left\| H \widehat{k_\lambda} \right\| \longrightarrow 0$$

statistically radially, and similarly for $T_{\overline{\varphi}} H^* \widehat{k_\lambda}$. So, the corresponding assertions for $HT_\varphi \widehat{k_\lambda}$ are immediate from the following fact. □

Proposition 1. *Let $\varphi \in L^\infty(\mathbb{T})$, and denote, as before, by $\widetilde{\varphi}$ its harmonic extension (by the Poisson formula) into $\mathbb{D}$. Then $T_\varphi \widehat{k_\lambda} - \widetilde{\varphi}(\lambda)\widehat{k_\lambda} \longrightarrow 0$ statistically radially, i.e.,*

$$st - \lim_{r \nearrow 1} \left\| T_\varphi \widehat{k}_{re^{i\theta}} - \widetilde{\varphi}(re^{i\theta}) \widehat{k}_{re^{i\theta}} \right\| = 0$$

for almost all $t \in [0, 2\pi)$.

The proof of this proposition is immediate from Theorem 6 in [4] and Corollary 1.1 in Section 1.

Denote

$$\mathcal{A}_1 := \left\{ T_\varphi + H : \varphi \in L^\infty(\mathbb{T}),\ H \in \mathcal{E} \right\}.$$

The following theorem, which generalizes the Douglas theorem, can be proved by using Corollary 1.1 and the method of the proof of Theorem 7 in (Engliš, 1995) and therefore its proof is omitted.

Theorem 3.2. *We have;*
(i)$\mathcal{A}_1$ *is a C^*-algebra.*
(ii) *For any $T \in \mathcal{A}_1$, there exists a statistical radial limit , denoted $\sigma_{st}(T)$, of $\widetilde{T}(\lambda)$:*

$$\widetilde{T} \longrightarrow \sigma_{st}(T) \in L^\infty(\mathbb{T}) \text{ statistically radially.}$$

(iii) *$\sigma_{st} : \mathcal{A}_1 \longrightarrow L^\infty(\mathbb{T})$ induces a C^*-isomorphism of $\mathcal{A}_1/\mathcal{E}$ onto $L^\infty(\mathbb{T})$ which maps T_φ into φ, for any $\varphi \in L^\infty(\mathbb{T})$.*

The following result shows that Theorem 3.2 can also be used for the characterization of the class $\mathcal{T}$.

Proposition 2. *Let $A \in \mathcal{B}\left(H^2\right)$ be an operator. If $A \in \mathcal{T}$, then*
(i) *$\widetilde{A}$ has statistical radial limit : $\widetilde{A} \longrightarrow \varphi$ statistically radially for some $\varphi \in L^\infty(\mathbb{T})$; and*
(ii) *$\left\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\|$ and $\left\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\right\| \longrightarrow 0$ statistically radially.*

Proof. Let $A \in \mathcal{T} \subset \mathcal{A}_1$. Then clearly $A = T_\varphi + H$ for some $\varphi \in L^\infty(\mathbb{T})$ and $H \in \mathcal{E}$. Hence $\widetilde{A} \longrightarrow \varphi$ statistically radially, which proves *(i)*, and $\left\|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\right\| = \left\|H\widehat{k}_\lambda\right\| \longrightarrow 0$ statistically radially. By virtue of Proposition 1, this is equivalent to $\left\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\| \longrightarrow 0$ statistically radially, because

$$\begin{aligned}\left\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\| &\leq \left\|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\right\| + \left\|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\| \\ &\leq \left\|H\widehat{k}_\lambda\right\| + \left\|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\| \longrightarrow 0\end{aligned}$$

statistically radially. Similarly, it can be proved that $\left\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\right\| \longrightarrow 0$ statistically radially, which completes the proof of *(ii)*. So, the proposition is proved. □

Below we give some results further extending the result of Engliš from (Engliš, 1995) by means of statistical Banach limits. The proofs of them are slight modification of analogous results of the paper (Engliš, 1995), and therefore omitted.

Following by (Engliš, 1995) , recall that if $BC[0, 1)$ is the C^*-algebra of all bounded continuous functions on the half-open interval $[0, 1)$. It is known by Gelfand theory that $BC[0, 1) \simeq C(M)$, where M is the maximal ideal space of $BC[0, 1)$. M consist of a homeomorphic copy of $[0, 1)$ plus a certain fiber, denoted M_1, over the point 1. Each multiplicative linear functional $Lim \in M_1$ will be called a Banach limit. For $f \in L^\infty(\mathbb{D})$ and $\varphi \in L^\infty(\mathbb{T})$, we say that f tends to φ statistically radially with respect to Lim, written $f \overset{Lim}{\longrightarrow} \varphi$ statistically radially, when

$$Lim(r \overset{st}{\mapsto} f(re^{i\theta})) = \varphi(e^{i\theta})$$

for all $\theta \in [0, 2\pi)$ except for a set of measure zero. Define

$\mathcal{E}_{st-Lim} := \{H \in \mathcal{B}\left(H^2\right) : \left\|H\widehat{k}_\lambda\right\|$ and $\left\|H^*\widehat{k}_\lambda\right\| \overset{Lim}{\longrightarrow} 0$ statistically radially$\}$. It can be easily shown that all of assertions in Theorem 3.1 remain in force when $\mathcal{E}$ is replaced by $\mathcal{E}_{st-Lim}$, and thus we obtain the following analogs of Theorem 3.2 and Proposition 2.

Theorem 3.3. *Let $\mathcal{A}_{st-Lim} := \left\{T_\varphi + H : \varphi \in L^\infty(\mathbb{T}), H \in \mathcal{E}_{st-Lim}\right\}$. Then*
(i) *$\mathcal{A}_{st-Lim}$ is a C^*-algebra;*
(ii) *$\forall T \in \mathcal{A}_{st-Lim}\ \exists\varphi \in L^\infty(\mathbb{T})$ such that $\widetilde{T} \overset{Lim}{\longrightarrow} \varphi$ statistically radially;*
(iii) *The mapping $T \longmapsto \varphi$ is a C^* morphism of $\mathcal{A}_{st-Lim}/\mathcal{E}_{st-Lim}$ onto $L^\infty(\mathbb{T})$ which maps T_φ onto φ.*

Proposition 3. *Let A be an operator on H^2. A necessary and sufficient condition for $A \in \mathcal{A}_{st-Lim}$ is that*
(i) *there is $\varphi \in L^\infty(\mathbb{T})$ such that $\widetilde{A} \overset{Lim}{\longrightarrow} \varphi$ statistically radially;*
(ii) *$\left\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\right\| \overset{Lim}{\longrightarrow} 0$ statistically radially and $\left\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\right\| \overset{Lim}{\longrightarrow} 0$ statistically radially.*

In conclusion we define one more generalized Engliš algebra (which in case of usual radial limits is defined by Engliš (Engliš, 1995)) and study its some properties.

Define

$$\mathcal{A}_{st} := \left\{ T \in \mathcal{B}\left(H^2\right) : \left\| T\widehat{k}_\lambda \right\|^2 - \left| \left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \text{ and} \right.$$
$$\left. \left\| T^*\widehat{k}_\lambda \right\|^2 - \left| \left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \longrightarrow 0 \text{ statistically radially} \right\} \tag{3}$$

or, in other words , $\left\| T\widehat{k}_\lambda \right\|^2 - \left| \widetilde{T}(\lambda) \right|^2 \longrightarrow 0$ statistically radially and similarly for T^*. Following by the arguments of the paper (Engliš, 1995), if we decompose $T\widehat{k}_\lambda$ as

$$T\widehat{k}_\lambda = c_\lambda \widehat{k}_\lambda + d_\lambda,\ c_\lambda \in \mathbb{C},\ d_\lambda \bot \widehat{k}_\lambda \tag{4}$$

then $\widetilde{T}(\lambda) = \left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle = c_\lambda$ and $\left\| T\widehat{k}_\lambda \right\|^2 = |c_\lambda|^2 + \|d_\lambda\|^2$, so the first condition in (3) reads just

$$d_\lambda \rightarrow 0 \text{ statistically radially} \tag{5}$$

Proposition 4. *$\mathcal{A}_{st}$ is a C^*-algebra.*

Proof. In fact, it follows from (4) and (5) that $\mathcal{A}_{st}$ is linear, and from (3) that (see (Engliš, 1995)) it is self-adjoint and statistically norm closed. If $T, T' \in \mathcal{A}_{st}$ and $T\widehat{k}_\lambda = c\widehat{k}_\lambda + d_\lambda$, $T'\widehat{k}_\lambda = c'\widehat{k}_\lambda + d'_\lambda$ are the decomposition (4), then

$$TT'\widehat{k}_\lambda = Td'_\lambda + c'_\lambda c_\lambda \widehat{k}_\lambda + c'_\lambda d_\lambda.$$

Let $TT'\widehat{k}_\lambda = c''_\lambda \widehat{k}_\lambda + d''_\lambda$ be the decomposition (4) for TT'. Then

$$(c'_\lambda c_\lambda - c''_\lambda)\widehat{k}_\lambda = (d''_\lambda - c'_\lambda d_\lambda) - Td'_\lambda. \tag{6}$$

Since $d_\lambda \perp \widehat{k}_\lambda$ and $d''_\lambda \perp \widehat{k}_\lambda$, taking the inner product with $\widehat{k}_\lambda$ on both sides gives

$$\begin{aligned} \left| c'_\lambda c_\lambda - c''_\lambda \right| &= \left| \langle Td'_\lambda, \rangle \widehat{k}_\lambda \right| \\ &\leq \|T\| \left\| d'_\lambda \right\| \rightarrow \text{statistically radially (by (5)).} \end{aligned} \tag{7}$$

Now putting this back into (6) shows that

$$d''_\lambda - (c'_\lambda d_\lambda + Td'_\lambda) \rightarrow 0 \text{ statistically radially.}$$

Now $\left| c'_\lambda \right| \leq \|T'\|$, so by (5) and the boundedness of $T$and T' it follows that

$$c'_\lambda d_\lambda + Td'_\lambda \rightarrow 0 \text{ statistically radially.}$$

Consequently, $d''_\lambda \rightarrow 0$ statistically radially. By a similar arguments for T'^*T^* it can be easily proved that $TT' \in \mathcal{A}_{st}$, that is $\mathcal{A}_{st}$ is also closed under multiplication. So, $\mathcal{A}_{st}$ is a C^*-algebra, as desired. □

Proposition 5. *For $\varphi \in L^\infty(\mathbb{T})$, $T_\varphi \in \mathcal{A}_{st}$.*

Proof. Immediate from Proposition 1 and the fact that $\widetilde{T_\varphi} = \widetilde{\varphi}$. □

Proposition 6. *If $H \in \mathcal{E}$, then $H \in \mathcal{A}_{st}$.*

Proof. The assertion that $\left\| H\widehat{k}_\lambda \right\| \to 0$ statistically radially implies that $\left\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \to 0$ statistically radially (because $\left| \left\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \leq \left\| H\widehat{k}_\lambda \right\|$), and hence $\left\| H\widehat{k}_\lambda \right\|^2 - \left| \left\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \to 0$ statistically radially as well; similarly for H^*, and thus $\mathcal{E} \subset \mathcal{A}_{st}$, as desired. □

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On a Subclass of Harmonic Univalent Functions Based on Subordination

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Abstract

In this paper, we introduce a new class of harmonic univalent functions defined by subordination with a linear operator. Certain properties of this class are discussed.

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1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small (Clunie & Sheil-Small, 1984)). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k. \tag{1.1}$$

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = 0$.

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In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Also note that SH reduces to the class S of normalized analytic univalent functions in U, if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^n $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of f was introduced by Sălăgean (Sălăgean, 1983). For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. (Jahangiri *et al.*, 2002) defined the modified Sălăgean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k.$$

Next, for functions $f \in A$, Al-Oboudi (Al-Oboudi, 2004) defined multiplier transformations. For $n \in \mathbb{N}_0$, $\lambda \geq 1$ and $f \in SH^0$ of the form (1.1), Yaşar and Yalçın (Yaşar & Yalcin, 2012) defined the modified Al-Oboudi operator $D_\lambda^n : SH^0 \to SH^0$ by

$$D_\lambda^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D_\lambda^1 f(z) = (1-\lambda) D^0 f(z) + \lambda D^1 f(z), \quad \lambda \geq 1, \tag{1.2}$$

$$D_\lambda^n f(z) = D_\lambda^1 \left(D_\lambda^{n-1} f(z) \right). \tag{1.3}$$

If f is given by (1.1), then from (1.2) and (1.3) we see that (see (Yaşar & Yalcin, 2012))

$$D_{1,\lambda}^n h(z) = z + \sum_{k=2}^{\infty} [\lambda (k-1) + 1]^n a_k z^k,$$

$$D_{2,\lambda}^n g(z) = \sum_{k=2}^{\infty} [\lambda (k+1) - 1]^n b_k z^k,$$

$$D_\lambda^n f(z) = D_{1,\lambda}^n h(z) + (-1)^n \overline{D_{2,\lambda}^n g(z)}$$

or

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [\lambda (k-1) + 1]^n a_k z^k + (-1)^n \sum_{k=2}^{\infty} [\lambda (k+1) - 1]^n \overline{b_k z^k}. \tag{1.4}$$

When $\lambda = 1$, we get modified Sălăgean differential operator (Jahangiri *et al.*, 2002). If we take the co-analytic part of $f = h + g$ of the form (1.1) is identically zero, $D_\lambda^n f$ reduces to the Al-Oboudi operator (Al-Oboudi, 2004).

The Hadamard product (or convolution) of functions $f_1$and f_2 of the form

$$f_t(z) = z + \sum_{k=2}^{\infty} a_{t,k} z^k + \sum_{k=2}^{\infty} \overline{b_{t,k} z^k} \quad (z \in U, t = \{1, 2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k}a_{2,k}z^k + \sum_{k=2}^{\infty} \overline{b_{1,k}b_{2,k}z^k} \quad (z \in U)$$

Also if f is given by (1.1), then we have

$$\begin{aligned} D_\lambda^n f(z) \quad : \quad &= f(z) * \underbrace{\left(\phi_1(z) + \overline{\phi_2(z)}\right) * \ldots * \left(\phi_1(z) + \overline{\phi_2(z)}\right)}_{n \text{ times}}, \\ &= h(z) * \underbrace{\phi_1(z) * \ldots * \phi_1(z)}_{n \text{ times}} + \overline{g(z)} * \underbrace{\overline{\phi_2(z)} * \ldots * \overline{\phi_2(z)}}_{n \text{ times}}. \end{aligned}$$

where

$$\phi_1(z) = \frac{(\lambda - 1) z^2 + z}{(1-z)^2}, \quad \phi_2(z) = \frac{(\lambda-1) z^2 + (1-2\lambda) z}{(1-z)^2}.$$

We say that a function $f : U \to \mathbb{C}$ is subordinate to a function $g : U \to \mathbb{C}$, and write $f(z) \prec g(z)$, if there exists a complex valued function w which maps U into itself with $w(0) = 0$, such that

$$f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Denote by $SH^0(\lambda, n, A, B)$ the subclass of SH^0 consisting of functions f of the form (1.1) that satisfy the condition

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \prec \frac{1+Az}{1+Bz}, \quad -B \le A < B \le 1 \tag{1.5}$$

where $D_\lambda^n f(z)$ is defined by (1.4).

By suitably specializing the parameters, the classes $SH^0(\lambda, n, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH^0(1, \lambda, A, B) = H_\lambda(A, B),\ \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (Dziok *et al.* (2016)),
(ii) $SH^0(1, 1, A, B) = S_H^*(A, B) \cap SH^0$ ((Dziok, 2015*a*)),
(iii) $SH^0(\lambda, n, 2\alpha - 1, 1) = SH(\lambda, n, \alpha) \cap SH^0$ (Yaşar & Yalcin, 2012),
(iv) $SH^0(1, n, 2\alpha - 1, 1) = H^0(n, \alpha)$ (Jahangiri *et al.*, 2002),
(v) $SH^0(1, 0, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ((Jahangiri, 1999), (Silverman, 1998), (Silverman & Silvia, 1999)),
(vi) $SH^0(1, 1, 2\alpha - 1, 1) = S_{H^0}^c(\alpha)$ ((Jahangiri, 1999)),
(vii) $SH^0(\lambda, n, 2\alpha - 1, 1) = \overline{SH}(\lambda, 1 - \lambda, n, \alpha)$ ((Bayram & Yalcin, 2017)).

Making use of the techniques and methodology used by Dziok (see (Dziok, 2015*a*), (Dziok, 2015*b*)), Dziok et al. (Dziok *et al.*, 2016), in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $SH^0(\lambda, n, A, B)$.

2. Main Results

First, we provides a necessary and sufficient convolution condition for the harmonic functions in $SH^0(\lambda, n, A, B)$.

Theorem 2.1. *For $z \in U \backslash \{0\}$, let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if*

$$D_\lambda^n f(z) * \varphi(z;\zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1),$$

where

$$\begin{aligned}\varphi(z;\zeta) &= \frac{[(A-B)\zeta + \lambda(1+B\zeta)]z^2 + (B-A)\zeta z}{(1-z)^2} \\ &\quad -(-1)^n \frac{[-\lambda(1+B\zeta) + (B-A)\zeta]\overline{z}^2 + [2\lambda(1+B\zeta) + (A-B)\zeta]\overline{z}}{(1-\overline{z})^2}.\end{aligned}$$

Proof. Let $f \in SH^0$. Then $f \in SH^0(\lambda, n, A, B)$ if and only if the condition (1.5) holds or equivalently

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1). \tag{2.1}$$

Now for

$$D_\lambda^n f(z) = D_\lambda^n f(z) * \left(\frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}}\right),$$

and

$$D_\lambda^{n+1} f(z) = D_\lambda^n f(z) * \left(\phi_1(z) + \overline{\phi_2(z)}\right),$$

the inequality (2.1) yields

$$\begin{aligned}&(1+B\zeta) D_\lambda^{n+1} f(z) - (1+A\zeta) D_\lambda^n f(z) \\ &= D_\lambda^n f(z) * \left\{\frac{(1+B\zeta)\left[(\lambda-1)z^2 + z\right]}{(1-z)^2} + \frac{(1+B\zeta)\left[(\lambda-1)\overline{z}^2 + (1-2\lambda)\overline{z}\right]}{(1-\overline{z})^2}\right\} \\ &\quad - D_\lambda^n f(z) * \left\{\frac{(1+A\zeta)z}{1-z} + \frac{(1+A\zeta)\overline{z}}{1-\overline{z}}\right\} \\ &= D_\lambda^n f(z) * \left\{\frac{[(A-B)\zeta + \lambda(1+B\zeta)]z^2 + (B-A)\zeta z}{(1-z)^2}\right. \\ &\quad \left. - \frac{[2-\lambda(1+B\zeta) + (B+A)\zeta]\overline{z}^2 + [-2+2\lambda(1+B\zeta) + (B-A)\zeta]\overline{z}}{(1-\overline{z})^2}\right\} \\ &= D_\lambda^n f(z) * \varphi(z;\zeta) \neq 0.\end{aligned}$$

$\square$

A sufficient coefficient for the functions in $f \in SH^0(\lambda, n, A, B)$ is provided in the following.

Theorem 2.2. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1).Then $f \in SH^0(\lambda, n, A, B)$, if*

$$\sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \leq B - A \tag{2.2}$$

where

$$M_k = [(k-1)\lambda + 1]^n [\lambda (k-1)(B+1) + B - A] \tag{2.3}$$

and

$$N_k = [(k+1)\lambda - 1]^n [\lambda (k+1)(B+1) + A - B]. \tag{2.4}$$

Proof. It is easy to see that the theorem is true for $f(z) = z$. So, we assume that $a_k \neq 0$ or $b_k \neq 0$ for $k \geq 2$. Since $M_k \geq k(B - A)$ and $N_k \geq k(B - A)$ by (2.2), we obtain

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} - \sum_{k=2}^{\infty} k |b_k| |z|^{k-1} \\ &\geq 1 - |z| \sum_{k=2}^{\infty} (k |a_k| + k |b_k|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (M_k |a_k| + N_k |b_k|) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Therefore f is sense preserving and locally univalent in U. For the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^k - z_2^k}{z_1 - z_2} \right| = \left| \sum_{m=1}^{k} z_1^{m-1} z_2^{k-m} \right| \leq \sum_{m=1}^{k} \left| z_1^{m-1} \right| \left| z_2^{k-m} \right| < k, \quad k \geq 2.$$

Hence

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=2}^{\infty} b_k \left(z_1^k - z_2^k \right)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k \right)} \right| \\ &> 1 - \frac{\sum_{k=2}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \geq 1 - \frac{\sum_{k=2}^{\infty} \frac{N_k}{B-A} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{M_k}{B-A} |a_k|} \geq 0 \end{aligned}$$

which proves univalence.

On the other hand, $f \in SH^0(\lambda, n, A, B)$ if and only if there exists a complex valued function w; $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$ such that

$$\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

or equivalently

$$\left| \frac{D_\lambda^{n+1} f(z) - D_\lambda^n f(z)}{BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z)} \right| < 1, \quad (z \in U). \tag{2.5}$$

The above inequality (2.5) holds, since for $|z| = r$ $(0 < r < 1)$ we obtain

$$\begin{aligned}
&\left|D_\lambda^{n+1} f(z) - D_\lambda^n f(z)\right| - \left|BD_\lambda^{n+1} f(z) - AD_\lambda^n f(z)\right| \\
= \ & \left| \sum_{k=2}^{\infty} (k-1)\lambda\left[(k-1)\lambda + 1\right]^n a_k z^k - (-1)^n \sum_{k=2}^{\infty} (k+1)\lambda\left[(k+1)\lambda - 1\right]^n \overline{b_k z^k} \right| \\
& - \Bigg| (B-A)z + \sum_{k=2}^{\infty} \left[(k-1)\lambda B + B - A\right]\left[(k-1)\lambda + 1\right]^n a_k z^k \\
& -(-1)^n \sum_{k=2}^{\infty} \left[(k+1)\lambda B + A - B\right]\left[(k+1)\lambda - 1\right]^n \overline{b_k z^k} \Bigg| \\
\leq \ & \sum_{k=2}^{\infty} (k-1)\lambda\left[(k-1)\lambda + 1\right]^n |a_k| r^k + \sum_{k=2}^{\infty} (k+1)\lambda\left[(k+1)\lambda - 1\right]^n |b_k| r^k - (B-A)r \\
& + \sum_{k=2}^{\infty} \left[(k-1)\lambda B + B - A\right]\left[(k-1)\lambda + 1\right]^n |a_k| r^k + \sum_{k=2}^{\infty} \left[(k+1)\lambda B + A - B\right]\left[(k+1)\lambda - 1\right]^n |b_k| r^k \\
\leq \ & r\left\{ \sum_{k=2}^{\infty} \left(M_k |a_k| + N_k |b_k|\right) r^{k-1} - (B-A) \right\} < 0,
\end{aligned}$$

therefore $f \in SH^0(\lambda, n, A, B)$, and so the proof is complete. □

Next we show that the condition (2.2) is also necessary for the functions $f \in H$ to be in the class $SH_T^0(\lambda, n, A, B) = T^n \cap SH^0(\lambda, n, A, B)$ where T^n is the class of functions $f = h + \overline{g} \in SH^0$ so that

$$f = h + \overline{g} = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k \quad (z \in U). \tag{2.6}$$

Theorem 2.3. *Let $f = h + \overline{g}$ be defined by (2.6). Then $f \in SH_T^0(\lambda, n, A, B)$ if and only if the condition (2.2) holds.*

Proof. The "if" part follows from Theorem 2.2. For the "only-if" part, assume that $f \in SH_T^0(\lambda, n, A, B)$, then by (2.5) we have

$$\left| \frac{\sum\limits_{k=2}^{\infty} (k-1)\lambda[(k-1)\lambda+1]^n |a_k| z^k + (k+1)\lambda[(k+1)\lambda-1]^n |b_k| \overline{z}^k}{(B-A)z - \sum\limits_{k=2}^{\infty} [(k-1)\lambda B+B-A][(k-1)\lambda+1]^n |a_k| z^k + [(k+1)\lambda B+A-B][(k+1)\lambda-1]^n |b_k| \overline{z}^k} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum\limits_{k=2}^{\infty} \{(k-1)\lambda[(k-1)\lambda+1]^n |a_k| + (k-+1)\lambda[(k+1)\lambda-1]^n |b_k|\} r^{k-1}}{B-A-\sum\limits_{k=2}^{\infty} \{[(k-1)\lambda B+B-A][(k-1)\lambda+1]^n |a_k| + [(k+1)\lambda B+A-B][(k+1)\lambda-1]^n |b_k|\} r^{k-1}} < 1.$$

Thus, for M_k and N_k as defined by (2.3) and (2.4), we have

$$\sum_{k=2}^{\infty} \left[M_k |a_k| + N_k |b_k| \right] r^{k-1} < B - A \quad (0 \le r < 1). \tag{2.7}$$

Let $\{\sigma_k\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty} \left[M_k |a_k| + N_k |b_k| \right].$$

Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.7) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} \left[M_k |a_k| + N_k |b_k| \right] = \lim_{k \to \infty} \sigma_k \le B - A.$$

This gives the condition (2.2). □

In the following we show that the class of functions of the form (2.6) is convex and compact.

Theorem 2.4. *The class $SH_T^0(\lambda, n, A, B)$ is a convex and compact subset of SH.*

Proof. Let $f_t \in SH_T^0(\lambda, n, A, B)$, where

$$f_t(z) = z - \sum_{k=2}^{\infty} \left| a_{t,k} \right| z^k + (-1)^n \sum_{k=2}^{\infty} \left| b_{t,k} \right| \overline{z^k} \quad (z \in U, \ t \in \mathbb{N}). \tag{2.8}$$

Then $0 \le \eta \le 1$, let $f_1, f_2 \in SH_T^0(\lambda, n, A, B)$ be defined by (2.8). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1-\eta) f_2(z) \\ &= z - \sum_{k=2}^{\infty} \left(\eta \left| a_{1,k} \right| + (1-\eta) \left| a_{2,k} \right| \right) z^k \\ &\quad + (-1)^n \sum_{k=2}^{\infty} \left(\eta \left| b_{1,k} \right| + (1-\eta) \left| b_{2,k} \right| \right) \overline{z^k} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=2}^{\infty}\left\{M_k\left[\eta\left|a_{1,k}\right|+(1-\eta)\left|a_{2,k}\right|\right]+N_k\left[\eta\left|b_{1,k}\right|+(1-\eta)\left|b_{2,k}\right|\right]\right\} \\
= \ & \eta\sum_{k=2}^{\infty}\left\{M_k\left|a_{1,k}\right|+N_k\left|b_{1,k}\right|\right\}+(1-\eta)\sum_{k=2}^{\infty}\left\{M_k\left|a_{2,k}\right|+N_k\left|b_{2,k}\right|\right\} \\
\leq \ & \eta(B-A)+(1-\eta)(B-A)=B-A.
\end{aligned}$$

Thus, the function $\kappa=\eta f_1+(1-\eta)f_2$ belongs to the class $SH_T^0(\lambda,n,A,B)$. This means that the class $SH_T^0(\lambda,n,A,B)$ is convex.

On the other hand, for $f_t\in SH_T^0(\lambda,n,A,B)$, $t\in\mathbb{N}$ and $|z|\leq r(0<r<1)$, we get

$$\begin{aligned}
|f_t(z)| &\leq r+\sum_{k=2}^{\infty}\left\{\left|a_{t,k}\right|+\left|b_{t,k}\right|\right\}r^k \\
&\leq r+\sum_{k=2}^{\infty}\left\{M_k\left|a_{t,k}\right|+N_k\left|b_{t,k}\right|\right\}r^k \\
&\leq r+(B-A)r^2.
\end{aligned}$$

Therefore, $SH_T^0(\lambda,n,A,B)$ is locally uniformly bounded. Let

$$f_t(z)=z-\sum_{k=2}^{\infty}\left|a_{t,k}\right|z^k+(-1)^n\sum_{k=2}^{\infty}\left|b_{t,k}\right|\overline{z^k}\quad(z\in U,\ t\in\mathbb{N})$$

and let $f=h+\overline{g}$ be so that h and g are given by (1.1). Using Theorem 2.3 we obtain

$$\sum_{k=2}^{\infty}\left\{M_k\left|a_{t,k}\right|+N_k\left|b_{t,k}\right|\right\}\leq(B-A). \tag{2.9}$$

If we assume that $f_t\to f$, then we conclude that $\left|a_{t,k}\right|\to|a_k|$ and $\left|b_{t,k}\right|\to|b_k|$ as $k\to\infty$ $(t\in\mathbb{N})$. Let $\{\sigma_k\}$ be the sequence of partial sums of the series $\sum_{k=2}^{\infty}\{M_k|a_k|+N_k|b_k|\}$. Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.9) it is bounded above by $B-A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty}\{M_k|a_k|+N_k|b_k|\}=\lim_{k\to\infty}\sigma_k\leq B-A.$$

Therefore $f\in SH_T^0(\lambda,n,A,B)$ and therefore the class $SH_T^0(\lambda,n,A,B)$ is closed. In consequence, the class $SH_T^0(\lambda,n,A,B)$ is compact subset of SH, which completes the proof. □

We continue with the following lemma due to Jahangiri (Jahangiri, 1999).

Lemma 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} \leq 1 \quad (z \in U)$$

where $0 \leq \alpha < 1$. Then f is harmonic, orientation preserving, univalent in U and f is starlike of order α.

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $SH_T^0(\lambda, n, A, B)$.

Theorem 2.5. *Let $0 \leq \alpha < 1$, M_k and N_k be defined by (2.3) and (2.4). Then*

$$r_\alpha^*(SH_T^0(\lambda, n, A, B)) = \inf_{k \geq 2} \left[\frac{1-\alpha}{B-A} \min \left\{ \frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha} \right\} \right]^{\frac{1}{k-1}}. \tag{2.10}$$

Proof. Let $f \in SH_T^0(\lambda, n, A, B)$ be of the form (2.6). Then, for $|z| = r < 1$, we get

$$\begin{aligned} &\left| \frac{D_1 f(z) - (1+\alpha) f(z)}{D_1 f(z) + (1-\alpha) f(z)} \right| \\ &= \left| \frac{-\alpha z - \sum\limits_{k=2}^{\infty} (k-1-\alpha) |a_k| z^k - (-1)^n \sum\limits_{k=2}^{\infty} (k+1+\alpha) |b_k| \overline{z}^k}{(2-\alpha) z - \sum\limits_{k=2}^{\infty} (k+1-\alpha) |a_k| z^k - (-1)^n \sum\limits_{k=2}^{\infty} (k-1+\alpha) |b_k| \overline{z}^k} \right| \\ &\leq \frac{\alpha + \sum\limits_{k=2}^{\infty} \{(k-1-\alpha) |a_k| + (k+1+\alpha) |b_k|\} r^{k-1}}{2-\alpha - \sum\limits_{k=2}^{\infty} \{(k+1-\alpha) |a_k| + (k-1+\alpha) |b_k|\} r^{k-1}}. \end{aligned}$$

Note (see Lemma 2.1) that f is starlike of order α in U_r if and only if

$$\left| \frac{D_1 f(z) - (1+\alpha) f(z)}{D_1 f(z) + (1-\alpha) f(z)} \right| < 1, \; z \in U_r$$

or

$$\sum_{k=2}^{\infty} \left\{ \frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right\} r^{k-1} \leq 1. \tag{2.11}$$

Moreover, by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{M_k}{B-A} |a_k| + \frac{N_k}{B-A} |b_k| \right\} r^{k-1} \leq 1.$$

The condition (2.11) is true if

$$\frac{k-\alpha}{1-\alpha} r^{k-1} \leq \frac{M_k}{B-A} r^{k-1},$$

$$\frac{k+\alpha}{1-\alpha}r^{k-1} \leq \frac{N_k}{B-A}r^{k-1} \quad (k = 2,3,\ldots)$$

or if

$$r \leq \frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha}\right\}^{\frac{1}{k-1}} \quad (k = 2,3,\ldots).$$

It follows that the function f is starlike of order α in the disk $U_{r^*_\alpha}$ where

$$r^*_\alpha := \inf_{k\geq 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k-\alpha}, \frac{N_k}{k+\alpha}\right\}\right]^{\frac{1}{k-1}}.$$

The function

$$f_k(z) = h_k(z) + \overline{g_k(z)} = z - \frac{B-A}{M_k}z^k + (-1)^n\frac{B-A}{N_k}\overline{z}^k$$

proves that the radius r^*_α cannot be any larger. Thus we have (2.10). □

Using a similar argument as above we obtain the following.

Theorem 2.6. *Let* $0 \leq \alpha < 1$ *and* M_k *and* N_k *be defined by (2.3) and (2.4). Then*

$$r^c_\alpha(SH^0_T(\lambda,n,A,B)) = \inf_{k\geq 2}\left[\frac{1-\alpha}{B-A}\min\left\{\frac{M_k}{k(k-\alpha)}, \frac{N_k}{k(k+\alpha)}\right\}\right]^{\frac{1}{k-1}}.$$

Our next theorem is on the extreme points of $SH^0_T(\lambda,n,A,B)$.

Theorem 2.7. *Extreme points of the class* $SH^0_T(\lambda,n,A,B)$ *are the functions* f *of the form (1.1) where* $h = h_k$ *and* $g = g_k$ *are of the form*

$$\begin{aligned} &h_1(z) = z, \quad h_k(z) = z - \tfrac{B-A}{M_k}z^k, \\ &g_k(z) = (-1)^n\tfrac{B-A}{N_k}\overline{z^k} \quad (z \in U,\ k \geq 2). \end{aligned} \tag{2.12}$$

Proof. Let $g_k = \eta f_1 + (1-\eta)f_2$ where $0 < \eta < 1$ and $f_1, f_2 \in SH^0_T(\lambda,n,A,B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty}\left|a_{t,k}\right|z^k + (-1)^n\sum_{k=2}^{\infty}\left|b_{t,k}\right|\overline{z^k} \quad (z \in U,\ t \in \{1,2\}).$$

Then, by (2.12), we have

$$\left|b_{1,k}\right| = \left|b_{2,k}\right| = \frac{B-A}{N_k},$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2,3,\ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2,3,\ldots\}\setminus\{k\}$. It follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme points of the function class $SH^0_T(\lambda,n,A,B)$. Similarly, we can verify that the functions $h_k(z)$ are the extreme points of the class $SH^0_T(\lambda,n,A,B)$. Now, suppose that a function f of the form (1.1) is in the family of extreme points of the class $SH^0_T(\lambda,n,A,B)$ and f is not of the form (2.12). Then there exists $m \in \{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{B-A}{[(m-1)\lambda+1]^n[\lambda(m-1)(B+1)+B-A]}$$

or

$$0 < |b_m| < \frac{B-A}{[(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]}.$$

If

$$0 < |a_m| < \frac{B-A}{[(m-1)\lambda + 1]^n [\lambda (m-1)(B+1) + B - A]},$$

then putting

$$\eta = \frac{|a_m| [(m-1)\lambda + 1]^n [\lambda (m-1)(B+1) + B - A]}{B-A}$$

and

$$\phi = \frac{f - \eta h_m}{1-\eta},$$

we have $0 < \eta < 1,\ h_m \neq \phi$.

Therefore, f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$. Similarly, if

$$0 < |b_m| < \frac{B-A}{[(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]},$$

then putting

$$\eta = \frac{|b_m| [(m+1)\lambda - 1]^n [\lambda (m+1)(B+1) + A - B]}{B-A}$$

and

$$\phi = \frac{f - \eta g_m}{1-\eta},$$

we have $0 < \eta < 1,\ g_m \neq \phi$.

It follows that f is not in the family of extreme points of the class $SH_T^0(\lambda, n, A, B)$ and so the proof is completed. □

Therefore, by Theorem 2.7, we have the following corollary.

Corollary 2.1. *Let $f \in SH_T^0(\lambda, n, A, B)$ and $|z| = r < 1$. Then*

$$r - \frac{B-A}{(\lambda+1)^n [\lambda (B+1) + A - B]} r^2 \leq |f(z)| \leq r + \frac{B-A}{(\lambda+1)^n [\lambda (B+1) + A - B]} r^2.$$

The following covering result follows from Corollary 2.1.

Corollary 2.2. *If $f \in SH_T^0(\lambda, n, A, B)$ then $U_r \subset f(U)$ where*

$$r = 1 - \frac{B-A}{(\lambda+1)^n [\lambda (B+1) + A - B]}.$$

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Non-Computable, Indiscernible and Uncountable Mathematical Constructions. Sub-Cardinals and Related Paradoxes

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Abstract

One of the most important achievements of the last century is the knowledge of the existence of non-countable sets. The proof by Cantor's diagonal method requires the assumption of actual infinity. By two paradoxes we show that this method sometimes proves nothing because of it can involve self-referential definitions. To avoid this inconvenient, we introduce another proving method based upon the information in the involved object definitions. We also introduce the concepts of indiscernible mathematical construction and sub-cardinal. In addition, we show that the existence of indiscernible mathematical constructions is an unavoidable consequence of uncountability.

Keywords: Remote-cardinals, sub-cardinals, uncountability, indiscernibility, paradoxes.
2010 MSC: 03E10, 03E35, 03D15, 68Q30, 03A10.

1. Introduction and preliminaries

In this article that two sets X and Y have different cardinalities only means that it is impossible to define one bijection between them. The existence of a bijection between two sets leads to size equality, but the converse implication need not be true, at least, with respect to infinite sets. The term "size" is ambiguous when applied to infinite sets. Recall that there is always a bijection between every infinite set and some proper subset of it. Bijections not only compare set-sizes but information and complexity in the definitions of their members. In particular, we consider those mathematical constructions that cannot be defined or determined by any finite expression. We say these objects to be *indiscernible*.

If $f : X \to Y$ is a bijection, the expression $y = f(x)$ specifies the object y uniquely. Thus, we can consider $y = f(x)$ as a definition for y, whenever both f and x are definable. As a consequence, if the information lying in any definition for y is always greater than the one in both definitions of f and x, we cannot assume the local equality $y = f(x)$ at x. We can find this situation when y is

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an indiscernible object. This method leads to non-existence of a bijection $f : X \to Y$ when there is, at least, one indiscernible object in Y. Even singletons can satisfy this condition. However, it is trivial to define a bijection between two singletons, what suggests that this proving method is wrong. Fortunately, in Theorem 2.2 we show that, if a set E contains one indiscernible object, then E must be infinite, and satisfies the following relation.

$$\text{“}E \textit{ contains an indiscernible object”} \Rightarrow (\text{“}E \textit{ is infinite”} \wedge \text{“}E \textit{ is uncountable.”})$$

Analogously, by virtue of Theorem 2.1,

$$\text{“}E \textit{ is uncountable”} \Rightarrow \text{“}E \textit{ contains an infinite subset each member of which is indiscernible.”}$$

We can find similar facts using complexity instead of information. For instance, let Ω denote the Chaitin's constant (Chaitin, 2012). If $f : \mathbb{N} \to E \subseteq \mathbb{R}$ is a bijection, being computable by a Turing machine T, then E cannot contain Ω; otherwise, T could compute Ω, simply, as the value of f at some positive integer $n \in \mathbb{N}$. Accordingly, if $\Omega \in E$, then there is no bijection between $\mathbb{N}$ and E, being computable by any Turing machine. Nevertheless, depending on their nature, we could define a non-computable bijection F between $\mathbb{N}$ and E. This fact allows us to define the concept of sub-cardinal. Thus, comparing sets by computable bijections, we define sub-cardinals as sub-equivalences of those that we can state by definable one-to-one maps.

1.1. Preliminaries

To avoid any ambiguity, we use the concepts of “extensible language,” “identifier” and “discernible mathematical construction” with particular meanings that we explain below.

Let A be a finite alphabet. We term “vocabulary generated by A” every nonempty subset of the class $\mathrm{Voc}(A)$ of all finite and infinite sequences of symbols in A. Likewise, we term “language” every partial (syntactic) free-monoid $\mathcal{L}(\mathrm{Voc}(A))$, generated by $\mathrm{Voc}(A)$, provided that we assign a meaning to each of its members. Thus, $\mathcal{L}(\mathrm{Voc}(A))$ consists of sentences. Two sentences s_1 and s_2 are equivalent when both denote the same concept.

Definition 1.1. We say that a language $\mathcal{L}(\mathrm{Voc}(A))$ is “extensible” when it satisfies the following conditions.

1. If there is a set E of sentences in some language $\mathfrak{L}$ that have no equivalent in $\mathcal{L}(\mathrm{Voc}(A))$, we can add sentences and symbols to $\mathcal{L}(\mathrm{Voc}(A))$ denoting all members of E.
2. When we extend $\mathcal{L}(\mathrm{Voc}(A))$ with a set W of new symbols and sentences, the way that we assign meanings and interpret each member of W can be described by “finite” sentences in $\mathcal{L}(\mathrm{Voc}(A))$.

The second condition is important for our purposes. For instance, consider the sentence: S = *“We can compute the area of a polygon of three angles.”* Using the word “triangle” we can transform S into the shorter sentence: *“We can compute the area of a triangle.”* Since we can describe the meaning of the term “triangle” by a finite expression, this substitution satisfies the former definition.

Definition 1.2. We say that a sentence S in an extensible language is an "identifier" for a mathematical construction X, provided that S specifies X uniquely. Thus, S can denote a definition, a predicate, or any procedure that can be associated with X in an ambiguity-free way.

Identifiers can be infinite, for instance, the binary expansion $0.d_1d_2d_3\ldots$ of an irrational number consists of an infinite symbol sequence.

Definition 1.3. We say that a mathematical construction X is discernible when there is, at least, one finite identifier for it; otherwise, we say that X is indiscernible.

To illustrate the concept in the former definition, consider the real number π. There are endless expressions that identify π. For instance, the infinite digit sequence $3.14159\ldots$ of its decimal expansion. In spite of being an endless symbol sequence, the number π is discernible because it can be specified by finite sentences. For instance,

$$S = \text{"}\textit{the ratio of a circle's circumference to its diameter.}\text{"}$$

Although the symbol π denotes the infinite expression $3.141592\ldots$, the language extension obtained adding π satisfies the second condition in Definition 1.1 because its meaning can be described by the finite sentence S. Likewise, every algebraic number is discernible by any equation that it satisfies. Every computable mathematical construction X is discernible because any algorithm (finite) that calculates X also identifies it.

A noticeable property related with indiscernible mathematical constructions is the impossibility of handling them individually. We show this topic in Theorem 2.2. There are sets of indiscernible objects that can be denoted by finite expressions. For instance, *"the subset K of all indiscernible members of* $[0, 1]$*"* is a finite definition; hence K is discernible, but each of its members is not. Likewise, we show in Theorem 3.1 that, if there is one non-computable number in $[0, 1]$, there is also an infinite subset no member of which is computable.

From now on, we frequently use the term "infinite," that can be regarded either as potential or actual. By potential infinity, we understand the concept stated by Poincaré (1854-1912) in the following quotation.

> *Actual infinity does not exist. What we call infinite is only the endless possibility of creating new objects no matter how many exist already.*

When we do not state the concept of infinity as "potential," we implicitly mean that it is actual.

2. Discernible mathematical constructions

In this section, we show that every uncountable set contains an infinite subset no member of which is discernible. In addition, indiscernible mathematical constructions cannot be identified and compared to each other by finite procedures. This property gives rise to some undecidable statements together with the impossibility of proving that a map satisfies the one-to-one property. By undecidable, we mean that either its truthfulness or falsehood cannot be proved by some finite procedure. Thus, if we assume any undecidable statement, there is no finite procedure to reject or confirm our assumption.

Lemma 2.1. *Every computable mathematical construction is discernible.*

Proof. Let O be a mathematical construction that is computable by a Turing machine T. The transition map of T is determined by a finite matrix T_M. We can denote T_M by a finite symbol sequence in any extensible language $\mathcal{L}(\mathrm{Voc}(A))$, simply, adding some end-row symbol to A, whenever it does not contain one.

Now, suppose that there is no Turing machine computing O, but we can compute it by some more complex algorithm Alg, for instance, a super-recursive one. In this case, we need a finite definition Def for Alg. Accordingly, in some extensible language there is a finite symbol sequence S that denotes Def. Thus, S determines Def; this denotes Alg; the later determines O. As a consequence, O is discernible through S. □

The concept of discernibility stated in Definition 1.3 can be either intrinsic or circumstantial. For instance, the expression E = *"the positive solution of the equation* $x^2 - 2 = 0$" determines $\sqrt{2}$. Since E is finite, $\sqrt{2}$ is discernible. Likewise, every algebraic number is discernible. The discernibility of $\sqrt{2}$ is intrinsic because arises from an intrinsic property of $\sqrt{2}$. By contrast, the expression

$$E = \text{“\textit{the second object in the sequence} } S = \square, \sqrt{2}, 6, \gamma\text{”}$$

determines $\sqrt{2}$ by a circumstance (position) that can be satisfied by any object O, simply, substituting $\sqrt{2}$ by O in S. This is a positional determination in the scenario denoted by S. If S can be defined by a finite symbol sequence, then this description together with the expression S is also an identifier for $\sqrt{2}$.

Theorem 2.1. *Every uncountable set X contains an uncountable subset each member of which is indiscernible. In addition, the subset of all discernible members of X is countable.*

Proof. Let X be a nonempty set each member of which is discernible, hence, for every $x \in X$, there is a finite expression $S_x = \alpha_1\alpha_2\dots\alpha_{k_x}$ in some extensible language $\mathcal{L}(\mathrm{Voc}(A))$ that determines x. Let $\beta : A \to \mathbb{N}$ be an injective map and $\{p_1, p_2, p_3 \dots\}$ the set of all positive prime integers. We can define a numbering function γ that sends each symbol sequence $\alpha_1\alpha_2\dots\alpha_{k_x}$ in $\mathcal{L}(\mathrm{Voc}(A))$ into the positive integer

$$\gamma(\alpha_1\alpha_2\dots\alpha_{k_x}) = \prod_{i=1}^{k_x} p_i^{\beta(\alpha_i)} \in \mathbb{N} \tag{2.1}$$

By construction, γ is injective. Since, by definition, there is a finite expression determining each discernible object x, the set X is countable because its image under γ is a subset of $\mathbb{N}$. As a consequence, to be uncountable, X must contain a nonempty set U of indiscernible objects. Since each member of $X \setminus U$ is discernible, as we have just seen, it is a countable subset of X. If U were also countable, $X = (X \setminus U) \cup U$ would be the union of two countable sets. Consequently, U must be uncountable. □

A straightforward consequence of the former theorem is the existence of indiscernible real numbers in the unit interval $[0, 1]$ because it is uncountable. This is an example of discernible set that contains indiscernible members. The set $[0, 1]$ is discernible because it can be defined by a finite sequence of finite expressions in any extensible language. For instance, *"the set of limits of all convergent sequences of rational numbers that are greater than or equal to* 0 *and less than or equal to* 1*."*

Axiom 2.1. *Let O_1 and O_2 be two mathematical constructions. If $O_1 \neq O_2$, there is, at least, one finitely representable predicate $p(x)$, in some extensible language, that O_1 satisfies and O_2 does not.*

The former axiom is widely satisfied. As the lemma below shows, the real number set satisfies it.

Lemma 2.2. *The real number set $\mathbb{R}$ satisfies Axiom 2.1.*

Proof. Let r_1 and r_2 be two real numbers. If

$$r_1 = c_1 c_2 \cdots c_k \,.\, c_{k+1} c_{k+2} c_{k+3} \cdots$$

and

$$r_2 = d_1 d_2 \cdots d_j \,.\, d_{j+1} d_{j+2} d_{j+3} \cdots$$

are their decimal expansions and $r_1 \neq r_2$, for some $n \in \mathbb{N}$, the inequality $c_n \neq d_n$ holds. Thus, r_1 is the only of them that satisfies the following predicate:

$p(x)$=*"The n-th digit of the decimal expansion of x is c_n."*

□

Lemma 2.3. *Let $E = \{O_n \mid n \in \{0, 1, \ldots k\}\}$ be a finite subset of $\mathbb{N}$. If the members of E satisfy Axiom 2.1 pairwisely, for every positive integer $n \leq k$, there is a predicate $q_n(x)$ that O_n satisfies and any other member of E does not.*

Proof. By Axiom 2.1, for every positive integer $m \leq k$, if $m \neq n$, there is a predicate $p_{n,m}(x)$ that O_n satisfies and O_m does not. Thus, O_n is the only member of E that satisfies the conjunction

$$q_n(x) = \bigwedge_{\substack{m \neq n \\ m \in \{0,1\ldots k\}}} p_{n,m}(x) \tag{2.2}$$

□

Now, we show that every finitely representable predicate that is satisfied by one indiscernible member of a finitely definable set K, it is also satisfied by every member of an infinite subset U of K; therefore K must be infinite.

Theorem 2.2. *Let X be a set the members of which satisfy Axiom 2.1 pairwisely. If X is finitely definable in an extensible language $\mathcal{L}(\mathrm{Voc}(A))$, and the subset U of all indiscernible members of X is nonempty, the following statements are true.*

1. *Let $p(x)$ be a finitely representable predicate in $\mathcal{L}(\mathrm{Voc}(A))$. If a member O_0 of U satisfies $p(x)$, then there is an infinite subset U_0 of U each member of which satisfies $p(x)$ too; therefore U must be infinite.*

2. *With the same assumptions as in the preceding statement, there is no finitely definable bijective map from any nonempty subset $\mathbf{K}$ of $\mathbb{N}$ onto U_0.*

Proof.

1. Suppose that O_0 is the only member of U that satisfies $p(x)$. In this case, $p(x)$ determines O_0 uniquely. Since, by assumption, $p(x)$ can be represented by a finite expression E_1 in $\mathcal{L}(\mathrm{Voc}(A))$, then O_0 is discernible and belongs to $X \setminus U$; which contradicts our assumption. Thus, O_0 cannot be the only object that satisfies $p(x)$. Since we assume that X is finitely definable in $\mathcal{L}(\mathrm{Voc}(A))$ by a finite expression E_0, the subset U of X is also finitely representable in $\mathcal{L}(\mathrm{Voc}(A))$ by the expression

   E_2 = *"The subset of all indiscernible members of the set denoted by E_0."*

   If U does not contain any other member of the set $\{O \in X \mid p(O)\}$, both finite expressions *"x belongs to the set denoted by E_2"* and *"x satisfies the predicate denoted by E_1"* form a finite expression that determines O_0, and both expressions E_1 and E_2 consist of finite symbol sequences in A. As in the previous case, O_0 would be discernible. Accordingly, there is at least one $O_1 \in U$ that satisfies $p(x)$ too. By Axiom 2.1 and taking into account Lemma 2.3, there is a finitely representable predicate $p_1(x)$ that O_0 satisfies and O_1 does not. The conjunction of both predicates $p(x) \wedge p_1(x)$ determines O_0, unless there is $O_2 \in U$ that also satisfies this conjunction. Iterating the procedure, we obtain an infinite subset U_0 of U each member of which satisfies $p(x)$.

2. If there is a bijection $f : \mathbb{N} \to U_0$ that can be defined by a finite expression E_f in $\mathcal{L}(\mathrm{Voc}(A))$, for every $n \in \mathbf{K}$, the expression

   "$f(n)$ is the image of the integer n under the map f defined by E_f"

   determines $f(n) \in U_0$ uniquely. Since, by hypothesis, E_f can be denoted by a finite symbol sequence, the former expression is finite, and $f(n)$ is discernible, for every $n \in \mathbb{N}$. Thus, f cannot be surjective. Accordingly, the existence of any finitely definable bijection f is not compatible with the indiscernibility of the members of U_0.

□

Corollary 2.1. *With the same assumptions as in the preceding theorem, if a discernible set E contains one indiscernible member O, then there is an infinite subset U of E each member of which is also indiscernible; hence E must be infinite. In addition, if E is uncountable, so is U.*

Proof. If E is discernible, there is at least one identifier Q for E that can be described by a finite symbol sequence in some extensible language. Now, let $P(x)$ denote the predicate:

$P(x)$ = *"x is a member of the set that Q specifies."*

It is straightforward that an object O satisfies $P(x)$ if and only if it is a member of E. Since $P(x)$ can be denoted by a finite expression, by virtue of the preceding theorem, there is an infinite set U each of its members satisfies also $P(x)$; hence $U \subseteq E$. Thus, E must be also infinite.

Finally, if E is uncountable, by virtue of Theorem 2.1, the subset $V \subseteq E$ of all discernible members of E is countable. As a consequence, U must be uncountable; otherwise $E = V \cup U$ is the union of two countable sets, which contradicts our assumption of the uncountability of E. □

Remark. As a consequence of Statement 1 in the former theorem, we can only handle infinite sets of indiscernible objects. Procedures involving indiscernible singletons require endless expressions. For instance, the short sentence

$$S = \text{“\textit{the subset of all undiscernible real numbers,}”}$$

denotes a subset $\mathbf{K}$ of $\mathbb{R}$. In spite of being discernible, because we can denote $\mathbf{K}$ by the short sentence S, each of its members requires an endless expression to be handled or identified. As a consequence, when $\mathbf{K}$ belongs to the image of a map f, by no finite method or procedure is possible to discern whether f is a one-to-one map because of the indiscernibility of the members of $\mathbf{K}$.

Each non-finitely definable map $f : X \to Y$ between infinite sets can be stated through a two column table, each of its rows consists of a member of X in the first column followed by its image under f in the second one. Its indiscernibility can be a consequence of its infinite size whenever by no predictable pattern we can determine its values. If for every finite subset K of X the restriction of f to K is finitely representable, we say f to be “first-kind indiscernible”. In this case, each restriction of f to any finite subset of its domain is discernible. By contrast, if for some $x_0 \in X$ the image $f(x_0)$ can only be denoted by an infinite symbol sequence in any language, then we say that f is “second-kind indiscernible”.

Theorem 2.3. *Let X be a set that contains a nonempty subset of indiscernible objects. If the members of X satisfy Axiom 2.1 pairwisely, for every one-to-one map $f : \mathbb{N} \to X$, the following statements are true.*

1. *If f is a finitely definable map in some language $\mathcal{L}(\mathrm{Voc}(A))$, $\forall K \in \mathbb{N}$, there is no positive integer $n \leq K$ such that its image $y = f(n)$ is indiscernible.*
2. *If f is a first-kind indiscernible map, the image of f does not contain any indiscernible member of X consequently f cannot be surjective.*

Proof.

1. By hypothesis, the map f is finitely definable. Thus, the predicate $p(n, y)$ denoted by the expression
$$p(n, y) = \text{“\textit{y is the image of n under f,}”}$$
satisfies Definition 1.1; therefore it is also finitely definable, for every $n \in \mathbb{N}$. As a consequence of Statement (1) in Theorem 2.2, there is an infinite subset of indiscernible members of X each member of which satisfies $p(n, y)$ too, which contradicts our hypothesis because f would be a relation, but not a map.

2. Let T be a table that describes f, and M any positive integer. The restriction of f to the finite set $\{1, 2, 3 \ldots M\}$ consists of a finite sub-table T_M of T. Since, by hypothesis, f is first-kind indiscernible, the finite sub-table T_M can be described by a finite sequence S_M of symbols in an extensible language $\mathcal{L}(\mathrm{Voc}(A))$. The result of substituting S_M by the corresponding finite symbol sequence in the expression

   E =*"x is the member of X denoted by the expression lying in the second column and the n-th row of the table denoted by* S_M*"*

   is also finite and determines the image of f at n, for each $n \leq M$. Thus, the image of every positive integer n under f is discernible. As a consequence, f cannot be surjective, whenever its codomain contains some nonempty subset of indiscernible members. By hypothesis, X satisfies this condition.

□

It is worth pointing out that Theorem 2.3 is very similar to the well-known Cantor's Theorem. However, the non-existence of a first-kind indiscernible bijection is a consequence of indiscernible members of X, which is a local property. By contrast, the diagonal proof is built under the assumption of actual infinity and depends on the complete bijection domain. In the proof of Cantor's Theorem, there is no mention of the nature of the table denoting the map $f : \mathbb{N} \to [0, 1]$. We suppose that, at most, f must be a first-kind indiscernible map; otherwise, assuming that f cannot be injective, to reject this claim we need an endless procedure.

To proceed more accurately, we say that a set X is ω-countable to denote the cardinality equivalence between X and $\mathbb{N}$ is based on the existence of a *first-kind indiscernible* bijection $f : \mathbb{N} \to X$.

Theorem 2.3 is built by a pointwise method, which does not depend on the set sizes. We are aware that this method can seem dark because, in finite sets, cardinality strongly depends on size. This is why, in the section below, we show similar results from another viewpoint. In any case, it is worth mentioning that, by virtue of Corollary 2.1, every finitely definable set that contains some indiscernible member must be infinite. The following result shows the existence of indiscernible numbers in the unit interval $[0, 1]$.

Theorem 2.4. *If there is one non-computable number* ϖ *in the unit interval* $[0, 1]$*, there is also an infinite subset* $U \subseteq [0, 1]$*, each member of which is indiscernible.*

Proof. For each $n \in \mathbb{N}$, let $p_n(x, d_n)$ denote the predicate

$$p_n(x, d_n) = \text{“}\textit{The n-th digit of the decimal mantissa of x is } d_n\text{.”}$$

It is straightforward that, for each map $\lambda : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, the conjunction

$$\bigwedge_{n \in \mathbb{N}} p_n(x, \lambda(n))$$

together with the predicate $Q(x) =$ *"x belongs to* $[0, 1]$*"* define a unique member of $[0, 1]$.

Now, for each $r \in [0, 1]$, let $\Upsilon(n, r, \bigwedge_{m \in \mathbb{N}} p_m(x, c_m))$ be the conjunction obtained by substituting $p_n(x, c_n)$ by $p_n(x, d_n)$ in $\bigwedge_{m \in \mathbb{N}} p_m(x, c_m)$, where the members of $\{d_n \mid n \in \mathbb{N}\}$ are the digits of the decimal expansion of r.

Let $r_1 = 0.c_1c_2c_3\cdots \in [0,1]$ be a number the figures of which are chosen at random. Taking into account Theorem 2.2, if r is indiscernible the theorem is true. If it is not, consider the number r_2 defined by the conjunction $P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m\in\mathbb{N}} p_m(x, c_m))$. If there is no finitely representable predicate being equivalent to $P_1(x)$, then the number r_2 that it defines is indiscernible; otherwise, let $P_2(x)$ be the conjunction $\Upsilon(4, \varpi, P_1(x))$. Once again, if there is no finitely representable predicate being equivalent to $P_2(x)$, the number r_3 that it defines is indiscernible, and the theorem is true. Iterating the process, we obtain a sequence $\mathbf{S} = P_1(x), P_2(x), P_3(x)\dots$ defined recursively as follows.

$$\begin{cases} P_1(x) = \Upsilon(2, \varpi, \bigwedge_{m\in\mathbb{N}} p_m(x, c_m)) \\ P_{n+1} = \Upsilon\left(2^{(n+1)}, \varpi, P_n(x)\right) \end{cases} \tag{2.3}$$

The process stops when, for some $n \in \mathbb{N}$, the predicate $P_n(x)$ defines an indiscernible number, and the theorem is true. Otherwise, the process becomes infinite. In this case, if the infinite sequence $\mathbf{S}$ defines a member r of $[0,1]$, since this is an endless definition, r is indiscernible. Notice, that by no finite method is possible to build the sequence $\mathbf{S}$ because it involves the digits of ϖ and, by hypothesis, this is not computable. Now, we show that $\mathbf{S}$ defines a real number in $[0,1]$.

Let $S = r_1, r_2, r_3\dots$ be the sequence that the members of $\mathbf{S}$ define. As a consequence of (2.3), for every positive integer $m > 0$, and each pair $(i, j) \in \mathbb{N}\times\mathbb{N}$, if $i, j > 2^m$, the 2^m-th first digits of the mantissas of both r_i and r_j are the same. Accordingly, for each couple r_i and r_j in $[0,1]$, the relation $i, j > 2^m$ leads to

$$|r_i - r_j| \le \frac{1}{10^{2^m}}. \tag{2.4}$$

Thus, S is a Cauchy sequence and converges to some $r \in \mathbb{R}$. Since $[0,1]$ is closed, $r \in [0,1]$. Taking into account Theorem 2.2, the subset U of indiscernible members of $[0,1]$ is infinite. □

It is worth pointing out, that the information in the sequence $\mathbf{S}$ is infinite. No finite expression can contain the same information as $\mathbf{S}$. Since the definition of $\mathbf{S}$ involves figures obtained at random, in each instance determines different numbers. As a consequence, $\mathbf{S}$ defines a class of indiscernible objects. Recall that, by finite methods, we only can handle infinite sets of indiscernible objects (Theorem 2.2). Finally, by virtue of Theorem 2.3, the former result leads to the ω-uncountability of $[0,1]$.

2.1. Remote-cardinals and identifier supports

Let $\mathfrak{R}$ be a class of discernible maps and relations, and X the union of all domains and parameter sets of all members of $\mathfrak{R}$. We assume that X can contain hidden parameters too (Palomar Tarancón, 2016). From now on, we denote by $\mathfrak{C}[X, \mathfrak{R}]$ the class of all mathematical constructions such that, for each of which, we can build an identifier consisting of "finite compositions" of maps and relations in $\mathfrak{R}$. Likewise, for each mathematical construction K, we say each class $\mathfrak{C}[X, \mathfrak{R}]$ that contains K, to be an identifier support for it, provided that $\mathfrak{R}$ is finite and X is nonempty.

Notation. For every mathematical construction K, we denote by $\mathrm{Supp}(K)$ the class of all identifier supports for K.

Definition 2.1. We say that $\mathfrak{E}[X, \mathcal{R}] \in \mathrm{Supp}(K)$ is a *basic identifier support for K* provided that X satisfies the relation

$$\forall \mathfrak{E}(Y, \mathfrak{S}) \in \mathrm{Supp}(K): \quad \#(X) \leq \#(Y) \tag{2.5}$$

In this case, we also say that $\#(X)$ is the remote-cardinal of K, that we denote by the symbol $\flat$. In this case, $\flat(K) = \#(X)$.

Lemma 2.4. *The remote-cardinal of the set $\mathbb{Z}$ of all integers and the one of every of its members is* 1.

Proof. We can build every positive integer n in $\mathbb{Z}$ iterating n-times the map $\mathrm{suc}: m \to (m+1)$ with the only argument 0. Likewise, we can build each negative integer iterating the inverse map suc^{-1}. To obtain 0 we can apply the identity map. Thus, if $X = \{0\}$ and $\mathfrak{R} = \{\mathrm{suc}, \mathrm{suc}^{-1}, \mathrm{id}_{\mathbb{Z}}\}$, then $\mathfrak{E}[X, \mathfrak{R}]$ is an identifier support for $\mathbb{Z}$, and so is also for every of its members. Since $\#(\{0\}) = 1$ is the smallest possible cardinal of any nonempty set, $\flat(\mathbb{Z}) = \#(\{0\}) = 1$. Analogously, $\forall n \in \mathbb{Z}$: $\flat(n) = \#(\{0\}) = 1$. □

Theorem 2.5. *If $f: X \to Y$ is a discernible bijection, and for every $x \in X$ there are both remote-cardinals $\flat(x)$ and $\flat(f(x))$, then*

$$\forall x \in X: \quad \flat(x) = \flat(f(x)). \tag{2.6}$$

Proof. First, we show that $\flat(f(x)) \leq \flat(x)$. Let $\mathfrak{E}(A_0, \mathfrak{R}_0)$ be a basic identifier support for x, and $\mathfrak{E}(A_1, \mathfrak{R}_1)$ a basic one for $f(x)$; hence $\flat(x) = \#(A_0)$, and $\flat(f(x)) = \#(A_1)$. Since we can obtain $f(x)$, simply, applying the map f to x, and by assumption f is discernible, the expression $\mathfrak{E}(A_0, \mathfrak{R}_0 \cup \{f\})$ is also an identifier support for $f(x)$. Now, taking into account (2.5), the following relation holds.

$$\flat(f(x)) = \#(A_1) \leq \#(A_0) = \flat(x). \tag{2.7}$$

Likewise, because of f is a bijection there is the inverse f^{-1}; therefore we can also show that

$$\#(A_0) = \flat(x) = \flat\left(f^{-1}(f(x))\right) \leq \flat(f(x)) = \#(A_1) \tag{2.8}$$

Both equations (2.7) and (2.8) lead to (2.6). □

Definition 2.2. We say that a prime factorization $p_{k_1}^{n_1} p_{k_2}^{n_2} \cdots p_{k_j}^{n_j}$ of an integer N is exhaustive if it contains every prime smaller than p_{k_j}, perhaps with exponent 0.

For instance, $20 = 2^2 \cdot 5^1$, but the exhaustive factorization is $20 = 2^2 \cdot 3^0 \cdot 5^1$, which contains every prime smaller than 5.

Corollary 2.2. *The remote-cardinal of every discernible mathematical construction is* 1.

Proof. If X is a discernible mathematical construction, the map γ defined in (2.1) specifies X by an integer $M = \prod_{i=1}^{m} p_i^{\beta(\alpha_i)}$ because we can obtain the parameters $\alpha_1\alpha_2 \ldots \alpha_m$, simply, through the prime factorization of M as follows.

Let Φ denote the map that sends each integer M into the m-tuple of its factors

$$\left(p_1^{k_1}, p_2^{k_2}, \cdots p_m^{k_m}\right) = \left(p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \cdots p_m^{\beta(\alpha_m)}\right)$$

in its exhaustive prime factorization, and ω the map

$$\left(p_1^{\beta(\alpha_1)}, p_2^{\beta(\alpha_2)}, \cdots p_m^{\beta(\alpha_m)}\right) \mapsto \alpha_1\alpha_2 \ldots \alpha_m.$$

Accordingly, $\mathfrak{E}[\{M\}, \{\Phi, \omega\}]$ is an identifier support for X. Since $\{M\}$ is a singleton, $\flat(X) = \#(\{M\}) = 1$.

We can also show this result as a corollary of Theorem 2.5. By Theorem 2.1 we know that, for every nonempty set E, if every of its members is discernible, the discernible injective map γ sends E into a subset K of $\mathbb{N}$. Since the restriction of γ to its image is bijective, by virtue of Theorem 2.5, for each X in E, $\flat(X) = 1$. □

Lemma 2.5. *The remote-cardinal of every indiscernible real number is* $\aleph_0$.

Proof. On the one hand, a real number r is defined as the limit of a sequence $(a_n)_{n\in\mathbb{N}}$ of rationals. On the other hand, since $\mathbb{R}$ with the standard topology T is a Hausdorff space, the binary relation Lim_T between each converging sequence and its limit is a map. If $f : \mathbb{N} \to \mathbb{Q}$ is the map $n \mapsto a_n$, then $\mathfrak{E}\left[\mathrm{img}(f), \{\mathrm{Lim}_T, f\}\right]$ is an identifier support for r. Now, we show that, for every indiscernible real number, this support is basic.

If r is indiscernible, every of its identifiers must be infinite. By definition, for every $\mathfrak{E}[A, \mathfrak{R}] \in \mathrm{Supp}(r)$, each identifier E for r, associated with $\mathfrak{E}[A, \mathfrak{R}]$, must be built by finite compositions of members of $\mathfrak{R}$, and each of which is discernible. Under this condition, E can only be infinite if so is A. Accordingly, $\#(\mathbb{N}) = \aleph_0 \leq \#(A)$. Since $\mathrm{img}(f)$ is countable, then $\#(\mathrm{img}(f)) = \aleph_0$, and $\mathfrak{E}\left[\mathrm{img}(f), \{\mathrm{Lim}_T, f\}\right]$ is basic. Thus, $\flat(r) = \#\left(\mathrm{img}(f)\right) = \#(\mathbb{N}) = \aleph_0$. □

Corollary 2.3. *If the set* $[0, 1]$ *contains any indiscernible number, there is no discernible bijection between* $\mathbb{N}$ *and* $[0, 1]$.

Proof. Let $f : \mathbb{N} \to E \subseteq [0, 1]$ be a bijection. As a consequence of Theorem 2.5 and Lemma 2.5, for every n in $\mathbb{N}$, $\flat(n) = 1$ and $\flat\left(f(n)\right) = 1$; hence $\mathrm{img}(f)$ does not contain any indiscernible real number. Consequently, $E = \mathrm{img}(f)$ is a proper subset of $[0, 1]$ and f cannot be surjective. □

3. Non-computable real numbers

In this section, we say that a real number r is non-computable, whenever there is no Turing machine computing r. We show that the existence of any non-computable number in $[0, 1]$ leads to an infinite set of them. To this end, we assign an automaton to each positive integer, which accepts every binary representation of integers. We simplify their structures using one-way automaton classes. Consider the equivalence between two-way and one-way finite automata (Hulden, 2015).

Definition 3.1. Let $\mathbf{Aut}(\mathbb{N})$ be the class of all automata each of its members is a 7-tuple $\mathbf{A}_n = (Q_n, \Gamma, \varnothing, \Sigma, \delta, q_0, F)$; where

- $Q_n = \{p_1, p_2, \ldots p_n\} \cup \{HALT\}$ is the set of states, for some positive integer n; where each member is associated with a prime integer p_k, and $q_0 = p_1$ is the initial state.
- $\Gamma = \{0, 1, \varnothing\}$ is the tape alphabet, and $\varnothing$ is the blank symbol.
- $\Sigma \subset \Gamma$ is the set $\{0, 1\}$ of input symbols.
- $\delta : (Q_n \setminus F) \times \Gamma \to Q_n \times \Gamma \times \{R\}$ is the transition map, the codomain of which only contains the move R (right).
- $F = \{HALT\} \subseteq Q_n$ is the subset of final states.

From now one, we assume that each member of $\mathbf{Aut}(\mathbb{N})$ satisfies the following conditions.

1. For every $k \in \mathbb{N}$, p_k denotes the k-th prime integer. Thus, the initial state is $q_0 = p_1 = 2$. Likewise, $p_2 = 3$, $p_3 = 5$ and so on.
2. For every state $s \in (Q_n \setminus F)$: $\quad \delta(s, \varnothing) = (HALT, 1, R)$.
3. The image $(p_j, y, R) = \delta(p_k, x)$ under δ of every pair $(p_k, x) \in (Q_n \setminus F) \times \Gamma$ satisfies the following conditions.

$$p_j = \begin{cases} p_{k+1} \text{ if } k < n \text{ and } x \neq \varnothing, \\ p_n \text{ if } k = n \text{ and } x \neq \varnothing, \\ HALT \text{ if } x = \varnothing. \end{cases} \tag{3.1}$$

Now, we assign a Turing machine to each non-negative integer through the map $\Psi : \mathbb{N} \to \mathbf{Aut}(\mathbb{N})$ defined as follows.

- If $n \leq 2$, then $\Phi(n)$ is the Turing-machine in $\mathbf{Aut}(\mathbb{N})$ with the only states p_1 and $HALT$, and with the transition map

$$\delta(p_1, x) = \begin{cases} (p_1, x, R) \text{ if } x \in \{0, 1\} \\ (HALT, 1, R) \text{ otherwise.} \end{cases} \tag{3.2}$$

- If $n > 2$ and $p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m} = n$ is its exhaustive factorization into prime integers, then $\Phi(n)$ is the member of $\mathbf{Aut}(\mathbb{N})$ with the transition map δ defined in the following table.

$$\left| \begin{array}{c|c|c|c|c} x & \text{state } p_1 & \text{state } p_2 & \ldots & \text{state } p_m \\ \hline 0 & (p_2, f_{k_1}(0), R) & (p_3, f_{k_2}(0), R) & \ldots & (p_m, f_{k_m}(0)) \\ 1 & (p_2, f_{k_1}(1), R) & (p_3, f_{k_2}(1), R) & \ldots & (p_m, f_{k_m}(1)) \\ \varnothing & (HALT, 1, R) & (HALT, 1, R) & \ldots & (HALT, 1, R) \end{array} \right| \tag{3.3}$$

where for each $k \in \mathbb{N}$, $f_k : \{0,1\} \to \{0,1\}$ is the function defined as follows.

$$f_m(x) = \begin{cases} 0 \text{ if } m \equiv 0 \mod (4) \\ x \text{ if } m \equiv 1 \mod (4) \\ 1 \text{ if } x = 0 \text{ and } m \equiv 2 \mod (4) \\ 0 \text{ if } x = 1 \text{ and } m \equiv 2 \mod (4) \\ 1 \text{ if } m \equiv 3 \mod (4). \end{cases} \tag{3.4}$$

Notation. For every positive integer n and each finite tape-symbol sequence $c_1c_2\dots c_j$, we denote by $\Phi(n)[c_1c_2\dots c_j]$ the output that $\Phi(n)$ carries out from the initial tape configuration $c_1c_2\dots c_j$.

Lemma 3.1. *For every $n \in \mathbb{N}$, the Turing machine $\Phi(n)$ accepts every finite tape-symbol sequence.*

Proof. Let $s_1, s_2, s_3 \dots s_m$ be any finite symbol sequence in the tape of $\Phi(n)$. Since the only move is R, after a finite step sequence the focused cell is the one containing the symbol s_m. In the following step, the head reads the $(m+1)$-th cell that must contain the blank symbol $\varnothing$. According to (3.3), $\varnothing$ is substituted by 1, and the process gets the final state HALT. □

Theorem 3.1. *The set $[0,1] \subset \mathbb{R}$ contains an infinite subset of non-computable discernible numbers.*

Proof. We show that if there is one non-computable number in $[0,1]$, it also contains an infinite subset $\mathbf{K} \subset [0,1]$ of them. Let $r = 0.d_1d_2d_3\dots d_j\dots$ a non-computable real number in $[0,1]$ written in the binary numeration system. For each integer $k > 0$, let n_k be the positive one the binary figures of which are

$$d_{10k+1}d_{10k+2}\dots d_{(10k+10)} \tag{3.5}$$

Likewise, let $c_{k,1}c_{k,2}\dots c_{k,j_k}$ be the result of the computation

$$\Phi(n_k)[d_{10k+1}d_{10k+2}\dots d_{(10k+10)}] = c_{k,1}c_{k,2}\dots c_{k,j_k} \tag{3.6}$$

The real number

$$r^* = 0.c_{1,1}c_{1,2}\dots c_{1,j_1}c_{2,1}c_{2,2}\dots c_{2,j_2}\dots{}_{k,1}\, c_{k,2}\dots c_{k,j_k}\dots \tag{3.7}$$

is non-computable because its binary figures depend on the ones of r, and by hypothesis so is r. In addition, the set of Turing machines

$$\{\Phi(n_1), \Phi(n_2)\dots\Phi(n_k)\dots\}$$

by which we compute r^*, is infinite and requires to know previously the binary expansion of r.

Iterating the process over r^*, we obtain an infinite sequence of non-computable real numbers in $[0,1]$. Although in (3.5) we take sequences of ten binary figures, with figure sequences of other length we can also build infinite sets of non-computable real numbers.

Finally, the non-computable numbers involved in this proof can be determined by the finite expressions (3.5), (3.6) and (3.7); hence each of them is discernible. □

4. Sub-cardinals

From now on, we denote by **Dsc** the subset of all discernible members of $[0, 1]$. Likewise, by **Cmp** we denote the subset of all computable members of **Dsc**. As we have just seen in Theorem 3.1, there exist non-computable numbers that are discernible; hence, **Cmp** is a proper subset of **Dsc**.

Lemma 4.1. *There is no computable bijective map between any nonempty subset* $\mathbf{K}$ *of* $\mathbb{N}$ *and* $\complement_{[0,1]}\mathbf{Cmp}$.

Proof. Suppose that there is a computable bijection $f : \mathbf{K} \to \complement_{[0,1]}\mathbf{Cmp}$. For every positive integer n_0 in $\mathbf{K}$, $f(n_0)$ would be computable, which contradicts $f(n_0) \in \complement_{[0,1]}\mathbf{Cmp}$. □

Both results, Statement (2) in Theorem 2.2 and the former lemma, allow us to compare infinite sets through several criteria based upon the nature of the comparing bijections. According to Theorem 2.1 and Lemma 2.1, both sets $\mathbf{Dsc} \subset [0, 1]$ and $\mathbf{Cmp} \subset [0, 1]$ are countable. However, there is no "computable" bijection between these subsets of $[0, 1]$. This peculiarity allows us to state the following definitions.

Definition 4.1. Two sets X and Y are of the same λ-cardinality if there is, at least, one "computable" bijection between them (Radó, 1962). Likewise, we say X and Y to have the same δ-cardinal whenever there is a "discernible" bijection $f : X \to Y$.

Remark. The set **Cmp** of all computable real numbers in $[0, 1]$ is a proper subset of the set **Dsc** of all finitely definable members of $[0, 1]$. As we have seen in Lemma 2.1 and Theorem 2.1, both are countable subsets of $[0, 1]$, hence there is, at least, one finitely definable bijection $f : \mathbf{Cmp} \to \mathbf{Dsc}$. However, as a consequence of Lemma 4.1, there is no computable bijection $k : \mathbf{Cmp} \to \mathbf{Dsc}$. In other words, although both sets have the same cardinal $\aleph_0$, they are not of the same λ-cardinality. That is a consequence of the information that each member of **Cmp** and **Dsc** contains. Both, the information and complexity of non-computable numbers are greater than that lying in any computable one (Li & Vitányi, 2008).

Taking into account these properties, we can consider the λ-cardinal of **Cmp** as a sub-cardinal of $\aleph_0 = \#(\mathbb{N})$, which we denote by $\aleph_{-2}$. Since $\aleph_{-2} \neq \aleph_0$, then $\aleph_{-2}$ is a "proper" sub-cardinal of $\aleph_0$. By sub-cardinal we do not mean "smaller than." According to both Theorem 2.2 and Lemma 4.1, cardinalities are, simply, equivalence relations based on the comparing bijection nature. This viewpoint needs some explanation. The existence of an isomorphism between structured sets leads to size equality (Hume's principle). The converse statement need not be true. The size equality between structured sets need not lead to the existence of isomorphisms. Nevertheless, we can define bijections between the underlying sets, which are structure-free constructions. However, we cannot always forget the structure of any mathematical construction. For instance, real numbers are defined as limits of rational sequences. If we forget the topology of $\mathbb{R}$, the concept of limit vanishes, and we lose the real number definition. Indiscernibility is also an unforgettable structure property because it is an intrinsic object-definition attribute.

We denote by the expression $\aleph_{-1}$ the cardinal of **Dsc**. Since **Cmp** is a proper subset of **Dsc**, $\aleph_{-1}$ is a sub-cardinal of $\aleph_0$; and $\aleph_{-2}$ is a sub-cardinal of $\aleph_{-1}$. Finally, we denote by $\aleph_\omega$ the cardinal of each set E such that there is a first-kind indiscernible bijection between $\mathbb{N}$ and E.

Conjecture 4.1. *Each of the following incompatible statements is undecidable.*

1) $\aleph_\omega$ *is a proper sub-cardinal of* $\aleph_0$.
2) $\aleph_\omega = \aleph_0$.

In the former conjecture, by undecidable we mean the impossibility of rejecting or proving any of these statements by finite procedures. Take into account that, in general, by no finite procedure we can discern the equality or inequality of two indiscernible objects. Discerning the equality is necessary to know whether any map is a one-to-one correspondence. This is a consequence of Theorem 2.2.

5. Some paradoxes arising from Cantor's method

In this section, we introduce three paradoxes to show that Cantor's methods can involve inadequate self-referential definitions. The first paradox is built through an instance of the diagonal method, which leads to a contradiction. This situation occurs if the method involves any self-referential definition that fits into the following pattern.

$$X = \text{“}\textit{Expression that contains } X\text{”} \tag{5.1}$$

The former expression is an abstract equation. Since there are unsolvable equations, it is possible that some self-referential definitions define nothing. Self-referential definitions can be implicitly stated as in the following pattern.

$$X = \text{“}\textit{Expression that contains a class } E \textit{ containing } X \textit{ implicitly.}\text{”} \tag{5.2}$$

This is the case of Cantor's diagonal method under the assumption of actual infinity. For instance, let the following table denote a bijection $f : \mathbb{N} \to X \subseteq [0, 1]$.

$$\begin{cases} f(1) = 0.d_{11}d_{12}d_{13}\dots \\ f(2) = 0.d_{21}d_{22}d_{23}\dots \\ \quad\dots\dots\dots\dots\dots\dots\dots \\ f(n) = 0.d_{n1}d_{n2}d_{n3}\dots \\ \quad\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{5.3}$$

By the diagonal method we define a member $r = 0.c_1c_2c_3\dots$ of $[0, 1]$ that satisfies the condition

$$\forall n \in \mathbb{N} : \quad c_n \neq d_{nn} \tag{5.4}$$

If $r \notin X$ the statement above is an adequate definition, but if $r \in X$, is implicitly self-referential.

Paradox 5.1. According to Theorem 2.1, the subset **Dsc** of all discernible members or $[0, 1]$ is countable. Let $\gamma : \mathbf{Dsc} \to \mathbb{N}$ be the injective map finitely defined in (2.1), and $\gamma_0 : \mathbf{Dsc} \to \mathrm{img}(\gamma)$ the restriction to its image, which is bijective; hence there exists its inverse

$$\gamma_0^{-1} : \mathrm{img}(\gamma) \subseteq \mathbb{N} \to \mathbf{Dsc}.$$

Since γ is finitely defined in (2.1), so are both γ_0 and γ_0^{-1}; consequently, the three maps are discernible.

Now, for every positive integer n, let $0.c_{n1}c_{n2}c_{n3}\ldots$ be the decimal mantissa of $\gamma_0^{-1}(n)$. By Cantor's diagonal method we can build a real number $r = 0.d_1d_2d_3\ldots$ in $[0, 1]$ as follows.

$$\forall n \in \mathbb{N}: \quad d_n = \begin{cases} c_{nn} + 1 \text{ if } c_{nn} < 9 \\ 0 \text{ otherwise.} \end{cases} \tag{5.5}$$

The former equation is a finite expression that determines r uniquely; hence it is discernible, and the relation

$$r \in \mathbf{Dsc}. \tag{5.6}$$

is true. Nevertheless, equation (5.5) leads to $\forall n \in \mathbb{N} : r \neq \gamma_0^{-1}(n)$, because $\forall n \in \mathbb{N} : c_{nn} \neq d_n$; consequently

$$r \notin \mathbf{Dsc}, \tag{5.7}$$

which contradicts (5.6).

We can solve the former paradox easily. Recall that, in the scope of actual infinity, every infinite set is a construction that can be completed. In this case, when the image of an indiscernible bijection $f : \mathbb{N} \to E$ contains every number r_0 that can be obtained by Cantor's diagonal method, this method involves a self-referential definition for r_0. This is the case in the former paradox. To better understand this topic, we state the paradox below by the inverse method. We choose the number $r_0 \in [0, 1]$ and build a bijection that does not contain it.

Paradox 5.2. Let $f : \mathbb{N} \to \mathbf{M} \subseteq [0, 1]$ be a bijection that satisfies the following conditions.

1. Let $\mathbb{E}$ denote the set $\{2 \cdot n \mid n \in \mathbb{N}\}$ of all positive even integers. The restriction $f|_{\mathbb{E}} : \mathbb{E} \to \mathbf{M}$ of f to $\mathbb{E}$ is the subset $\mathbb{Q} \cap [0, 1]$ of all rationals in $[0, 1]$. Recall that both $\mathbb{E}$ and $\mathbb{Q}$ are countable; so then we can build the map f under the condition $\mathrm{img}(f|_{\mathbb{E}}) = \mathbb{Q} \cap [0, 1]$.
2. The image $f(n)$ of each odd integer n is an irrational member of $\mathbf{M}$. Thus, $\mathrm{img}(f)$ contains every rational in $[0, 1]$ and an infinite and countable subset of irrationals.

Let $r \in \mathbb{Q} \cap [0, 1] \subset \mathrm{img}(f)$ be a rational number and $0.d_1d_2d_3\ldots$ the figures of its decimal expansion. Suppose that, for every $n \in \mathbb{N}$, the expansion of $f(n)$ is $0.c_{n1}c_{n2}c_{n3}\ldots$. Now, we define a new bijection f_1 as follows. If $d_1 \neq c_{11}$, then $f_1 = f$; otherwise, if n_1 is the smallest integer such that $c_{n1} \neq d_1$, then

$$f_1(n) = \begin{cases} f(n_1) \text{ if } n = 1 \\ f(1) \text{ if } n = n_1 \\ f(n) \text{ otherwise;} \end{cases} \tag{5.8}$$

We obtain the bijection f_1, simply, by the transposition of $f(1)$ and $f(n_1)$ in the image of f, hence $\mathrm{img}(f) = \mathrm{img}(f_1)$, besides, the first figure in the mantissa of $f_1(1)$ differs from d_1.

Now, we define another bijection f_2 by a similar procedure. If the second figure in the mantissa of $f_1(2)$ is not equal to d_2, then $f_2 = f_1$; otherwise, if n_2 is the smallest integer such that $n_2 > 2$ and $c_{n2} \neq d_2$, then

$$f_2(n) = \begin{cases} f(n_2) \text{ if } n = 2 \\ f(2) \text{ if } n = n_2 \\ f(n) \text{ otherwise;} \end{cases} \tag{5.9}$$

As in the preceding case, the bijection f_2 is obtained, simply, by the transposition of $f(2)$ and $f(n_2)$; hence $\text{img}(f) = \text{img}(f_1) = \text{img}(f_2)$, besides, the second figure in the mantissa of $f_1(2)$ differs from d_2.

Iterating the procedure m-times, we obtain a bijection $f_m : \mathbb{N} \to \mathbf{M}$ such that $\text{img}(f_m) = \text{img}(f)$, and $\forall n \leq m$: $r \neq f_m(n)$. Under the scope of potential infinity, the iteration never ends. By contrast, assuming infinity as an actual entity, the iteration can be completed obtaining a bijection $f_\infty : \mathbb{N} \to \mathbf{M}$ with $\text{img}(f_\infty) = \text{img}(f) = \mathbf{M}$, and $\forall n \in \mathbb{N} : r \neq f_\infty(n)$, hence $r \notin \mathbf{M}$; which contradicts that $r \in \mathbb{Q} \cap [0, 1] \subseteq \mathbf{M}$, that we assume implicitly in statement (1), and f satisfies this condition.

Paradox 5.3. Let $\mathcal{F}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N})$ be the subset of all finitely definable subsets of $\mathbb{N}$. Since we can denote every member of $\mathcal{F}(\mathbb{N})$ by a finite symbol sequence in some extensible language, we can also define a map $\zeta : \mathcal{F}(\mathbb{N}) \to \mathbb{N}$ similar to the one γ defined in (2.1). Let $\zeta_0 : \mathcal{F}(\mathbb{N}) \to \text{img}(\zeta)$ be the restriction of ζ to its image, which is a bijection.

Now, we define the subset $\mathbf{K}$ of $\mathbb{N}$ by the finite expression

$$\mathbf{K} = \{n \in \text{img}(\zeta) \subseteq \mathbb{N} \mid n \notin \zeta_0^{-1}(n)\} \tag{5.10}$$

If $\zeta(\mathbf{K}) \in \mathbf{K}$, by virtue of (5.10), this relation leads to $\zeta(\mathbf{K}) \notin \mathbf{K}$, and vice versa. We can solve this contradiction supposing that ζ is not defined at $\mathbf{K}$. In other words: $\mathbf{K} \notin \mathcal{F}(\mathbb{N})$. However, $\mathbf{K}$ is a subset of $\mathbb{N}$ finitely defined in (5.10); hence $\mathbf{K} \in \mathcal{F}(\mathbb{N})$.

It is worth pointing out that, as in both paradoxes 5.1 and 5.2, the former one requires the assumption of actual infinity. Thus, if we assume that every infinity is potential, these paradoxes vanish.

6. Conclusion

On the one hand, we should not ignore the existence of indiscernible mathematical constructions, unless we reject the existence of uncountable sets. Unfortunately, as a consequence of Theorem 2.2, we can only handle infinite sets of indiscernible objects. Thus, when working with real world problems, we never can be involved with any "finite" set of indiscernible numbers.

On the other hand, in the scope of actual infinity, Cantor's diagonal method sometimes proves nothing. Fortunately, taking into account Corollary 2.1 and Theorem 2.3, we can deduce the existence of uncountable sets from indiscernible mathematical constructions. In addition, if Conjecture 4.1 is true, a set theory including it becomes incomplete, and we can apply Gödel's incompleteness theorems to it.

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Proximal Čech Complexes in Approximating Digital Image Object Shapes. Theory and Application

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Abstract

This article introduces proximal Čech complexes in approximating object shapes in digital images. The theoretical framework is based on Čech complexes and proximity spaces. Several topological structures are defined for the Čech nerve based covers of a finite region of Euclidean plane. We define k-petals and k-corollas which are the generalizations of spokes and maximal nuclear clusters. We extend the classical notion of a proximity as a binary relation, to arbitrary number of sets. A new shape signature based on the distribution of orders of Čech nerves is defined. A practical application of this framework in approximating object shapes in digital images is given.

Keywords: Čech complex, Digital image, Nerve, Object shape
2010 MSC No: Primary 54E05 (Proximity), Secondary 68U05 (Computational Geometry)

1. Introduction

Understanding of shapes of objects in images is an important aspect of artificial intelligence, with numerous applications. Some of the applications detection of specific patterns in images (Hettiarachchi *et al.*, 2014), quantifying information content of images using Vorono ï tessellations (A-iyeh & Peters, 2016) and detecting outliers from images of different classes (Shamir, 2013). In this paper we aim to understand the shape of the objects in a digital image, by covering it with sets of known geometrical properties. Moreover, we aim at enriching the conventional notion of shape

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as its contour (as used in the prevalent literature) to include the interior as well. For this purpose we will combine the notions of topology and geometry with proximity spaces.

E. Čech introduced Čech complexes during a seminar at Brno(1936-39) (Čech, 1966, §A.5). Recent works define the Čech complexes as collections of intersecting closed geometric balls of a radius of r (Edelsbrunner & Harer, 2010); (Peters, 2017*a*). In this study we focus on approximating shapes of objects in planar digital images. This leads to the restriction of the Čech complexes to a finite bounded region of the Euclidean Plane. In a recent study some open problems regarding planar shapes(shapes of objects in Euclidean plane) have been posed (Peters, 2017*a*) . The viability of Čech complexes as a method to approximate image object shapes has been shown in (Peters, 2017*b*).

A previous study formulates the notion of an object in the digital image as a topological space (Ahmad & Peters, 2017*a*). The notion of a nerve introduced by Alexandroff (Alexandroff, 1965) has been generalized to the notion of spoke complexes. The basic building blocks of the topology are assumed to be curvilinear triangulations. Another study builds on this notion and studies the covering properties of the curvilinear triangulation using the notion of area and geodesic diameter (Ahmad & Peters, 2017*b*). Moreover, a notion of the frequency of nerves of different order as a possible signature of the image objects was also introduced. The notion of proximity was also defined on triangulated spaces (Ahmad & Peters, 2017*a*) and Čech nerves (Peters, 2017*b*).

The subject of the current study is to extend the topological framework for object spaces formulated in (Ahmad & Peters, 2017*a*) to Čech complexes. In addition to this the framework of assessing the covering properties of nerves and object spaces similar to (Ahmad & Peters, 2017*b*) is also developed. A notion of proximity will be introduced on the resulting object spaces. This yields a relator space of objects in the digital images.

2. Basic Definitions

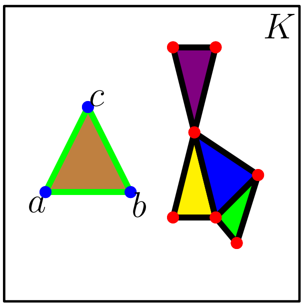
In this section, we will present some basic definitions. Let $X = \mathbb{R}^2$ be the Euclidean plane then $K \in 2^X$ is a finite bounded region in X. The basic building block of Čech complex is the closed geometric ball,$B_r(x)$, with center $x \in X$ and radius $r > 0$ defined as $B_r(x) = \{y \in X : \|x - y\| \leq r\}$.

In this paper, we will study the topology of the objects in a planar digital image. Let us first define **abstract simplicial complex**.

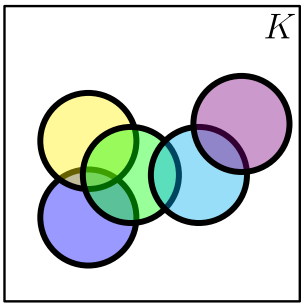
Definition 1. *(Ghrist, 2014, §2.1) Let S be a discrete set. Then the collection of finite subsets of S represented by X is called an abstract simplicial complex, provided for each $\sigma \in X$, all subsets of σ are also in X. This means that X is closed under this restriction.*

Example 1. *Consider a geometric realization of an abstract simplicial complex in form of collections of triangles. Let us first look at the set $X = \{a, b, c\}$, representing the vertices of a triangle as shown in Fig. 1.1. Then the powerset of all the subsets of the set X is,*

$$2^X = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$



1.1: Collections of triangles



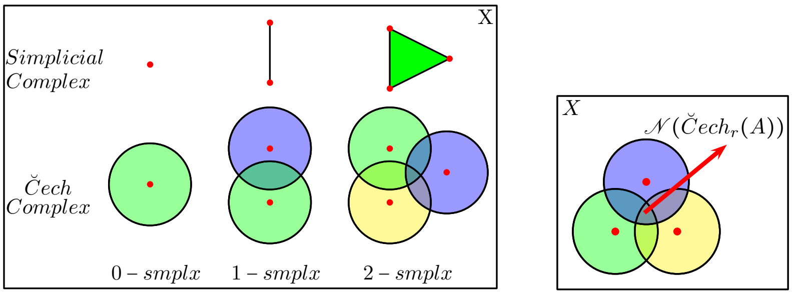
1.2: Collections of Geometric Balls

Figure 1. This figure shows the geometric realizations of abstract simplicial complexes as collections of connected triangles and intersecting closed geometric balls

It can be seen that the set 2^X is closed under restriction as per Def.1. The singleton elements of the set 2^X, i.e. $\{a\}, \{b\}$ and $\{c\}$ represent each of the vertices or the 0-simplices. These are shown in blue on the isolated triangle. The elements $\{a, b\}, \{b, c\}$ and $\{a, c\}$ represent the edges or the 1-simplices. These are shown in green on the isolated triangle. The element $\{a, b, c\}$ represents the filled triangle or the 2-simplex. This is shown with brown color on the isolated triangle. One can easily verify that the connected triangles in Fig. 1.1, also form a simplicial complex. This follows from the fact that each of the triangles in itself is a simplicial complex determined by the power set of the set of its vertices. The whole collection of the triangles is represented by the set Y, which is the union of their respective power sets. Every possible subset of the elements in set Y is contained in itself. Thus set Y is also a simplicial complex as per Def. 1. ■

Keeping in view the previous example it can be seen that the collection of geometric balls($B_r(x)$ for $x \in K$) with a finite number of intersections can be seen as a realization of the abstract simplicial complex. For the purpose of visualization consider the Fig. 1.2 The individual balls are the 0-simplices. The pairs of balls with non-empty intersection are the 1-simplices. The examples are the blue and the violet ball, green and the blue ball etc. If three balls have a common intersection, they lead to a 2-simplex. The yellow, green and blue balls in Fig. 1.2 are an example of a 2-simplex. It can be seen that every pair of balls included in the 2-simplex form a 1-simplex and each constituent ball of this resulting 1-simplex is a 0-simplex. Hence, the collection of closed geometric balls ($B_r(x)$, $x \in K$) is also a geometric realization of an abstract simplicial complex. The 1-simplex which is a line segment in the simplicial complex, is analogous to two Čech balls having common intersection. The 2-simplex is a triangle in the simplicial complex and three Čech balls with a common intersection, in the Čech complex.

Moreover, it can be seen from Fig. 1.2 that one can draw analogs between the simplicial complex (generalization of a triangle to arbitrary dimensions) and the simplices in a Čech complex. It can be seen that each ball in a Čech complex is a 0-simplex and hence similar to the vertex in a simplicial complex. Let us define the notion of a Čech nerve.



2.1: Comparison of corresponding simplices in a simplicial complex and Čech complex

2.2: Nucleus of a Čech nerve

Figure 2. This figure explains the notion of a Čech complex(Def. 3) and compares it with the simplicial complex. Moreover, the notion of the nerve (Def. 11) of a Čech complex is illustrated.

Definition 2. *A **Čech nerve**, is a collection of sets having a non-empty intersection denoted by ($\check{C}ech_r(K)$) i.e.,*

$$\check{C}ech_r(K) = Nrv\{B_r(x) : x \in K\} = \{B_r(x) : \bigcap B_r(x) \neq \emptyset\}.$$

Then, we can define the notion of the **order** of a nerve. It is denoted by $|\check{C}ech_r(K)|$, and is defined as the number of balls in the nerve. We define a set with no non-empty intersections as being a nerve of order 1. Using this notion, we can formulate a definition of the Čech complex in a space X.

Definition 3. *Let X be a finite, bounded planar region $X \in \mathbb{R}^2$ and let $K \in X$ be a nonempty collection of points. A Čech complex is a collection of Čech nerves. The Čech nerves, $\check{C}ech_r(x \in K)$ of order k, form the $k-1$-simplices. The Čech complex is denoted by $cxK = 2^{\check{C}ech_r(K)}, K \in 2^{\mathbb{R}^2}$.*

The basic building block of a Čech complex is a closed geometric ball $B_r(x)$. The view of a digital image as a topological space has been detailed in (Peters, 2014). In current work we extend the conventional notion of a Čech nerve as follows.

Definition 4. *A **strong Čech nerve**, denoted by $\check{C}ech^s_r(K)$, is a collection of sets whose interiors have a non-empty intersection i.e.,*

$$\check{C}ech^s_r(K) = Nrv^s\{B_r(x)\text{''}x \in K\} = \{B_r(x) : \bigcap int(B_r(x)) \neq \emptyset\}.$$

In doing so we have also extended the classical notion of a **nerve** to **strong nerve**. A strong nerve is a collection of sets, whose interiors have a non-empty intersection. The notion of **order** remains unchanged. Similar to the notion of the Čech complex, we define the concept of a **strong** Čech complex on a space X.

Definition 5. *Let X be a finite, bounded planar region $X \in \mathbb{R}^2$ and let $K \in X$ be a nonempty collection of points. A strong Čech complex is a collection of strong Čech nerves. The strong Čech nerves, $\check{C}ech^s_r(x \in K)$ of order k form the $k-1$-simplices. The strong Čech complex is denoted by $cx^s K = 2^{\check{C}ech^s_r(K)}, K \in 2^{\mathbb{R}^2}$.*

In this paper we want to approximate the objects in the digital images as topological spaces. This will allow us to talk about shape (Segal & Dydak, 1978) and define invariant signatures for classification (Carlsson *et al.*, 2005); (Chazal *et al.*, 2009). The classical notion of a topology is defined using the notion of open sets.

Definition 6. *(Edelsbrunner & Harer, 2010) Topology is an ordered pair (X, τ), where X is a set and τ is the collection of subsets of X satisfying the following axioms:*
1^o *The empty set, $\varnothing$, and X belong to τ*
2^o *Union of sets in τ is also in τ*
3^o *Intersection of finite number of members of τ is also in τ*

We call this classical notion of topology as **open topology**. Using the De Morgan's laws, we can convert the notion of an open set topology to a closed set topology. Let us define the notion of a complement with respect to a set X. For a set $A \subset X$, the complement of a A denoted by $\overline{A}$ is defined as $\overline{A} = X \backslash A$. The laws state that for two sets A, B:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

A set A is closed in a topology τ, if its complement $\overline{A}$ is in τ. Using these notions we can extend the definition of a topology with open set as a primitive to using the closed set as the basis. Let us define the notion of topology using closed sets as primitive. To differentiate this we call it a **closed topology**.

Definition 7. *Topology is an ordered pair (X, η), where X is a set and η is the collection of subsets of X satisfying the following axioms:*
1^0 *The empty set, $\varnothing$, and X are closed and belong to η*
2^0 *The intersection of any collection of sets in η is closed*
3^0 *The union of a collection of finite number of sets in η is closed*

Example 2. *Let us examine the equivalence of the two definitions of topology based on open sets(Def. 6) and closed sets(Def. 7). It must be noted that as we have defined the set to be closed under the topology if its complement is in the topology. To establish the equivalence of the notions of open and closed topology we define the collections of subsets η to contain the complement of each subset in the collection τ. This complement is taken with respect to the set X. Let us begin with the first axiom in Def. 6. It states that both the $\varnothing$ and X are in τ. Thus, both are open sets. It can be seen that $\overline{\varnothing} = X \in \tau$ and $\overline{X} = \varnothing \in \tau$. Which means that both $\varnothing, X$ are closed and are in η. Thus, this statement is equivalent to the first axiom stated in Def. 7.*

The second axiom in the Def.6 can be stated as $\{\bigcup_{i\in\mathscr{I}} A_i : A_i \in \tau\} \in \tau$*, where* $\mathscr{I}$ *is an index set which can be infinite and* $A_i \in \tau$*. This means that* $\{\bigcup_{i\in\mathscr{I}} A_i\}$ *is an open set. Which is equivalent to* $\overline{\bigcup_{i\in\mathscr{I}} A_i} \in \eta$ *being a closed set. Using De Morgan's law we can write this as* $\bigcap_{i\in\mathscr{I}} \overline{A_i}$*, where* $\overline{A_i} \in \eta$ *and* $\mathscr{I}$ *is an index set which can be infinite. Since the complement of an open set must be closed,* $\bigcap_{i\in\mathscr{I}} \overline{A_i}$ *must be closed. We have the second axiom of Def. 7. Which states that intersection of any collection of sets in* η *is closed.*

The third axiom in Def. 6 is $\{\bigcap_{i\in\mathscr{I}} A_i : A_i \in \tau\} \in \tau$*, where* $\mathscr{I}$ *is a finite index set. This means that* $\bigcap_{i\in\mathscr{I}} A_i$ *is an open set and its complement is closed. Thus* $\overline{\bigcap_{i\in\mathscr{I}} A_i} = \{\bigcup_{i\in\mathscr{I}} \overline{A_i} : \overline{A_i} \in \eta\}$ *(using De Morgan's law), is a closed set. This statement is equivalent to the third axiom of Def. 7, which states that the union of a collection of finite number of sets in* η *is closed. Hence, the Def. 6 and Def. 7 are equivalent. The former uses the open sets as the primitive and the later uses the notion of a closed set.* ■

Let us define a topology on the Euclidean plane($\mathbb{R}^2$) using the closed geometric balls , $B_r(x)$.

Definition 8. *Let* U *be a closed subset of* $\mathbb{R}^2$ *and* τ_{std} *be a family of closed subsets of* $\mathbb{R}^2$*. Then,* $U \in \tau_{std}$ *if and only if for all* $p \in U$*, there exists a positive real number* r *such that* $B_r(p) \subseteq U$*.*

Now, let us verify that $(\mathbb{R}^2, \tau_{std})$ is a closed topological space.

Lemma 1. *The pair* $(\mathbb{R}^2, \tau_{std})$ *is a closed toplogical space.*

Proof. Let us verify that the family of subsets τ_{std} satisfies the axioms stated in Def. 7.

1^o : Let us verify that $\varnothing \in \tau_{std}$. From Def. 8, if $\varnothing \in \tau_{std}$, then the following condition must be satisfied.

$$\forall p \in \varnothing \Rightarrow \exists r \in \mathbb{R}^+ \ s.t.\ B_r(p) \subseteq \varnothing$$

It is evident from the definition of implication($\Rightarrow$), that $A \Rightarrow B$ is always true if A is false. Since, $p \in \varnothing$ is always false,thus the statement

$$\exists r \in \mathbb{R}^+ \ s.t.\ B_r(p) \in \varnothing,$$

is true. From this it follows that $\varnothing \in \tau_{std}$. Let us now verify that there exists a positive real number r such that $B_r(x) \subseteq \mathbb{R}^2$, for all $x \in \mathbb{R}^2$. It can be seen from the definition of $B_r(x) = \{y \in \mathbb{R}^2 : \|x - y\| \leq r\}$ that $B_r(x) \subseteq \mathbb{R}^2$ independent of x and r. Thus $\mathbb{R}^2 \in \tau_{std}$.

2^o : Let $\{C_i\}_i$ be a family of sets such that $\forall i, C_i \in \tau_{std}$. Then, if $p \in \bigcap C_i$ then there exists a positive real number r_i for each C_i, such that $B_{r_i}(p) \subseteq C_i$. It can be stated that,

$$B_{r_1}(p) \subseteq C_1 \wedge \cdots \wedge B_{r_i}(p) \subseteq C_i.$$

Let us define $r = \min_i(r_i)$. Then it can be seen that,

$$B_r(p) \subseteq B_{r_1}(p) \subseteq C_1 \wedge \cdots \wedge B_r(p) \subseteq B_{r_i}(p) \subseteq C_i.$$

From this we can conclude that $B_r(p) \in \bigcap C_i$. Therefore, $\bigcap C_i \in \tau_{std}$.

3^o : Let $\{C_i\}_i$ be a finite collection of sets such that $\forall i, C_i \in \tau_{std}$ and $p \in \bigcup C_i$. Then $p \in C_i$, for some i. Since, $C_i \in \tau_{std}$:

$$\exists r \in \mathbb{R}^+ \; s.t. \; B_r p \subseteq U \subseteq \bigcup C_i.$$

From this we can conclude that $\bigcup C_i \in \tau_{std}$.

Thus, it can be concluded that $(\mathbb{R}^2, \tau_{std})$. □

Remark 1. *The notion of a closed standard topology is important as it is the closed set analouge of the standard topology, used for analysis and the conventional signal and image processing.*

Moving on, we define the notion of the closed subset topology.

Definition 9. *Let (X, τ) be a closed topological space, and S be a closed subset of X, i.e. $S \subseteq X$. Let us define $\tau_S = \{S \cap U \; s.t. \; U \in \tau\}$. Then the pair (S, τ_S) is a topological space where τ_S is the subset topology.*

Let us verify that (S, τ_S) is indeed a topological space.

Lemma 2. *Let (X, τ) be a topological space and let S be a closed subset of X. Then the pair (S, τ_S), where $\tau_S = \{S \cap U \; s.t. \; U \in \tau\}$, is a closed topological space.*

Proof. To prove that the pair (S, τ_S) is a closed topological space it must satisfy the axioms detailed in Def. 9.

1^o : It can be verified from the definition of τ_S, that $S \cap (\emptyset \in \tau) = \emptyset$ is in τ_S. Moreover, it can also be verified from definition of τ_S, that $S \cap (X \in \tau) = S$ is also in τ_S. Thus, both $\emptyset$ and S are in τ_S.

2^o : Let $\{C_i\}_i$ be a collection of sets, such that $\forall i, C_i \in \tau_S$. Then from definition of τ_S it can be inferred that, for each $C_i \in \tau_S$ there exists a set D_i in τ, such that for all i, $S \cap D_i = C_i$. Since τ is a closed topology over the set X,then by Def. 9 the set $\bigcap D_i$ is also in τ. This leads to the conclusion that $(\bigcap D_i) \cap S$ is in τ_S by definition. We can see that $(\bigcap D_i) \cap S$ can be written as $\bigcap (D_i \cap S)$. As, we defined in this argument that $D_i \cap S = C_i$ we can conclude that $\bigcap C_i$ is in τ_S.

3^o : Let $\{C_i\}_i$ be a finite collection of sets, such that for all $i \, C_i \in \tau_S$. Then from the definition of τ_S, we can infer that for each $C_i \in \tau_S$ there exists a set $D_i \in \tau$, such that $D_i \cap S = C_i$. Since, τ is a closed topology then by definition it satisfies the condition that,$\bigcup D_i$ is in τ. From the definition of τ_S we can conclude that $S \cap (\bigcup D_i)$ is in τ_S. From the distributive laws of set algebra we can rewrite this as $\bigcup (D_i \cap S)$ is in τ_S. We know that $D_i \cap S = C_i$, hence $\bigcup C_i$ is in τ_S.

Since the pair (S, τ_S) satisfies all the axioms it is a closed topological space over S, with a subset topology τ_S. □

Now, let us talk about the Čech complex as a closed topological space.

Theorem 1. *Let $(\mathbb{R}^2, \tau)$, be a closed topological space and let cxK be a Čech complex. Then the pair (cxK, τ_{Cech}) is a closed topological subspace, where τ_{Cech} is the subset topology, defined as $\{cxK \cap U \text{ s.t. } U \in \tau\}$.*

Proof. Let us first ascertain that cxK is a subset of $\mathbb{R}^2$. This follows from definition as cxK is the union of Čech nerves($\check{C}ech_r(K)$) as defined in Def. 3. Moreover, the $\check{C}ech_r(K)$ is a union of closed geometric balls as per Def. 2. Next, we can see that the closed geometric balls are in standard closed topology τ_{std} by definition as there exists a positive real number r such that $B_r(p) \subseteq B_r(x)$ for all p in $B_r(x)$. Since, the closed geometric balls are in τ_{std}, thus by the definition of a closed topology, the union of a finite number of closed geometric balls must also be in τ_{std} i.e. closed. Thus, cxK is closed and by the definition it is a subset of $\mathbb{R}^2$. From this and Lemma 2, it can be concluded that given a closed topological space $(\mathbb{R}^2, \tau)$, the pair (cxK, τ_{Cech}), where $\tau_{Cech} = \{cxK \cap U \text{ s.t. } U \in \tau\}$, is a topological space, where τ_{Cech} is the subset topology. □

Now that we have a notion of Čech nerves as a topological space, we have to draw a correspondence to the objects in digital images. For this purpose, we use specific points from the image called keypoints. There are several algorithms for selecting such points based on their strength in an appropriate feature spaces (Lowe, 1999); (Bronstein & Kokkinos, 2010); (Aubry *et al.*, 2011); (Bruna & Mallat, 2013); (Alahi *et al.*, 2012). We construct Čech nerves with these keypoints as the centers of the geometric closed balls. We assume that the object under consideration, is the most important part of an image in the feature space. Let us now define the topological structures in the strong Čech nerves, that we are going to use as approximations of the objects in an image. We have used the notion of strong Čech nerves because they allow a more rich set of proximity relations, that we will define later on. The same can be done for the Čech nerves.

The first structure that we are going to define is the notion a petal.

Definition 10. *Let $\check{C}ech^s_r(K)$ be a strong Čech nerve,then each closed geometric ball in the nerve $\check{C}ech^S_r(K)$ is called a petal and is denoted by* **ptl***.*

Now, we define the notion of a nucleus.

Definition 11. *Let $\check{C}ech^s_r(K)$ be a strong Čech nerve. Then, the nucleus of the nerve is the common intersection of the constituent sets of the nerve. It is defined as $\mathcal{N}(\check{C}ech^s_r A) = \{\bigcap B_r(x) : B_r(x) \in \check{C}ech^s_r A\}$, where $\check{C}ech^s_r A$ is a strong Čech nerve.*

Example 3. *Let us explain the concept of a strong Čech nerve. For this consider Fig. 2.2. K is a set of points in the finite, bounded region of $\mathbb{R}^2$, shown as red points in the figure. The $\check{C}ech^s_r(K)$ in this illustration is the union of the three geometric balls(blue, green and yellow) centered at the points represented as red dots. Each of the individual balls is the petal(***ptl***) of the $\check{C}ech^s_r(K)$. This is represented as* **ptl**$(\check{C}ech^s_r(K))$*. The common intersection of the nerve is called the nucleus. This is represented as $\mathcal{N}(\check{C}ech^s_r(K))$.* ■

We now generalize the concept of a petal to the notion of a k-petal, defined using a recursive definition.

Definition 12. *Let cx^sA be a strong Čech complex on a finite, bounded region of the Euclidean plane $\mathbb{R}^2$,$\check{C}ech^s_rA$ is a strong Čech nerve,and $k > 0$, $k \in \mathbb{Z}$. Then k-petal is defined as the closed geometric ball $B_r(x)$ that has a nonempty intersection with a $(k-1)$-petal. The 0-petal is the nucleus. This can be formally written as:*

$$\mathbf{ptl}_k = \{B_r(x) \in cx^sA \backslash \{\bigcup \mathbf{ptl}_{k-1}\} : B_r(x) \cap \{\bigcup \mathbf{ptl}_{k-1}\} \neq \emptyset, \mathbf{ptl}_0 = \mathcal{N}(\check{C}ech^s_rA)\}.$$

Here, it is important to mention that the nerve of the highest order in the image is very important for object extraction in digital images (Peters & İnan, 2016). Such a nerve is called the **maximal strong Čech nerve** and is denoted as $\mathbf{max}\check{C}ech^s_r(K)$. The nucleus of the maximal strong Čech nerve is called the **maximal nucleus** and is denoted as $\mathbf{max}\mathcal{N}$. It must be noted that there can be multiple maximal strong Čech nerves in a digital image. In this case the the $\mathbf{max}\check{C}ech^s_r(k)$ is a set of each of the maximal strong Čech nerves, and the $\mathbf{max}\mathcal{N}$ is the set of the nuclei associated with the maximal nerves.

Next, we consider the analogue of a corolla, from Botany. A corolla is a cluster of petals of a flower, typically forming a whorl within the sepals and enclosing the reproductive organs. This construct works well in analyzing the interior of Čech complexes that cover an image object shape.

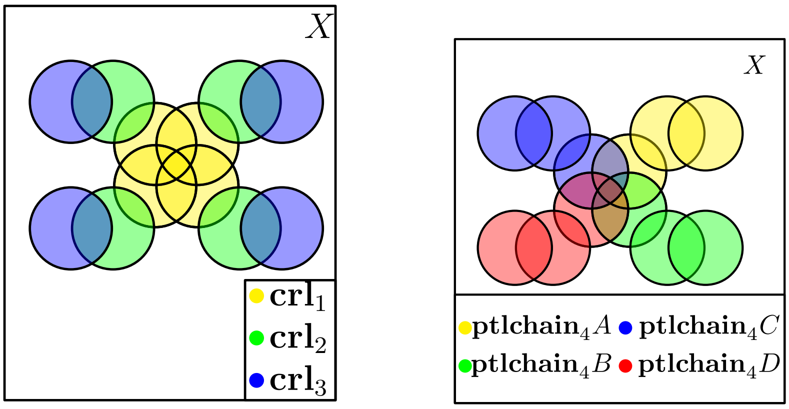
Definition 13. *Let cxA be a strong Čech complex on a finite, bounded region of the Euclidean plane $\mathbb{R}^2$. Then the k-corolla, denoted as $\mathbf{crl}_k$ is defined as: $\mathbf{crl}_k = \bigcup \mathbf{ptl}_k$.*

Another notion that we will define using the notion of a k-petal is the k-petal chain. It is a sequence of petals, one for each value of k. Each petal has a nonempty intersection with adjacent petals in the chain. This notion is formalized as follows.

Definition 14. *Let cx^sA be a strong Čech complex on a finite, bounded region of Euclidean plane $\mathbb{R}^2$. Then the k-petal chain denoted as $\mathbf{ptlchain}_k$ is defined as:*

$$\begin{aligned} \boldsymbol{k{>}1} : \mathbf{ptlchain}_k &= \{\bigcup_{\mathscr{I}} A_i : A_i \in \mathbf{crl}_i, A_i \cap A_{i-1} \neq \emptyset, \mathscr{I} = 1, \cdots, k\} \\ \boldsymbol{k{=}0} : \mathbf{ptlchain}_k &= \mathcal{N}(\check{C}ech^s_rA) \end{aligned}$$

Example 4. *Let us explain the topological structures that we have defined above. Let us first explain the notion of a k-petal($\mathbf{ptl}_k$) defined in Def. 12, using Fig. 3.1. The $\mathbf{ptl}_0$(0-petal) has not been explicitly marked on the figure. It is the intersection of the yellow balls and is also the nucleus of the strong Čech nerve, $\mathcal{N}(\check{C}ech^s_r(K))$. Each of the yellow balls is the $\mathbf{ptl}_1$(1-petal). It is evident form the illustration that each of the green balls has a non-empty intersection with the yellow balls ($\mathbf{ptl}_1$) and no intersection with the nucleus($\mathbf{ptl}_0$). Thus each of the green balls is a $\mathbf{ptl}_2$(2-petal) as per Def. 12. Each of the blue balls has a non-empty intersection with the green balls($\mathbf{ptl}_2$), while has no intersection with the yellow balls($\mathbf{ptl}_1$) and as per Def. 12 are the $\mathbf{ptl}_3$.*



3.1: Collections of triangles

3.2: Collections of Geometric Balls

Figure 3. This figure displays the geometric realizations of the k-corolla, $\mathbf{crl}_k$(Def. 13), and k-petal chain $\mathbf{ptlchain}_k$(Def. 14).

Let us now use the notion of a k-petal to define the notion of a k-corolla($\mathbf{crl}_k$). We are still using the Fig.3.1. The union of all the $\mathbf{ptl}_0$ in the image are the $\mathbf{crl}_0$. Since, there is one nucleus in the image and it has not been marked, hence the $\mathbf{crl}_0$ has not been marked. The union of all the yellow balls ($\mathbf{ptl}_1$) is the $\mathbf{crl}_1$(1-corolla). The union of all the green balls($\mathbf{ptl}_2$) is the $\mathbf{crl}_2$(2-corolla) and the union of all the blue balls($\mathbf{ptl}_3$) is the $\mathbf{crl}_2$(3-corolla) as per Def. 13.

Now using Figs. 3.1 and 3.2 in conjunction, we will explain the idea of a $\mathbf{ptlchain}_k$ (Def. 14). It must be noted that, similar to each of the structures explained in this example, each of the structures has the nucleus of the nerve($\mathcal{N}(\check{C}ech_r^s)$) as its generating point. The $\mathbf{ptlchain}_0$ is the nucleus of the nerve which is the intersection of the yellow balls in Fig. 3.1. It is not labeled in this image. It is important to note here that each of the yellow balls is the $\mathbf{ptlchain}_1$, from Def. 14. This is because it contains the nucleus and the $\mathbf{ptl}_1$(the yellow ball). Thus the $\mathbf{ptlchain}_1$ is the same as the $\mathbf{ptl}_1$. There are four possible $\mathbf{ptl}_1$ and similarly four possible $\mathbf{ptlchain}_1$. Now, by adding to this construction a ball from $\mathbf{crl}_2$, that has a non-empty intersection with the $\mathbf{ptl}_1 \in \mathbf{ptlchain}_1$, we can construct a $\mathbf{ptlchain}_2$ as per Def. 14. In a similar fashion we can construct $\mathbf{ptlchain}_3$ by adding to this construction a ball from $\mathbf{crl}_3$, having non-empty intersection with the $\mathbf{ptl}_2 \in \mathbf{ptlchain}_2$. It can be seen that we can have four possible $\mathbf{ptlchain}_3$ as illustrated in Fig. 3.2. ■

Using the above definitions we will now define the notion of an object space in the strong Čech complex.

Definition 15. *Let cx^sA be a strong Čech complex in a finite, bounded region of an Euclidean plane $\mathbb{R}^2$,and let $\hat{k} \in \mathbb{Z}^+$ such that $\mathbf{crl}_{\hat{k}} = \emptyset$. Then, the object space $\mathcal{O}_p^{\check{C}ech}$, for each $p \in \mathbf{max}\mathcal{N}$ is defined as:*

$$\mathcal{O}_p^{\check{C}ech} = \{\bigcup_k \mathbf{crl}_k : \mathbf{crl}_0 = p, k = 0, \cdots, \hat{k} - 1\}$$

We define the notion of a boundary petal **bdypt**.

Definition 16. *Let $\mathscr{O}_p^{\check{C}ech}$ A be an object space over a finite, bounded region of the Euclidean plane, $\mathbb{R}^2$. Then the boundary petal of the object space,* **bdpt**$(\mathscr{O}_p^{\check{C}ech})$ *is defined as:*

$$\boldsymbol{bdypt}(\mathscr{O}_p^{\check{C}ech}) = \{x : \forall k\, x \in \mathbf{crl}_k \text{ and } x \cap \mathbf{crl}_{k+1} = \emptyset\}$$

Now, we introduce a bounding corolla in the object space, denoted as **bdycrl**$(\mathscr{O}_p^{\check{C}ech})$.

Definition 17. *Let $\mathscr{O}_p^{\check{C}ech}$ A be an object space over a finite, bounded region of the Euclidean plane, $\mathbb{R}^2$. Then the bounding corolla of the object space,* **bdycrl**$(\mathscr{O}_p^{\check{C}ech})$ *is defined as:*

$$\boldsymbol{bdycrl}(\mathscr{O}_p^{\check{C}ech}) = \bigcup \boldsymbol{bdypt}(\mathscr{O}_p^{\check{C}ech})$$

Let us define the notion of a maximal petal chains.

Definition 18. *Let cx^sA be a strong Čech complex, and let* **ptlchain**$_k$*A be a petal chain in cxA. Then,* **ptlchain**$_k$*A is called a maximal chain if* **ptlchain**$_k A \cap \mathbf{crl}_{k+1} = \emptyset$*. The k for which a petal chain becomes maximal is called the* **length** *of the chain.*

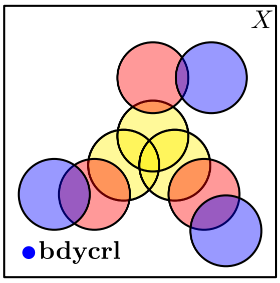
Remark 2. *Each maximal petal chain(***maxptlchain**$_k$ *contains a boundary petal(***bdpt***). This follows directly from Def. 16 and Def. 18.*

Using this notion of a **length** of petal chain, we define the regularity of an object space $\mathscr{O}_p^{\check{C}ech}$.

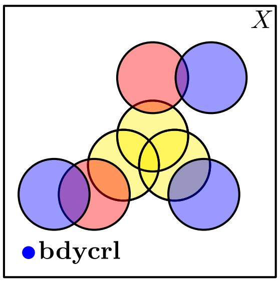
Definition 19. *Let $\mathscr{O}_p^{\check{C}ech}$ be an object space and A be the set of all the maximal petal chains in the object space. $\mathscr{O}_p^{\check{C}ech}$ is called* **regular***, provided the length of all the maximal petal chains in A is the same. The object space is called* **irregular** *otherwise.*

Example 5. *Two types of object spaces are shown in Fig. 4.1 and 4.2. According to Def. 16, in Fig. 4.1 each of the blue balls(in* **crl**$_3$*) is the boundary petal as it has a non-empty intersection with the previous corolla(in* **crl**$_2$*) but has no intersection with the with next corolla(***crl**$_4$*). This is due to the fact that the next corolla,* **crl**$_4$*, is ∅. In Fig. 4.2 each of the blue balls(in* **crl**$_k$*) is the boundary petal as it has an intersection with the previous corolla(***crl**$_{k-1}$*), but has no intersection with the next corolla(***crl**$_{k+1}$*). In contrast to the Fig. 4.1 the Fig. 4.2 has boundary petals in different corolla(***crl**$_k$*). As per Def. 17 the collection of all the blue balls is the bounding corolla(***bdycrl***). From Def. 18 we see that both the figures have three maximal petal chains(***maxptlchain***). In Fig. 4.1 all the maximal chains are of equal length 3, thus from Def. 19, the object space in this figure is regular. In Fig. 4.2 the maximal chains(***maxptlchain***) have different length, two chains having lengths of 3 and one chain of length 2. Thus, according to Def. 19 the object space in the figure is an irregular object.* ■

Let us consider the order of a nerve as a signature for the $\mathscr{O}_p^{\check{C}ech}$. Based on this notion we define the Čech spectrum.



4.1: A regular object space



4.2: An irregular object space

Figure 4. This figure illustrates the bounding petal(**bdypt**) defined in Def. 16 and the bounding corolla (**bdycrl**) of an object space defined in Def. 17. Moreover, all the maximal petal chains($\textbf{maxptlchain}_k$) can be identified using the Def. 18 and the accompanying remark. The regularity of an object can be determined using the Def.19

Definition 20. *Let cx^sA be a Čech complex over a finite, bounded region of $\mathbb{R}^2$, Then the Čech spectrum is defined as:*

$$\mathscr{C}(cx^sA) = \{|N^k| : \forall\ k, N^k = \{\check{C}ech^s_r(cx^sA) : |\check{C}ech^s_r(cx^sA)| = k\}\}.$$

It is important to be noted that we count only the unique nerves, i.e. the nerves that are not completely included in a nerve of higher order.

In this paper we study the **proximity** relations defined over the topological structures defined above. **Proximity** is a measure of nearness between non-empty sets, and a non-empty set endowed with a proximity is known as a **proximity space**. The proximity space is represented as a pair (X, δ), where X is a non-empty set and the δ is an arbitrary proximity relation on the set X. In this paper we will use the concept of spatial Lodato proximity(δ), strong proximity($\overset{\wedge\wedge}{\delta}$) and descriptive proximity(δ_Φ). Let us list the axioms of of each of the proximities. We start with the spatial Lodato proximity(δ).

Definition 21. *Let X be a non-empty set and $A, B, C \subset X$ be the subsets of X. Then the spatial Lodato proximity(δ) is a binary relation on the set X, that satisfies the following axioms:*

(P1) $\emptyset \not\delta\, A,\ \forall A \subset X.$

(P2) $A\,\delta\, B \Leftrightarrow\ B\,\delta\, A.$

(P3) $A \cap B \neq \emptyset \Rightarrow A\delta B.$

(P4) $A\,\delta\,(B \cup C) \Leftrightarrow A\,\delta\, B\, or\, A\,\delta\, C.$

(P5) $A\,\delta B\, and\, \forall\ b \in B\, \{b\}\,\delta\, C\ \Rightarrow A\,\delta\, C.$

Now we move on to the definition of strong proximity($\overset{\wedge\!\!\!\wedge}{\delta}$).

Definition 22. *Let X be a non-empty set and $A, B, C \subset X$ be the subsets of X. Then the strong proximity($\overset{\wedge\!\!\!\wedge}{\delta}$) is a binary relation on the set X, that satisfies the following axioms:*

(snN1) $\emptyset \, \overset{\wedge\!\!\!\wedge}{\not\delta} \, A, \; \forall \, A \subset X,$ *and* $X \, \overset{\wedge\!\!\!\wedge}{\delta} \, A, \; \forall A \subset X.$

(snN2) $A \, \overset{\wedge\!\!\!\wedge}{\delta} \, B \Leftrightarrow B \, \overset{\wedge\!\!\!\wedge}{\delta} \, A.$

(snN3) $A \, \overset{\wedge\!\!\!\wedge}{\delta} \, B \Rightarrow A \cap B \neq \emptyset.$

(snN4) *If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $A \, \overset{\wedge\!\!\!\wedge}{\delta} \, B_{i^*}$ for some $i^* \in I$ such that $int(B_{i^*}) \neq \emptyset$, then $A \, \overset{\wedge\!\!\!\wedge}{\delta} \, (\bigcup_{i \in I} B_i)$.*

(snN5) $intA \bigcap intB \neq \emptyset \Rightarrow A \, \overset{\wedge\!\!\!\wedge}{\delta} \, B.$

(snN6) $x \in int(B) \Rightarrow x \, \overset{\wedge\!\!\!\wedge}{\delta} \, A.$

(snN7) $\{x\} \, \overset{\wedge\!\!\!\wedge}{\delta} \, \{y\} \Leftrightarrow x = y.$

Moving on to the notion of a descriptive proximity(δ_Φ), we first define the notion of a **probe function**. For a set A, the function ϕ maps a feature vector to each of the elements of A. It is defined as $\phi(A) = \{\phi(x) \in \mathbb{R}^n : x \in A\}$. Based on this let us define the concept of a descriptive intersection ($\underset{\Phi}{\bigcap}$). It is defined as $A \underset{\Phi}{\bigcap} B = \{x \in A \bigcup B : \phi(x) \in \phi(A) \, and \, \phi(x) \in \phi(B)\}$. These concepts were introduced and formalized in (Peters, 2007*a*),(Peters, 2007*b*). Let us now formalize the notion of a descriptive proximity(δ_Φ).

Definition 23. *Let X be a non-empty set and $A, B, C \subset X$ be the subsets of the set X. Then the descriptive proximity(δ_Φ) is a binary relation on the set X, that follows the following axioms:*

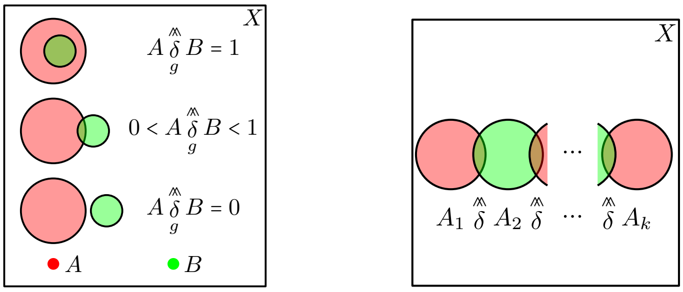
(dP1) $\emptyset \not\delta_\Phi A, \; \forall A \subset X.$

(dP2) $A \, \delta_\Phi \, B \Leftrightarrow B \, \delta_\Phi \, A.$

(dP3) $A \underset{\Phi}{\bigcap} B \neq \emptyset \Rightarrow A \, \delta_\Phi \, B.$

(dP4) $A \delta_\Phi B \, and \, \{b\} \, \delta_\Phi \, C, \; \forall b \in B \Rightarrow A \, \delta_\Phi \, C.$

Let us build upon the notion a strong proximity($\overset{\wedge\!\!\!\wedge}{\delta}$). It is a boolean valued relation i.e. it either tells whether two sets A, B are strongly near($\overset{\wedge\!\!\!\wedge}{\delta}$) or strongly far($\overset{\wedge\!\!\!\wedge}{\not\delta}$). Next, consider an extension of the Smirnov degree-of-closeness measure (Smirnov, 1952). We define a gradation of the strong proximity to make it a continuous valued function. This then gives a notion of the degree of nearness rather than just an indication of being near or not.



5.1: Gradation of a Strong Proximity 5.2: A sequence in a proximal space

Figure 5. This figure illustrates the gradation of proximity relation. It generalizes the usual binary proximity to a continuous valued relation.Moreover, the concept of a sequence in the procimity space is also illustrated.

Definition 24. *Let X be a non-empty set and $A, B \in X$ be two subsets of X, then we can define a graded strong proximity as a function $\underset{g}{\overset{\wedge\!\!\wedge}{\delta}} : X \times X \to [0,1]$.*

$$A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B = \frac{|A \cap B|}{min(|A|, |B|)},$$

where $min(a,b)$ returns the smaller of the two numbers.

Remark 3. *It can be seen from the Def. 24, that if $A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B = 0$ then then this means that $A \cap B = \varnothing$, or $A = \varnothing$ or $B = \varnothing$. Any of these conclusions leads to $A \overset{\wedge\!\!\wedge}{\not\delta} B$. Moreover, if $A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B = 1$ then either $A \subseteq B$ or $B \subseteq A$. Thus the graded strong proximity ranges from $\overset{\wedge\!\!\wedge}{\not\delta}$(for $\underset{g}{\overset{\wedge\!\!\wedge}{\delta}} = 0$) to $\subseteq$(for $\underset{g}{\overset{\wedge\!\!\wedge}{\delta}}$=1).*

Let us now talk about a sequence in a proximity space (X, δ).

Example 6. *The notion of proximity as defined in the Def. 22 is a boolean function. This tells us whether two sets are strongly near($\overset{\wedge\!\!\wedge}{\delta}$) or not($\overset{\wedge\!\!\wedge}{\not\delta}$). In Def. 24, we define a notion of a continuous valued proximity relation. An illustration of the graded strong proximity is presented in the Fig. 5.1. It can be seen that for the case when $A \cap B = \varnothing$, the strong graded proximity is, $A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B = \frac{0}{min(|A|,|B|)} = 0$. The other extreme of the proximity is when $B \subseteq A$, then $A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B = \frac{|B|}{|B|} = 1$. All the other cases are in between the two extremes, $0 < A \underset{g}{\overset{\wedge\!\!\wedge}{\delta}} B < 1$, as $0 < |A \cap B| < |B|$. Thus, the graded strong proximity($\underset{g}{\overset{\wedge\!\!\wedge}{\delta}}$) gives us a more detailed view of the nearness or proximity between two sets.* ■

Definition 25. *Let X be a non-empty set and δ be an arbitrary proximity relation on the set X. Then (X, δ) is a proximity space. Let $A_i \in X$ for $i \in I$, where I is an index set. The a δ-sequence x_n in the proximity space (X, δ) is defined as:*

$$x_n = \{A_i : A_i \delta A_j \ for \ |j - i| \leq 1, \ i, j \in I\}$$

where δ represents the proximity relation.

Let us explain the idea of a sequence in a proximal space.

Example 7. *Let (X, δ) be a proximity space equipped with an arbitrary proximity relation. We define the idea of a sequence which establishes an arrangement of subsets of X. For an illustration of this concept, let us consider Fig. 5.2. It shows a sequence of sets $A_i \subset X$, where $i \in I$, and I is an index set. An important point to note in this definition of a sequence is that a set A_i is proximal only to adjacent sets(A_{i-1}, A_{i+1}) in the sequence and far(not proximal) from all other sets in the sequence. The idea of a δ-sequence can be extended to other proximites such as the $\overset{\wedge\wedge}{\delta}$ and δ_Φ.* ■

Till now, we have been discussing the notion of proximity as a binary mapping on a set X, of the form $\delta : X \times X \to \{0, 1\}$. Now, let us extend this notion to a mapping of n sets, where n is an arbitrary number. This mapping is termed as a proximity of order n, denoted as δ^n. It is non-continuous surjective map of the form $\delta^n : X^n \to \{0, 1\}$. This generalized notion of a proximity is referred to as **hyper-connectedness**. Let us first define the axioms of a **Lodato hyper-connectedness**. We have used the notation $A\delta B$ to denote that A, B are proximal, and $A \not\delta B$ to denote otherwise. Let us introduce a new notation that will come in handy in the case of hyper-connectedness. If $\delta(A, B, C) = 1$, then A, B are hyper-connected, and if $\delta(A, B, C) = 0$ then the opposite is true.

Definition 26. *Let X be a non-empty set and $\{A_i \subset X : i \in I\}$, where I is an index set, and B, C be the non-empty subsets of X. Let $S(D)$ be the set of n-permutations of the elements of the set D, for $2 \leq n \leq |I|$. Then the spatial Lodato hyper-connectedness(δ^k) is a mapping on k subsets of the set X, that satisfies the following axioms for $k \geq 1$:*

(hP1) $\forall A_k \subset X, \delta^k(A_1, \cdots, A_k) = 0, \ if \ any \ A_1, \cdots, A_k = \emptyset.$

(hP2) $\delta^k(A_1, \cdots, A_k) = 1 \Leftrightarrow \delta^k(Y) = 1, \ \forall Y \in S(\{A_1, \cdots, A_k\}).$

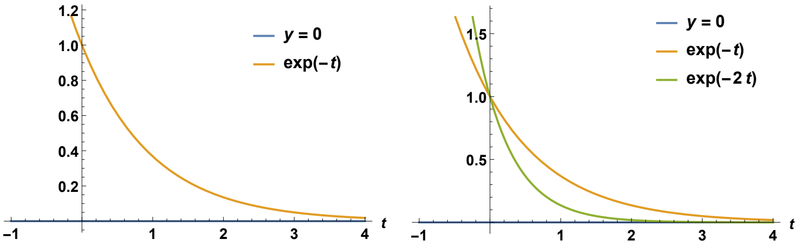
(hP3) $\bigcap_{i=1}^{k} A_i \neq \emptyset \Rightarrow \delta^k(A_1, \cdots, A_k) = 1.$

(hP4) $\delta^k(A_1, \cdots, A_{k-1}, B \cup C) = 1 \Leftrightarrow \delta^k(A_1, \cdots, A_{k-1}, B) = 1 \ or \ \delta^k(A_1, \cdots, A_{k-1}, C) = 1.$

(hP5) $\delta^k(A_1, \cdots, A_{k-1}, B) = 1 \ and \ \forall b \in B, \ \delta^2(\{b\}, C) = 1 \Rightarrow \delta^k(A_1, \cdots, A_{n-1}, C) = 1.$

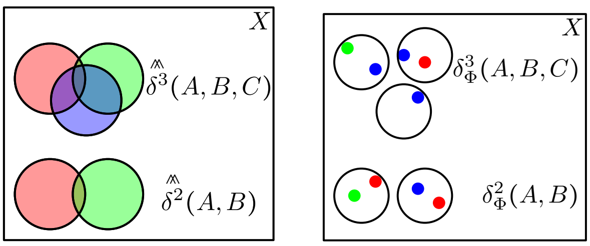
(hP6) $\forall A \subset X, \ \delta^1(A) = 1$, *a constant map.*

Now, we use an example to better understand the spatial Lodato hyper-connectedness denoted by δ^k.



6.1: Lodato Proximity 6.2: Lodato hyper-connectedness

Figure 6. This figure illustrates the spatial Lodato proximity in the form of convergence of exponential to the x-axis. It then generalizes this proximity to a hyper proximity, by considering two exponentials converging to x-axis. Further generalization to n such functions follow naturally.



7.1: Strong hyper-connectedness 7.2: Descriptive hyper-connectedness

Figure 7. The Fig. 7.1 illustrates the extension of the classical binary relation of spatial Lodato proximity to three sets. Such a relation can be generalized to *n* sets. Fig. 7.2 displays the same for the strong proximity.

Example 8. *Let us consider the idea of a spatial Lodato hyper-connectedness(Def. 26) as illustrated in Fig.6.1. This figure uses two sets* $A = \{y(t) = 0 : \forall t \in \mathbb{R}\}$ *and* $B = \{y(t) = e^{-t} : \forall t \in \mathbb{R}\}$*. It can be observed that these two sets are proximal in the sense that as t approaches* ∞*, the function* e^{-t} *approaches* 0*. Thus* $A \cap B \neq \emptyset$ *in the limit sense. This spatial Lodato hyper-proximity,*δ^2*, for* $k = 2$ *is the classical notion of spatial Lodato proximity defined in Def. 21. Let us extend this idea to arbitrary values of k i.e. to the proximity of k non-empty subsets of a set X. For* $k = 1$*, the notion of a proximity is trivial.* $\delta^1(C) = 1\ \forall C \subset X$ *as every set is proximal to itself. It does not make sense to talk about* $k = 0$ *as there is nothing to quantify the nearness of. Now moving on to* $k = 3$ *as shown in Fig. 6.2. The sets used in this illustration are* $A = \{y(t) = 0 : t \in \mathbb{R}\}$*,* $B = \{y(t) = e^{-t} : t \in \mathbb{R}\}$ *and* $C = \{y(t) = e^{-2t} : t \in \mathbb{R}\}$*. It is obvious that all the three sets are proximal in the limit sense as the functions* e^{-t} *and* e^{-2t} *approach* 0 *as t approached* ∞*. Thus* $\bigcap(A, B, C) \neq \emptyset$*, which implies that* $\delta^3(A, B, C) = 1$*. Using a similar approach we can generalize this notion for* $k > 3$*.*

We generalize the notion of strong proximity($\overset{\wedge\!\wedge}{\delta}$) defined in Def. 21 to the notion of strong hyper-connectedness.

Definition 27. *Let X be a non-empty set and* $A_i, B, C \subset X$ *be subsets of X, where* $i \in I$*,I is an index* *set. Let $S(D)$ be the set of n-permutations of set D, where $2 \leq n \leq |I|$.Then the strong hyper-connectedness($\overset{\wedge\!\!\wedge}{\delta}^k$) is a mapping on k subsets of the set X, that satisfies the following axioms for $k \geq 1$:*

(snhN1) $\forall A_k \subset X, \overset{\wedge\!\!\wedge}{\not\delta}^k(A_i, \cdots, A_k) = 0\, if\, any\, A_1, \cdots, A_k = \varnothing\, and\, \overset{\wedge\!\!\wedge}{\delta}^k(X, A_1, \cdots, A_{k-1}) = 1, \forall A_i \subset X.$

(snhN2) $\overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_k) = 1 \Leftrightarrow \overset{\wedge\!\!\wedge}{\delta}^k(Y) = 1,\ \forall Y \in S(\{A_1, \cdots, A_k\}).$

(snhN3) $\overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_k) = 1 \Rightarrow \bigcap_{i=1}^{k} A_i \neq \varnothing.$

(snhN4) *If $\{B_i\}_{i\in I}$ is an arbitrary family of subsets of X and $\overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_{k-1}, B_{i^*}) = 1$ for some $i^* \in I$ such that $int(B_{i*}) \neq \varnothing$, then $\overset{\wedge\!\!\wedge}{\delta}^n(A_1, \cdots, A_{k-1}, (\bigcup_{i\in I} B_i)) = 1$.*

(snhN5) $\bigcap_{i=1}^{k} intA_i \neq \varnothing \Rightarrow \overset{\wedge\!\!\wedge}{\delta}^k(A_1, \cdots, A_k) = 1.$

(snhN6) $x \in \bigcap_{i=1}^{k-1} int(A_i) \Rightarrow \overset{\wedge\!\!\wedge}{\delta}^k(x, A_1, \cdots, A_{k-1}) = 1.$

(snhN7) $\overset{\wedge\!\!\wedge}{\delta}^k(\{x_1\}, \cdots, \{x_k\}) = 1 \Leftrightarrow x_1 = x_2 = \cdots = x_n.$

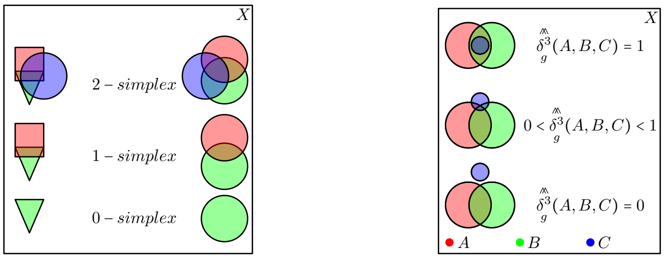
(snhN8) $\forall A \in X, \overset{\wedge\!\!\wedge}{\delta}^1(A) = 1$ *is a constant map.*

To explain the notion of strong hyper-connectedness we present an example.

Example 9. *Let us consider the illustration of strong hyper-connectedness,$\overset{\wedge\!\!\wedge}{\delta}^k$, in the Fig. 7.1. First we consider the case of $k = 2$. The two geometric balls (red and green) have a non-empty intersection and thus $\overset{\wedge\!\!\wedge}{\delta}^2(red\, ball, green\, ball) = 1$. Let us now generalize this notion to arbitrary values of k. We consider the case of $k = 1$, which is a trivial case. It can be seen that each ball is strongly near it self. The case of $k = 0$ does not make any sense as the notion of proximity requires a set or an object for definition. Let us now consider the case of $k = 3$. We can see that three geometric balls(red, blue and green) have a non-empty intersection,$\bigcap(red\, ball, green\, ball, blue\, ball) \neq \varnothing$, and thus are strongly proximal. Thus we can write that $\overset{\wedge\!\!\wedge}{\delta}^3(red\, ball, green\, ball, blue\, ball)$. Using a similar argument we can extend this notion of strong hyper-connectedness($\overset{\wedge\!\!\wedge}{\delta}^3$) to the case of $k > 3$.* ■

Now, we extend the notion of a descriptive proximity to descriptive hyper-connectedness, denoted by δ^k_Φ.

Definition 28. *Let X be a non-empty set and $A_i, B, C \subset X$ be the subsets of the set X, where $i \in I$,I is an index set. Let $S(D)$ be the set of all the n-permutations of set D, where $2 \leq n \leq |I|$. Then the descriptive hyper-connectedness(δ^k_Φ) is a mapping on the set X, that satisfies the following axioms for $k \geq 1$:*



8.1: Hyper-connected complexes

8.2: Graded strong hyper-connectedness

Figure 8. The Fig. 8.1 displays two different hyper-connected complexes defined in a proximity space X. Fig. 8.2 is a graphical visualization of the graded strong proximity for different values in the interval $[0, 1]$.

(dhP1) $\forall A_i \subset X,\ \delta^k_\Phi(A_1, \cdots, A_k) = 0$ *if any of the* $A_1, \cdots, A_k = \emptyset$.

(dhP2) $\delta^k_\Phi(A_1, \cdots, A_k) = 1 \Leftrightarrow \delta^k_\Phi(Y) = 1\ \forall Y \in S(\{A_1, \cdots, A_k\})$.

(dhP3) $\bigcap_\Phi A_i \neq \emptyset \Rightarrow \delta^k_\Phi(A_1, \cdots, A_k) = 1$.

(dhP4) $\delta^k_\Phi(A_1, \cdots, A_{k-1}, B) = 1$ *and* $\forall b \in B,\ \delta^2_\Phi(\{b\}, C) = 1 \Rightarrow \delta^k_\Phi(A_1, \cdots, A_{k-1}, C) = 1$.

(dhP5) $\forall A \subset X, \delta^1_\Phi(A) = 1$ *a constant map.*

To help explain the notion of descriptive hyper-connectedness δ^k_Φ.

Example 10. *We use the illustration in Fig. 7.2 to aid in understanding the concept of descriptive hyper proximity(δ^k_Φ). To describe the notion of descriptive proximity we require a probe function $\phi : A \to \mathbb{R}$. In this example we define the probe function such that it returns the hue value(color) of each element of the set. Based on similarity in the hue domain we define the notion of the descriptive proximity(Def. 23). Let us first discuss the case of $k = 2$. Consider the two sets at the bottom of the figure. We can see that each has two elements. Now we see that both sets have an element with red color. Thus the sets have a non-empty descriptive intersection i.e. $A \underset{\phi}{\cap} B \neq \emptyset$. This leads to $\delta^2_\Phi(A, B) = 1$. Let us now generalize this concept to arbitrary values of k. Now we consider the case of $k = 1$. It is obvious that every set is descriptively proximal to itself. Now we can generalize to the case of $k = 3$. We can see that the three sets on the top each have an element with blue color. Thus the sets have a non-empty intersection, $\bigcap_\phi(A, B, C) \neq \emptyset$, which implies that $\delta^3_\Phi(A, B, C) = 1$. Using a similar argument we can extend this notion to arbitrary values of k.*

Using the notion of hyper-connectedness, let us define the notion of a hyper-connected complex.

Definition 29. *Let S be a discrete set and X be a collection of non-empty subsets of S. X is a hyper-connected complex,$\delta - cx$, if every collection of non-empty subsets $\{A_1, \cdots, A_n\} \in X$, of cardinality $|n|$, is hyper-connected with respect to a proximity δ i.e. $\delta^n(A_1, \cdots, A_n) = 1$. Every subset of size k is the $k - 1$ simplex in the resulting hyper-connected complex,$\delta - cx$.*

To clarify the notion of a hyper-connected complex we present the following example.

Example 11. *Let us consider the illustration of a strong hyper-connected complex as depicted in the Fig. 8.1. It can be seen that the bottom row of the figure represents the* 0*-simplex as per Def. 27, we can see that every set is strongly connected to itself. Thus* $\overset{\wedge\wedge}{\delta}^1(A) = 1 \forall A$.

Let us move on to the middle row of the illustration. It can be seen that the two sets are strongly connected thus $\overset{\wedge\wedge}{\delta}^2(A,B) = 1$ *and the remaining constituent sets* $\{A\}$ *and* $\{B\}$ *are also strong hyper-connected(*$\overset{\wedge\wedge}{\delta}^1(A) = \overset{\wedge\wedge}{\delta}^1(B) = 1$*). Thus, the collection* $\{A,B\}$ *is a* 1*-simplex of the strong hyper-connected complex.*

Let us now move to the top row of the illustration. It can be seen that due to the common intersection the $\overset{\wedge\wedge}{\delta}^3(A,B,C) = 1$ *and all of the collections of its subsets are also strongly connected(*$\overset{\wedge\wedge}{\delta}^2(A,B) = \overset{\wedge\wedge}{\delta}^2(A,C) = \overset{\wedge\wedge}{\delta}^2(B,C) = \overset{\wedge\wedge}{\delta}^1(A) = \overset{\wedge\wedge}{\delta}^1(B) = \overset{\wedge\wedge}{\delta}^1(C) = 1$*).Thus the collection of sets* $\{A,B,C\}$ *forms a* 2*-simplex as per Def. 29.*

It must be noted that this concept can be generalized to spatial Lodato hyper-connectedness and descriptive hyper-connectedness.

Let us now define the graded strong hyper-connectedness, denoted by $\underset{g}{\overset{\wedge\wedge}{\delta}}{}^k$.

Definition 30. *Let* X *be a non-empty set and* $\{A_1, A_2, \cdots, A_n\}$ *be a family of subsets of* X*. Then the graded strong hyper-connectedness is a continuous surjection of the form* $\underset{g}{\overset{\wedge\wedge}{\delta}} : X^n \to [0,1]$ *and is defined as:*

$$\underset{g}{\overset{\wedge\wedge}{\delta}}{}^k = \frac{|\bigcap(A_1, A_2, \cdots, A_n)|}{min(|A_1|, |A_2|, \cdots, |A_n|)},$$

where $min(a_1, a_2, \cdots, a_n)$ *is the smallest of all the arguments.*

Example 12. *To explain the notion of graded strong hyper-connectedness, denoted as* $\underset{g}{\overset{\wedge\wedge}{\delta}}{}^k$*, let us consider the Fig. 8.2. Since we are considering three sets we are going to use* $k = 3$ *in the following discussion. First consider the case illustrated towards the bottom of the figure. It illustrates the case when* $\bigcap(A,B,C) \neq \emptyset$ *or any of the* A, B *or* C *is an empty set. In this case the* $\underset{g}{\overset{\wedge\wedge}{\delta}}{}^3 = 0$ *as per Defn. 30 and the three sets* A, B, C *are not strong hyper-connected,* $\overset{\wedge\wedge}{\delta}^3(A,B,C) = 0$.

Let us move on to the case when there is a partial intersection among the three sets as shown in the illustration in the middle of the figure. We can see that the number of elements in the intersection is smaller than the elements in smallest set i.e. C*. Thus the value of the graded strong hyper-connectedness is between* 0 *and* 1*. The sets* A, B *and* C *are strong hyper-connected(*$\overset{\wedge\wedge}{\delta}^3(A,B,C) = 1$*).*

Let us now consider the third case in which the smallest of the three sets,namely C *is contained in the intersection. This is represented as* $C \subset \bigcap(A,B,C)$*. For this case it can be seen from the*

Def. 30 that $\overset{\wedge\!\!\wedge}{\delta}{}^3_g(A,B,C) = 1$. *We can extend this framework to higher values of k using a similar argument.* ■

Last, but not the least we present the Borsuk's nerve theorem.

Theorem 2. *(Borsuk, 1948) If U is a collection of subsets in a topological space, the nerve complex is homotopy equivalent to the union of the subsets.*

This theorem will be helpful in the study of Čech object spaces($\mathscr{O}_p^{\check{C}ech}$) undertaken in this paper.

3. Main Results

Lemma 3. *Let* $\mathscr{O}_p^{\check{C}ech}$ *be an object space,* $\boldsymbol{crl}_i \in \mathscr{O}_p^{\check{C}ech}$ *be a corolla,* $\boldsymbol{ptlchain}_k \in \mathscr{O}_p^{\check{C}ech}$ *be a petal chain and* $\hat{k} \in \mathbb{Z}^+$ *such that* $\boldsymbol{crl}_{\hat{k}} = \emptyset$. *Then,* $\bigcup_i \boldsymbol{crl}_i = \bigcup \boldsymbol{maxptlchain}_k$, *where* $i = 0, 1, 2, \cdots, \hat{k} - 1$.

Proof. From Def. 13 it is obvious that $\mathbf{crl}_i = \bigcup \mathbf{ptl}_i$, thus i-corolla is the union of all the i-petals in $\mathscr{O}_p^{\check{C}ech}$. Thus for $i = 0, 1, 2, \cdots, \hat{k}-1$, the $\bigcup_i \mathbf{crl}_i$ is equivalent to $\bigcup_i \bigcup \mathbf{ptl}_i$. Thus, $\bigcup_i \mathbf{crl}_i$, $i = 1, 2, \cdots, \hat{k}-1$, is equal to the union of all the possible $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$($i$-petals in the object space).

Let us now look at Def. 14, from which it can be concluded that each k-petal chain in the object space contains one instance of $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$($i$-petal) for each value $i = 0, 1, \cdots k$. Thus if we consider the $\bigcup \mathbf{ptlchain}_k$, for $k = 0, 1, \cdots, \hat{k} - 1$, we get the union of all the $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$($i$-petal), where $i = 0, 1, \cdots, k$. Which means that union of all $\mathbf{ptlchain}_k \in \mathscr{O}_p^{\check{C}ech}$ consists of all the $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$ for all values of $i = 0, 1, \cdots, k$. Thus, from Def. 13 we get $\bigcup \mathbf{ptlchain}_k = \bigcup_i \mathbf{crl}_i$ for $i = 0, 1, \cdots, k$. From Def. 18 it can be seen that any $\mathbf{ptlchain}_k$ with the largest value of k is called the **maxptlchain**. From Def. 14 it can be seen that every member of the $\mathbf{ptlchain}_k$ has to be a member of the $\mathbf{crl}_i$ in the object space, for all values of $i = 0, 1, \cdots, k$. Thus, the maximum possible value of k for any petal chain can be $\hat{k} - 1$, as $\mathbf{crl}_{\hat{k}} = \emptyset$. It is possible for different maximal petal chains in $\mathscr{O}_p^{\check{C}ech}$ to have different values of k depending on the regularity, as per Def. 19. Keeping this in mind, it is obvious that the $\bigcup \mathbf{maxptlchain}$ equals the union of all the $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$ for $i = 0, 1, \cdots, \hat{k} - 1$.

Thus, we can conclude that both $\bigcup_i \mathbf{crl}_i$ for $i = 0, 1, \cdots, \hat{k} - 1$, and $\bigcup \mathbf{maxptlchain}$ are equal to the union of all the $\mathbf{ptl}_i \in \mathscr{O}_p^{\check{C}ech}$. Hence we can conclude that,$\bigcup_i \mathbf{crl}_i = \bigcup \mathbf{maxptlchain}_k$. □

Using this we can formulate an other definition of the object space($\mathscr{O}_p^{breveCech}$), equivalent to the Def. 15.

Theorem 3. *Let* cx^sA *be a strong Čech complex and an arbitrary positive integer* $\hat{k} \in \mathbb{Z}^+$ *such that* $\boldsymbol{crl}_k \in cx^sA = \emptyset$.*The object space, denoted by* $\mathscr{O}_p^{\check{C}ech}$*(Def. 15), can be defined as:*

$$\begin{aligned}\mathscr{O}_p^{\check{C}ech} &= \{\bigcup_k \boldsymbol{crl}_k : \boldsymbol{crl}_0 = p,\ k = 0, \cdots, \hat{k} - 1\} \\ &= \bigcup \boldsymbol{maxptlchain}.\end{aligned}$$

Proof. From Lemma 3 it can be seen that for $i = 0, 1, \cdots, \hat{k} - 1$, $\bigcup_i \mathbf{crl}_i = \bigcup \mathbf{maxptlchain}_k$. Using this conclusion and Def. 15 it is evident that $\mathscr{O}_p^{\check{C}ech} = \bigcup \mathbf{maxptlchain}$. □

Using this new definition of an object space($\mathscr{O}_p^{\check{C}ech}$), we can comment on the homotopy type of $\mathscr{O}_p^{\check{C}ech}$.

Theorem 4. *The object space $\mathscr{O}_p^{\check{C}ech}$ has the same homotopy type as the union of all the maximal petal chains(**maxptlchain**).*

Proof. From Def. 3 the object space is the union of all the maximal petal chains i.e. $\bigcup \mathbf{maxptlchain}$. From the Def. 14 we can conclude that every $\mathbf{ptlchain}_k \in \mathscr{O}_p^{\check{C}ech}$ contains the nucleus p which is $\mathscr{N}(\check{C}ech_r^s(A))$. Thus, we can conclude that $\mathscr{O}_p^{\check{C}ech}$ is a nerve as it is union of sets $\bigcup \mathbf{maxptlchain}$, and the sets have a common intersection $\bigcap \mathbf{maxptlchain} = p$. From Thm. 2, it follows that the homotopy type of the $\mathscr{O}_p^{\check{C}ech}$ is the same as the union of all its **maxptlchain**(maximal petal chains). □

Let us formulate some results for hyper-connectedness relations.

Lemma 4. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of non-empty subsets of a non-empty set X. If $\delta^n(A_1, A_2, \cdots, A_n) = 1$, then all the possible non-empty sub-collections are also spatial Lodato hyper-connected. This is represented as $\delta^{|T|}(T) = 1$, where T is a sub-collection of the set C.*

Proof. From axiom **(hP2)** of Def. 26, it can be seen that given $\delta^n(A_1, A_2, \cdots, A_n) = 1$(i.e. is Lodato hyper-connected), then all the k-permutations of the elements in the set $\{A_1, A_2, \cdots, A_n\}$ are also Lodato hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied, provided we introduce an integer $\acute{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \acute{n} \leq n$. Thus, the axiom is satisfied for all such $\acute{n}$. We have shown that if $\delta^n(A_1, A_2, \cdots, A_n) = 1$, then there exists $\acute{n} \in \mathbb{Z}^+$ *and* $2 \leq \acute{n} \leq n$, such that $\delta^{\acute{n}}(\acute{Y} \in \acute{S}(C)) = 1$. $\acute{S}(C)$ is the set of all the permutations of elements of set C, taken $\acute{n}$ at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom **(hP6)** of Def. 26, stating that for all $A \in C$, $\delta^1(A) = 1$. Hence, if $\delta^n(A_1, A_2, \cdots, A_n) = 1$, then all non-empty sub-collections($\tilde{C}$) of $C = \{A_1, A_2, \cdots, A_n\}$ also satisfy $\delta^{|\tilde{C}|}(\tilde{C}) = 1$. □

Using the lemma we have just formulated, consider next the following result for spatial Lodato hyper-connected complexes..

Theorem 5. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of subsets of a non-empty set X, and $\delta^n(C) = 1$. Then C is a spatial Lodato hyper-connected complex($\delta - cx$).*

Proof. From Lemma 4 it follows, that if a collection of sets $C \subseteq X$ satisfyies $\delta^n(C) = 1$, then $\forall T, \delta^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C. Thus all possible sub-collections of the set C are also Lodato hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a spatial Lodato hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k - 1$ simplex in the resulting spatial Lodato hyper-connected complex($\delta - cx$). □

Lemma 5. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of non-empty subsets of a non-empty set X. If $\overset{\wedge\!\!\!\wedge}{\delta}^n(A_1, A_2, \cdots, A_n) = 1$, then all the possible sub-collections are also strong hyper-connected. This is represented as $\overset{\wedge\!\!\!\wedge}{\delta}^{|T|}(T) = 1$, where T is a sub-collection of the set C.*

Proof. From axiom **(snhN2)** of Def. 27, it can be seen that given $\overset{\wedge\!\!\!\wedge}{\delta}^n(A_1, A_2, \cdots, A_n) = 1$(i.e. is strongly hyper-connected), then all the k-permutations of the elements in the set $\{A_1, A_2, \cdots, A_n\}$ are also strongly hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied, provided we introduce an integer $\acute{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \acute{n} \leq n$. Thus, the axiom is satisfied for all such $\acute{n}$. We have shown that if $\overset{\wedge\!\!\!\wedge}{\delta}^n(A_1, A_2, \cdots, A_n) = 1$, then there exists $\acute{n} \in \mathbb{Z}^+$ *and* $2 \leq \acute{n} \leq n$, such that $\overset{\wedge\!\!\!\wedge}{\delta}^{\acute{n}}(\acute{Y} \in \acute{S}(C)) = 1$. $\acute{S}(C)$ is the set of all the permutations of elements of set C, taken $\acute{n}$ at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom **(snhN8)** of Def. 22, such that for all $A \in C$, $\overset{\wedge\!\!\!\wedge}{\delta}^1(A) = 1$. Hence, if $\overset{\wedge\!\!\!\wedge}{\delta}^n(A_1, A_2, \cdots, A_n) = 1$, then all non-empty sub-collections($\tilde{C}$) of $C = \{A_1, A_2, \cdots, A_n\}$ also satisfy $\overset{\wedge\!\!\!\wedge}{\delta}^{|\tilde{C}|}(\tilde{C}) = 1$. □

Example 13. *To explain the Lemma 5, let us consider the Fig.7.1. There are two cases of strong hyper-connectedness shown here, namely the $\overset{\wedge\!\!\!\wedge}{\delta}^2(A, B) = 1$ and $\overset{\wedge\!\!\!\wedge}{\delta}^3(A, B, C) = 1$.*

*Let us first look at the case of $\overset{\wedge\!\!\!\wedge}{\delta}^2(A, B) = 1$. Here the set $C = \{A, B\}$ and all the possible sub-collections are $\{A\}$ and $\{B\}$. From axiom **(snhN8)** of Def. 27 it can be seen that $\overset{\wedge\!\!\!\wedge}{\delta}^1(A) = 1$ and $\overset{\wedge\!\!\!\wedge}{\delta}^1 B = 1$. Thus, Lemma 5 is satisfied.*

*Now moving on to the case of $\overset{\wedge\!\!\!\wedge}{\delta}^3(A, B, C) = 1$. Here the set $C = \{A, B, C\}$ and all possible sub-collections can be listed as $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{A, C\}, \{A, B, C\}\}$. We can see that from axiom **(snhN8)** of Def. 27, that $\overset{\wedge\!\!\!\wedge}{\delta}^1(A) = \overset{\wedge\!\!\!\wedge}{\delta}^1(B) = \overset{\wedge\!\!\!\wedge}{\delta}^1(C) = 1$. Moreover, from the figure it can be established that as all the three disks have interior points in common. This leads to the fact that any two of the disks, also have interior points in common. Hence, from axiom **(snhN5)** of Def. 27 $\overset{\wedge\!\!\!\wedge}{\delta}^2(A, B) = \overset{\wedge\!\!\!\wedge}{\delta}^2(B, C) = \overset{\wedge\!\!\!\wedge}{\delta}^2(A, C) = 1$. Hence, Lemma 5 is satisfied.*

The same argument can be extended to higher values of k. ■

From this lemma we obtain the following result.

Theorem 6. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of subsets of a non-empty set X, and $\overset{\wedge\!\!\!\wedge}{\delta}^n(C) = 1$, then C is a strong hyper-connected complex($\overset{\wedge\!\!\!\wedge}{\delta}-cx$).*

Proof. From Lemma 5 it follows, that if the for a collection of sets $C \subseteq X$ satisfying $\overset{\wedge\!\!\!\wedge}{\delta}^n(C) = 1$, then $\forall T$, $\overset{\wedge\!\!\!\wedge}{\delta}^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C. Thus all possible sub-collections of the set C are also strong hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a strong hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k - 1$ simplex in the resulting strong hyper-connected complex($\overset{\wedge\!\!\!\wedge}{\delta}-cx$). □

Lemma 6. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of non-empty subsets of a non-empty set X. If $\delta_\Phi^n(A_1, A_2, \cdots, A_n) = 1$, then all the possible sub-collections are also descriptivly hyper-connected. This is represented as $\forall\, T$, $\delta_\Phi^{|T|}(T) = 1$, where T is a sub-collection of the set C.*

Proof. From axiom **(dhP2)** of Def. 28, it can be seen that given $\delta_\Phi^n(A_1, A_2, \cdots, A_n) = 1$(i.e. is Lodato hyper-connected), then all the k-permutations of the elements in the set $\{A_1, A_2, \cdots, A_n\}$ are also Lodato hyper-connected, for $2 \leq k \leq n$. One can see that this axiom is satisfied if we introduce an integer $\acute{n} \in \mathbb{Z}^+$, such that $2 \leq k \leq \acute{n} \leq n$. Thus, the axiom is satisfied for all such $\acute{n}$. We have shown that if $\delta_\Phi^n(A_1, A_2, \cdots, A_n) = 1$, then there exists $\acute{n} \in \mathbb{Z}^+$ *and* $2 \leq \acute{n} \leq n$, such that $\delta_\Phi^{\acute{n}}(\acute{Y} \in \acute{S}(C)) = 1$. $\acute{S}(C)$ is the set of all the permutations of elements of set C, taken $\acute{n}$ at a time. Now, we need to prove that this equation is also satisfied for subsets of C of size 1. This follows directly from the axiom **(dhP6)** of Def. 28, that for all $A \in C$, $\delta_\Phi^1(A) = 1$. Hence, if $\delta_\Phi^n(A_1, A_2, \cdots, A_n) = 1$, then all non-empty sub-collections($\tilde{C}$) of $C = \{A_1, A_2, \cdots, A_n\}$ also satisfy $\delta_\Phi^{|\tilde{C}|}(\tilde{C}) = 1$. □

Example 14. *To explain the Lemma 6, let us consider the Fig.7.2. There are two cases of descriptive hyper-connectedness shown here, namely the $\delta_\Phi^2(A, B) = 1$ and $\delta_\Phi^3(A, B, C) = 1$.*

*Let us first look at the case of $\delta_\Phi^2(A, B) = 1$. Here the set $C = \{A, B\}$ and all the possible sub-collections are $\{A\}$ and $\{B\}$. From axiom **(dhP5)** of Def. 28, it can be seen that $\delta_\Phi^1(A) = 1$ and $\delta_\Phi^1 B = 1$. Thus, Lemma 5 is satisfied.*

*Now moving on to the case of $\delta_\Phi^3(A, B, C) = 1$. Here the set $C = \{A, B, C\}$ and all possible sub-collections can be listed as $\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{B, C\}, \{A, C\}, \{A, B, C\}\}$. We can see from axiom **(dhP5)** of Def. 28, that $\delta_\Phi^1(A) = \delta_\Phi^1(B) = \delta_\Phi^1(C) = 1$. Moreover, from the figure it can be established that as all the three sets have constituent elements of the same color(blue). This leads to the fact that any two of the sets, will contain constituent elements of same color(blue). Thus, from axiom **(dhP3)** of Def. 28, $\delta_\Phi^2(A, B) = \delta_\Phi^2(B, C) = \delta_\Phi^2(A, C) = 1$. Hence, Lemma 6 is satisfied.*

The same argument can be extended to higher values of k. ■

We now give a result for descriptive hyper-connected complexes..

Theorem 7. *Let $C = \{A_1, A_2, \cdots, A_n\}$ be a collection of subsets of a non-empty set X, and $\delta_\Phi^n(C) = 1$. Then C is a descriptive hyper-connected complex($\delta_\Phi - cx$).*

Proof. From Lemma 6, it follows, that if the for a collection of sets $C \subseteq X$ satisfying $\delta_\Phi^n(C) = 1$, then $\forall T$, $\delta_\Phi^{|T|}(T) = 1$, where T is a non-empty subcollection of the set C. Thus all possible sub-collections of the set C are also descriptivly hyper-connected. From this conclusion and the Def. 29, it directly follows that C is a descriptivly hyper-connected complex. Every subset $S \subseteq X$ of size k forms the $k-1$ simplex in the resulting descriptvly hyper-connected complex($\delta_\Phi - cx$). □

Theorem 8. *Let $A_1, A_2, \cdots, A_n$ be a collection of subsets of a non-empty set X then:*

1^o : $\overset{\wedge\!\!\wedge\!\!\wedge}{\delta^n}(A_1, A_2, \cdots, A_n) \Rightarrow \delta^n(A_1, A_2, \cdots, A_n)$

2^o : $\overset{\wedge\!\!\wedge\!\!\wedge}{\delta^n}(A_1, A_2, \cdots, A_n) \Rightarrow \delta_\Phi^n(A_1, A_2, \cdots, A_n)$

Proof. 1^o : It is obvious from axiom **(snhN3)** of Def. 27, that if $\overset{\wedge\!\!\wedge\!\!\wedge}{\delta^n}(A_1, A_2, \cdots, A_n) = 1$, then $\bigcap_{i=1}^{n} A_i \neq \emptyset$. Using this conclusion and axiom **(hP3)** of Def. 26, from $\bigcap_{i=1}^{n} A_i \neq \emptyset$, we can conclude that $\delta^n(A_1, A_2, \cdots, A_n)$.

$2^o:$ From axiom **(snhN3)** of Def. 27, that if $\hat{\hat{\delta}}^n(A_1, A_2, \cdots, A_n) = 1$, then $\bigcap\limits_{i=1}^{n} A_i \neq \emptyset$. Considering $p \in \bigcap\limits_{i=1}^{n} A_i$, we know that $p \in A_i$ for $i = 1, \cdots, n$. If we define a probe function $\phi : X \to \mathbb{R}$, then it is obvious that $\phi(p) \in \phi(A_i)$ for $i = 1, \cdots, n$. Thus $\bigcap\limits_{\Phi} A_i \neq \emptyset$. From axiom **(dhP3)** of Def. 28 it can be concluded that $\delta^n_{\Phi}(A_1, A_2, \cdots, A_n) = 1$.

□

Example 15. *Let us consider the Fig. 7.1, to explain the Thm. 8. It illustrates two cases, namely* $\hat{\hat{\delta}}^2(A, B) = 1$ *and* $\hat{\hat{\delta}}^3(A, B, C) = 1$.

It can be seen that if $\hat{\hat{\delta}}^2(A, B) = 1$, *both the sets* A, B *have interior points in common. Thus, from axiom* ***(hP3)*** *of Def. 26, that* $\delta^2(A, B) = 1$. *The same is true for* $\hat{\hat{\delta}}^3(A, B, C) = 1$. *All the three sets have interior points in common which from axiom* ***(hP3)*** *of Def. 26, leads to* $\delta^3(A, B, C)$.

It can be seen that if $\hat{\hat{\delta}}^2(A, B) = 1$, *both the sets* A, B *have interior points in common. If we consider a probe function* ϕ, *which maps the elements of the sets to a description in* $\mathbb{R}$. *It can be seen that for a point* $p \in A \cap B$, $\phi(p) \in \phi(A) \cap \phi(B)$. *Which means that* $A \bigcap\limits_{\Phi} B \neq \emptyset$.*Thus, from axiom* ***(dhP3)*** *of Def. 28, it can be concluded that* $\delta^2_{\Phi}(A, B) = 1$. *The same is true for* $\hat{\hat{\delta}}^3(A, B, C) = 1$. *All the three sets have interior points in common. Thus for a probe function* ϕ, *if there is a point* $p \in \bigcap(A, B, C)$, *then* $\phi(p) \in \bigcap(A, B, C)$. *Thus* $\bigcap\limits_{\Phi}(A, B, C) \neq \emptyset$. *Thus, from axiom* ***(dhP3)*** *of Def. 28, it can be concluded that* $\delta^3_{\Phi}(A, B, C) = 1$.

This argument can be generalized for higher values of k.

Theorem 9. *Let* $C = \{A_1, A_2, \cdots, A_n\}$ *be a collection of subsets of a non-empty set* X, *and* C *is a strong hyper-connected complex i.e.* $\hat{\hat{\delta}}-cx$. *Then,*

$1^o:$ *C is a* $\hat{\hat{\delta}}-cx \Rightarrow$ *C is a* $\delta - cx$

$2^o:$ *C is a* $\hat{\hat{\delta}}-cx \Rightarrow$ *C is a* $\delta_{\Phi} - cx$

Proof. $1^o:$ From Thm. 6 it can be seen that if C is a strong hyper-complex ($sn - cx$), then $\hat{\hat{\delta}}^n(C) = 1$. From Thm. 8 it can be concluded that as $\hat{\hat{\delta}}^n(C) = 1$, then $\delta^n(C) = 1$. From this conclusion and the Thm. 5 it can be concluded that as $\delta^n(C) = 1$, C is a Lodato hyper-complex denoted $\delta - cx$.

$2^o:$ From Thm. 6 it can be seen that if C is a strong hyper-complex ($sn - cx$), then $\hat{\hat{\delta}}^n(C) = 1$. From Thm. 8 it can be concluded that as $\hat{\hat{\delta}}^n(C) = 1$, then $\delta^n_{\Phi}(C) = 1$. From this conclusion and the Thm. 7 it can be concluded that as $\delta^n_{\Phi}(C) = 1$, C is a descriptive hyper-complex denoted $\delta_{\Phi} - cx$.

□

Theorem 10. *A Čech complex is:*

$1^o:$ *spatial Lodato hyper-connected complex,* $\delta - cx$

$2^o:$ *descriptive hyper-connected complex,* $\delta_{\Phi} - cx$

Proof. 1^o: From Def. 3, we come to know that Čech complex is a collection of Čech nerves represented as $\check{C}ech_r(K), K \in \mathbb{R}^2$. Form Def. 2, it can be concluded that $\check{C}ech_r(K)$ is a collection of closed geometric balls,$B_r(x \in K)$, with a non-empty intersection. The $\check{C}ech_r K$ of order k forms the $k-1$ simplex in the Čech complex, as per Def. 3. From the axiom **(dP3)** of Def. 26,it can be concluded that every $k-1$ simplex(or the $\check{C}ech_r(K)$ of order k) is a spatial Lodato hyper-connected collection of sets, i.e. $\delta^{|\check{C}ech_r(K)|}(\check{C}ech_r(K)) = 1$. Using this conclusion and Thm. 5, we can conclude that each $\check{C}ech_r(K)$ is a $\delta - cx$, i.e. a spatial Lodato hyper-connected complex. Similar to the simplicial complex, the union of spatial Lodato hyper-connected complexes is also a spatial Lodato hyper-connected complex.

2^o: From Def. 3, we come to know that Čech complex is a collection of Čech nerves($\check{C}ech_r(K), K \in \mathbb{R}^2$). Form Def. 2, it can be concluded that $\check{C}ech_r(K)$ is a collection of closed geometric balls,$B_r(x \in K)$, with a non-empty intersection. The $\check{C}ech_r K$ of order k forms the $k-1$ simplex in the Čech complex, as per Def. 3. Consider a probe function $\phi : K \to \mathbb{R}$. From the definition of intersection, it can be concluded that a point $p \in \bigcap B_r(x \in K)$ exists in each of the individual $B_r(x \in K)$. Now, when we map each ball to a feature space using the probe function ϕ, it is evident that $\phi(p) \in \bigcap \phi(B_r(x \in K))$. From this it can be concluded that if $\bigcap B_r(x \in K) \neq \emptyset$, then $\bigcap_{\Phi} B_r(x \in K) \neq \emptyset$. Using this result and from the axiom **(dhP3)** of Def. 28, it can be concluded that every $k-1$ simplex(or the $\check{C}ech_r(K)$ of order k) is a descriptively hyper-connected collection of sets, i.e. $\delta_{\Phi}^{|\check{C}ech_r(K)|}(\check{C}ech_r(K)) = 1$. Using this conclusion and Thm. 5, we can conclude that each $\check{C}ech_r(K)$ is a $\delta_{\Phi} - cx$, i.e. a descriptively hyper-connected complex. Similar to the simplicial complex, the union of descriptively hyper-connected complexes is also a descriptively hyper-connected complex.

□

Remark 4. *The Čech complex is not guaranteed to be a strong hyper-connected complex, $\overset{\wedge\!\!\wedge}{\delta} - cx$. This is evident from the definition Def. 3, as it requires the intersection of closed geometric balls,$B_r(x)$, to be non-empty. Since the balls are closed it is possible for the intersection to be points lying on the boundary of the balls. This leads to the spatial Lodato proximity(δ), spatial Lodato hyper-connectedness(δ^n), descriptive proximity(δ_{Φ}) and descriptive hyper-connectedness (δ_{Φ}^n). However, this does not allow the notion of a strong proximity ($\overset{\wedge\!\!\wedge}{\delta}$) and as a consequence the notion of strong hyper-connectedness($\overset{\wedge\!\!\wedge}{\delta}{}^n$). For these two notions the intersection must comprise of the interior points only.*

Based on the above observation we restrict the notion of a Čech complex to the notion of a strong Čech complex as defined in Def. 5. Now, we present the following results for the strong Čech complex.

Lemma 7. *A strong Čech complex is a strong hyper-connected complex, $\overset{\wedge\!\!\wedge}{\delta} - cx$.*

Proof. It is obvious from Def.5, that a strong Čech complex is a collection of strong Čech nerves, which are represented as $\check{C}ech_r^s(K)$. From the Def.4, it is evident that the strong Čech nerve is a collection of closed geometric balls, $B_r(x \in K)$, such that their interiors have a non-empty intersection. This can be written as $\check{C}ech_r^s(K) = \bigcap \mathrm{int}(B_r(x \in K)) \neq \emptyset$. From the axiom **(snhN5)**, it can be concluded that, $\delta^{|\check{C}ech_r^s(K)|}_{\mathrel{\stackrel{\wedge\wedge}{}}}(\check{C}ech_r^s(K)) = 1$. Using this conclusion and from Thm. 6, it can be concluded that $\check{C}ech_r^s(K)$ is a strong hyper-connected complex, $\overset{\wedge\wedge}{\delta}-cx$. We have proved that, each strong Čech nerve, $\check{C}ech_r^s(K)$ is a $sn-cx$.The union of simplicial complexes is also a simplicial complex. From the Def. 29 that the hyper-connected complex is a case of a simplicial complex (Def. 1), where the simplices are strong hyper-connected collections of sets. Thus, it follows directly that the union of $\overset{\wedge\wedge}{\delta}-cx$ is also a $\overset{\wedge\wedge}{\delta}-cx$. Thus, the collections of strong Čech nerves, or a Čech complex, is a strong hyper-connected complex, $\overset{\wedge\wedge}{\delta}-cx$. ∎

Using this lemma, let us formulate the following hyper-connectedness relations for the strong Čech complex.

Theorem 11. *A strong Čech complex, $cx^sK$is:*
1^o : *spatial Lodato hyper-connected complex, $\delta-cx$*
2^o : *descriptive hyper-connected complex, $\delta_\Phi-cx$*

Proof. 1^o : From Lemma 7, it can be seen that a strong Čech complex, cx^sK is a strong hyper-connected complex, $\overset{\wedge\wedge}{\delta}-cx$. From Thm. 9 it can be concluded that cx^sK is a spatial Lodato hyper-connected complex, $\delta-cx$.
2^o : From Lemma 7, it can be seen that a strong Čech complex, cx^sK is a strong hyper-connected complex, $\overset{\wedge\wedge}{\delta}-cx$. From Thm. 9 it can be concluded that cx^sK is a descriptive hyper-connected complex, $\delta_\Phi-cx$.

∎

Now, we formulate some importatnt results regarding equipping an object space with proximity relations.

Lemma 8. *Let $(\mathscr{O}_p^{\check{C}ech},\{\delta,\overset{\wedge\wedge}{\delta},\delta_\Phi\})$ be a proximal relator space, and $\mathbf{crl}_k\in\mathscr{O}_p^{\check{C}ech}$, where $k\in\mathbb{Z}^+$. Assuming $\mathbf{crl}_a\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_b$, then*
1^o : $\mathbf{crl}_a\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_b \Rightarrow \mathbf{crl}_a\,\delta\,\mathbf{crl}_b$
2^o : $\mathbf{crl}_a\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_b \Rightarrow \mathbf{crl}_a\,\delta_\Phi\,\mathbf{crl}_b$

Proof. 1^o : From axiom **(snN3)** of Def. 22, it can be concluded that as $\mathbf{crl}_a\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_b$, $\mathbf{crl}_a\cap\mathbf{crl}_b\neq\emptyset$. Using the axiom **(P3)** of Def. 21, we can conclude that as $\mathbf{crl}_a\cap\mathbf{crl}_b\neq\emptyset$, then $\mathbf{crl}_a\delta\mathbf{crl}_b$.
2^o : From axiom **(snN3)** of Def. 22, it can be concluded that as $\mathbf{crl}_a\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_b$, $\mathbf{crl}_a\cap\mathbf{crl}_b\neq\emptyset$. Let us consider a point $p\in\mathbf{crl}_a\cap\mathbf{crl}_b$ and a probe function $\phi:\mathscr{O}_p^{\check{C}ech}\to\mathbb{R}$. Then it can be seen that $\phi(p)\in\phi(\mathbf{crl}_a)\cap\phi(\mathbf{crl}_b)$, which leads to the fact that $\mathbf{crl}_a\underset{\Phi}{\cap}\mathbf{crl}_b\neq\emptyset$. Using the axiom **(dP3)** of Def. 23, we can conclude that as $\mathbf{crl}_a\underset{\Phi}{\cap}\mathbf{crl}_b\neq\emptyset$, then $\mathbf{crl}_a\delta_\Phi\mathbf{crl}_b$.

∎

Theorem 12. *Let $\mathscr{O}_p^{\check{C}ech}$ be an object space and $\mathbf{crl}_k\in\mathscr{O}_p^{\check{C}ech}$, where $k\in\mathbb{Z}^+$. Let $\hat{k}$ be the value of k such that $\mathbf{crl}_{\hat{k}}=\emptyset$. Then, for $0\leq j<\hat{k}-1$,*
1^o : $\mathbf{crl}_{j+1}\ \overset{\wedge\wedge}{\delta}\ \mathbf{crl}_j$

2^o : $\boldsymbol{crl}_j \overset{\wedge\!\!\wedge}{\delta} \boldsymbol{crl}_{j-1}$

Proof. 1^o : From Def. 13 it can be concluded that $\mathbf{crl}_{j+1} = \bigcup \mathbf{ptl}_{j+1}$ and $\mathbf{crl}_j = \bigcup \mathbf{ptl}_j$. From the Def. 12 it can be seen that a $\mathbf{ptl}_{j+1}$ is a closed geometric ball $B_r(x) \in (cxK \setminus \bigcup \mathbf{ptl}_j)$, such that $B_r(x) \cap \bigcup \mathbf{ptl}_j \neq \emptyset$. Thus each $\mathbf{ptl}_{j+1}$ has a non-empty intersection with $\bigcup \mathbf{ptl}_j = \mathbf{crl}_j$(Def. 13). Since $\mathbf{crl}_{j+1} = \bigcup \mathbf{ptl}_{j+1}$(Def. 13) and each $\mathbf{ptl}_{j+1} \cap \mathbf{crl}_j \neq \emptyset$, we can conclude that $\mathbf{crl}_{j+1} \cap \mathbf{crl}_j \neq \emptyset$. Here, our choice to use the strong Čech complex(cx^s) rather than the Čech complex(cx) as the basis of topological approximation of the underlying space,comes in handy. From Def. 15 it can be seen that we consider the object space($\mathscr{O}_p^{\check{C}ech}$) to be defined over a strong Čech complex(cx^s). Moreover, we can see from the Def. 12 that the petals are the closed geometric balls in the cx^s, and thus the resulting corolla are a collection of these balls. Moreover, each of the intersections in the cx^s(strong Čech complex) is of the form $\bigcap \mathrm{int}(B_r(x))$, as per Def. 5. Based on this we can conclude that all the intersections in the above argument are also on the interiors of the closed geometric balls. Thus, we can conclude that as the object space is formulated on the cx^s(strong Čech complex), $\mathrm{int}(\mathbf{crl}_{j+1}) \cap \mathrm{int}(\mathbf{crl}_j) \neq \emptyset$. Thus from axiom **(snN5)** of Def. 22, we can conclude that $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$.

2^o : The argument for the previous case of $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$ extends directly to this case of $\mathbf{crl}_j \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_{j-1}$, just by considering a dummy variable $\hat{j} = j + 1$. Thus, the case of the $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$ for $0 \leq j < \hat{k} - 1$ becomes $\mathbf{crl}_{\hat{j}} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_{\hat{j}-1}$ for $1 \leq \hat{j} < \hat{k}$. This change in the inequality ensures that we only consider the k-corollas for $0 \leq k < \hat{k}$, thus covering a complete range of corollas.

□

Based on this theorem we formulate the follwing important result for the adjacent corollas in an an object space.

Theorem 13. *Let* $(\mathscr{O}_p^{\check{C}ech}, \{\delta, \overset{\wedge\!\!\wedge}{\delta}, \delta_\Phi\})$ *be a proximal relator space, where* $\boldsymbol{crl}_k \in \mathscr{O}_p^{\check{C}ech}$ *and* $k \in \mathbb{Z}^+$. *Let* $\hat{k}$ *be the value of* k *such that* $\boldsymbol{crl}_{\hat{k}} = \emptyset$. *Then, for* $0 < j < \hat{k}$*,*

1^o : $\boldsymbol{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \boldsymbol{crl}_j \Rightarrow \boldsymbol{crl}_{j+1}\, \delta\, \boldsymbol{crl}_j$

2^o : $\boldsymbol{crl}_j \overset{\wedge\!\!\wedge}{\delta} \boldsymbol{crl}_{j-1} \Rightarrow \boldsymbol{crl}_j\, \delta\, \boldsymbol{crl}_{j-1}$

3^o : $\boldsymbol{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \boldsymbol{crl}_j \Rightarrow \boldsymbol{crl}_{j+1}\, \delta_\Phi\, \boldsymbol{crl}_j$

4^o : $\boldsymbol{crl}_j \overset{\wedge\!\!\wedge}{\delta} \boldsymbol{crl}_{j-1} \Rightarrow \boldsymbol{crl}_j\, \delta_\Phi\, \boldsymbol{crl}_{j-1}$

Proof. 1^o : We can conclude from Thm. 12, that $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$, and then from Lemma 8 we can conclude that $\mathbf{crl}_{j+1} \delta \mathbf{crl}_j$.

2^o : We can conclude from Thm. 12, that $\mathbf{crl}_j \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_{j-1}$, and then from Lemma 8 we can conclude that $\mathbf{crl}_j \delta \mathbf{crl}_{j-1}$.

3^o : We can conclude from Thm. 12, that $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$, and then from Lemma 8 we can conclude that $\mathbf{crl}_{j+1} \delta_\Phi \mathbf{crl}_j$.

4^o : We can conclude from Thm. 12, that $\mathbf{crl}_j \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_{j-1}$, and then from Lemma 8 we can conclude that $\mathbf{crl}_j \delta_\Phi \mathbf{crl}_{j-1}$.

□

Let us move on to discuss some important results regarding sequences in a proximity space (X, δ).

Lemma 9. *Let $(X, \{\delta, \overset{\wedge\wedge}{\delta}, \delta_\Phi\})$ be a proximal relator space, and $A_1, A_2, \cdots, A_n$ be subsets of X. Then, if the collection of sets $C = \{A_1, A_2, \cdots, A_n\}$ is $\overset{\wedge\wedge}{\delta}$-sequence:*
1^o : *C is a $\overset{\wedge\wedge}{\delta}$-sequence $\Rightarrow$ C is a δ-sequence*
2^o : *C is a $\overset{\wedge\wedge}{\delta}$-sequence $\Rightarrow$ C is a δ_Φ-sequence*

Proof. 1^o : If C is a $\overset{\wedge\wedge}{\delta}$-sequence, then from Def. 25 it can be concluded that for all $i = 1, \cdots, n$, $A_i \overset{\wedge\wedge}{\delta} A_j$ for $|j - 1| \leq 1$. Using this conclusion and axiom **(snN3)** of Def. 22, we can establish that $A_i \cap A_j \neq \emptyset$ for $|j - 1| \leq 1$. Using axiom **(P3)** of Def. 21, it can be concluded that $A_i \delta A_j$ for $|j - 1| \leq 1$. Thus from Def. 25, we can conclude that C is a δ-sequence.

2^o : If C is a $\overset{\wedge\wedge}{\delta}$-sequence, then from Def. 25 it can be concluded that for all $i = 1, \cdots, n$, $A_i \overset{\wedge\wedge}{\delta} A_j$ for $|j - 1| \leq 1$. Using this conclusion and axiom **(snN3)** of Def. 22, we can establish that $A_i \cap A_j \neq \emptyset$ for $|j - 1| \leq 1$. Now let us consider a point $p \in A_i \cap A_j$ for $|j - i| \leq 1$ and a probe function $\phi : C \to \mathbb{R}$. It can be seen that $\phi(p) \in \phi(A_i) \cap \phi(A_j)$ for $|j - i| \leq 1$, thus $A_i \underset{\Phi}{\cap} A_j \neq \emptyset$. Using axiom **(dP3)** of Def. 23, it can be concluded that $A_i \delta_\Phi A_j$ for $|j - 1| \leq 1$. Thus from Def. 25, we can conclude that C is a δ_Φ-sequence.

□

Let us now consider the petal chain,**ptlchain**$_k$, as a sequence in a proximity space.

Theorem 14. *Let $(\mathscr{O}_p^{\check{C}ech}, \overset{\wedge\wedge}{\delta})$ be a proximal object space and* **ptlchain**$_k$ $\in \mathscr{O}_p^{\check{C}ech}$*,for $k \in \mathbb{Z}^+$, be a petal chain contained in it. Then* **ptlchain**$_k$ *is a $\overset{\wedge\wedge}{\delta}$-sequence.*

Proof. From Def. 14 it can be seen that a **ptlchain**$_k$ is a collection of subsets A_i of cx^s (a strong Čech complex). These subsets satify the condition that each $A_i \in$ **crl**$_i$ and that $A_i \cap A_{i-1} \neq \emptyset$. Since we consider the strong Čech complex(Def. 5), thus all the intersections of the interiors of the subsets of cx^s are non-empty. Thus we can conclude that $\mathrm{int}(A_i) \cap \mathrm{int}(A_{i-1}) \neq \emptyset$. From axiom **(snN5)** of Def. 22, it can be concluded that $A_i \overset{\wedge\wedge}{\delta} A_{i-1}$. Moreover, subsituting $\hat{i} = i - 1$, we get $A_{\hat{i}+1} \overset{\wedge\wedge}{\delta} A_{\hat{i}}$, thus $A_j \overset{\wedge\wedge}{\delta} A_i$ for $|j - i| \leq 1$. Thus, from the Def. 25 it can be seen that as **ptlchain**$_k$ = $\{\bigcup_{\mathscr{I}} A_i : A_i \in \textbf{crl}_i, A_j \overset{\wedge\wedge}{\delta} A_i \ for \ |j - i| \leq 1, i, j \in \mathscr{I}\}$. Hence, **ptlchain**$_k$ is a $\overset{\wedge\wedge}{\delta}$-sequence with an additional condition that each $A_i \in$ **crl**$_i$. □

Using these results we formulate proximity relations for petal chains(**ptlchain**$_k$) in an object space($\mathscr{O}_p^{\check{C}ech}$).

Theorem 15. *Let $(\mathscr{O}_p^{\check{C}ech}, \{\delta, \overset{\wedge\wedge}{\delta}, \delta_\Phi\})$ be a proximal relator space,* **ptlchain**$_k$ $\in \mathscr{O}_p^{\check{C}ech}$ *be a petal chain, for $k \in \mathbb{Z}^+$. Then.*
1^0 : ***ptlchain***$_k$ *is a δ-sequence*
2^0 : ***ptlchain***$_k$ *is a δ_Φ-sequence*

Proof. 1^o : From Thm. 14 we can see that $\mathbf{ptlchain}_k$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence and using Lemma 9 we can conclude that $\mathbf{ptlchain}_k$ is a δ-sequence.

2^o : From Thm. 14 we can see that $\mathbf{ptlchain}_k$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence and using Lemma 9 we can conclude that $\mathbf{ptlchain}_k$ is a δ_Φ-sequence.

□

Let, us now move on to defining proximity relations on the whole object space, $\mathscr{O}_p^{\check{C}ech}$.

Lemma 10. *Let* $(\mathscr{O}_p^{\check{C}ech}, \overset{\wedge\!\!\wedge}{\delta})$ *be a proximal object space and* $C = \{\mathbf{crl}_k : \mathbf{crl}_k \in \mathscr{O}_p^{\check{C}ech} \text{ and } k = 0, 1, 2, \cdots, j\}$ *is the set of k-corollas of the object space. Suppose* $\hat{k} \in \mathbb{Z}^+$ *such that* $\mathbf{crl}_{\hat{k}} = \emptyset$*. Then for* $0 < j < \hat{k}$*, the collection of subsets C is a* $\overset{\wedge\!\!\wedge}{\delta}$*-sequence.*

Proof. From Thm. 13 it can be seen that $\mathbf{crl}_{j+1} \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_j$ and $\mathbf{crl}_j \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_{j-1}$ for $0 < j < \hat{k}$. Thus it can be concluded that $\mathbf{crl}_j \overset{\wedge\!\!\wedge}{\delta} \mathbf{crl}_i$ for $|j - i| \leq 1$ where $0 \leq j, i \leq 1$. From Def. 25 it can be concluded that the set $C = \{\mathbf{crl}_k : \mathbf{crl}_k \in \mathscr{O}_p^{\check{C}ech} \text{ and } k = 0, 1, \cdots, \hat{k}\}$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence. □

Let us now extend this result to the object space, $\mathscr{O}_p^{\check{C}ech}$.

Theorem 16. *Let* $(\mathscr{O}_p^{\check{C}ech}, \overset{\wedge\!\!\wedge}{\delta})$ *be a proximal object space, then the object space i.e.* $\mathscr{O}_p^{\check{C}ech}$ *is a* $\overset{\wedge\!\!\wedge}{\delta}$*-sequence.*

Proof. From Def. 15 it can be seen that $\mathscr{O}_p^{\check{C}ech} = \{\bigcup_k \mathbf{crl}_k : \mathbf{crl}_0 = p, k = 0, 1, \cdots, \hat{k}\}$, which from the definition of union is equivalent to $\{\mathbf{crl}_i : \mathbf{crl}_i \in \mathscr{O}_p^{\check{C}ech}, k = 0, 1, \cdots, \hat{k}\}$. Thus, from Lemma 10 it is evident that $\mathscr{O}_p^{\check{C}ech}$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence. □

Using this theorem, we detail the following proximity relations on the object space, $\mathscr{O}_p^{\check{C}ech}$.

Theorem 17. *Let* $(\mathscr{O}_p^{\check{C}ech}, \overset{\wedge\!\!\wedge}{\delta})$ *be a proximal object space, and the* $\mathscr{O}_p^{\check{C}ech}$ *is a* $\overset{\wedge\!\!\wedge}{\delta}$*-sequence. Then,*

1^o : $\mathscr{O}_p^{\check{C}ech}$ *is a* δ*-sequence*

2^o : $\mathscr{O}_p^{\check{C}ech}$ *is a* δ_Φ*-sequence*

Proof. 1^o : From Thm. 16 it can be concluded that $\mathscr{O}_p^{\check{C}ech}$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence and from Lemma 9 it is evident that $\mathscr{O}_p^{\check{C}ech}$ is a δ-sequence.

2^o : $\mathscr{O}_p^{\check{C}ech}$ is a $\overset{\wedge\!\!\wedge}{\delta}$-sequence and from Lemma 9 it is evident that $\mathscr{O}_p^{\check{C}ech}$ is a δ_Φ-sequence.

□

4. Computational Experimentation

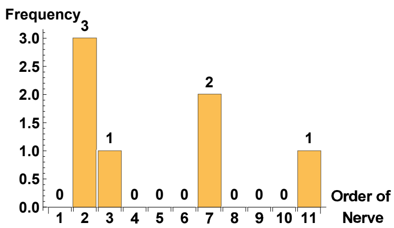
In this section, we will consider the applications of the topological structures defined in this paper. The aim of this study is to define a topological framework for approximating and extracting the shapes of objects in a digital image. As discussed earlier we require the choice of a selected keypoints from the image. First, we use the scale invariant feature transform(SIFT) based keypoints defined by Lowe (Lowe, 1999).

9.1: Original Image

9.2: Origianl image pre-processed

9.3: SIFT keypoints

9.4: Cech Nerves of different orders



9.5: Cech Spectrum

9.6: 1-Corolla($\mathbf{crl}_1$) of the maximal nerve

Figure 9. This figure displays the image of a boat(Fig. 9.1), and the same image after preprocessing to aid in extracting topological structures(Fig. 9.2). Fig. 9.3 shows the locations of SIFT keypoints. In Fig. 9.4, the Čech nerve are superimposed atop the image. Fig. 9.5 displays the Čech spectrum of the object space, denoted by $\mathscr{C}(\mathscr{O}_p^{\check{C}ech})$. Fig. 9.6 displays the 1-corolla($\mathbf{crl}_1$) of the object space or the maximal Čech nerve.

4.1. SIFT keypoints based Čech complex

Let us first consider the image shown in Fig. 9.1. The main object of focus in this image is a boat. There are many other objects e.g. mountain, beach, sea and people in the image, that can be considered as noise for current application. To aid in approximating the shape of the main object i.e. the boat, we pre-process the image to remove all other objects. The output of this pre-processing is displayed in Fig. 9.2. It can be observed that almost everything apart from the distinctive features of the boat have been removed. Some of the parts of the boat have also been removed because of the similarity with the objects that were considered to be noise.

After the pre-processing stage we extract the SIFT keypoints from the image, represented as a set S. Based on these points we can now construct a Čech complex(Def. 3) by considering a collection of balls of radius r, $\{B_r(s) : s \in S\}$ super-imposed on to the image. Every set of k balls with a non-empty intersection is the $k-1$ simplex in the resulting abstract simplicial complex.

Algorithm 1: Čech Complex represenatation of Image Objects

Input : digital image *img*, Keypoints S, Čech radius r
Output: Čech complex on the image $\check{C}ech_r(S)$,Čech nerve of maximal order **max**$\check{C}ech_r(S)$, Čech spectrum $\mathscr{C}$

foreach $s \in S$ **do**
  $\check{C}ech_r(S) := \check{C}ech_r S \cup B_r(s)$;
/*Calculating the Čech spectrum*/;
$\mathscr{S} := S$;
$Nrv(0) := S$;
Continue ← *True*; $k \leftarrow 1$;
while *Continue* $= True$ **do**
  $\mathscr{S} := (k+1)$-*Combination of* S;
  foreach $comb \in \mathscr{S}$ **do**
    bnddsk ← *minimal bounding disk of the* $(k+1)$ *points in comb*;
    rad ← *radius of bnddsk*;
    if $rad \leq r$ **then**
      nerve ← *comb*;
    else
      /*Continue*/;
  if $nerve \neq \emptyset$ **then**
    $\hat{S}$ ← *all the unique points* $s \in S$ *present in nerve*;
    $Nrv(k-1) := Nrv(k-1) \setminus \{(k) - Combination\ of\ \hat{S}\}$;
    $\acute{S} := S \setminus \hat{S}$;
    $k \leftarrow k+1$;
  else
    Continue ← *False*;
    maximalorder ← k;
$j \leftarrow 1$;
while $j \leq maximalorder$ **do**
  $Nrv(j) \longmapsto$ *number of elements*;
  $(j) \leftarrow$ *number of elements*
max$\check{C}ech_r(S) := Nrv(maximalorder)$;
$\check{C}ech_r(S) \longmapsto img$;

Moreover, it can be seen that every set of k geometric balls with a non-empty intersection is called a Čech nerve, denoted by $\check{C}ech_r(S)$. The common intersection is termed as the nucleus and the number of balls k in the nerve is termed its order. An overview of the algorithm to generate the Čech complex is presented in Alg. 1. In Fig. 9.4 all the $\{\check{C}ech_r s : s \in S\}$ in the Čech complex are illustrated on the image. The nerves have been color coded with respect to their order.

Here we can see that for the current choice of radius r, the collection of balls form two complexes as there is no intersection in between them. One of the complexes is on the front side of the boat while the other complex is on the mast and the rear end of the boat. If we increase the radius both these complexes will merge in to one complex. This is an important point to note and is common to this and other frameworks which aim to model the data using topological constructs (Ghrist, 2008). The topological features of the approximation built using Čech and related complexes is dependent on the radius. The appropriate choice of radius will be discussed in a future work.

Here we present a possible topological signature of the shape. It is termed as the Čech spectrum and is defined in Def. 20. It is defined as a sequence of numbers which represents the number of nerves of a particular order in an image. A similar shape signature was considered for the approximation of object space via curved and rectilinear triangulations (Ahmad & Peters, 2017*b*). It can be seen that this is related to the spatial distribution of the keypoint locations and the radius of the geometric balls. Nerves of higher order are a result of a large number of keypoints proximal to each other. Thus, based on how the keypoints are selected the nerves of different order represent different concentrations(or clusters) of specific features in an image. Based on this we can assume that the region in the image, where the highest number of keypoints are mutually proximal is the most important region.

This brings us to the concept of a maximal Čech nerve. For the image under consideration we can see from the Čech spectrum that the order of the maximal nerve is 11 and there is only one of them in the image. Let us look at the location of this nerve on the image. It is shown in Fig. 9.6. We can see that this nerve lies on the saftey tube hanging on the mast towards the rear end of the boat. The reason for this is the the tube is a compact structure that is highly differentiated from its background. Thus there is a high distribution of SIFT keypoints. Moreover, there are keypoints corresponding to the rear hull of the boat. The combination of these points results in the existence of the maximal nerve in this region of the image.

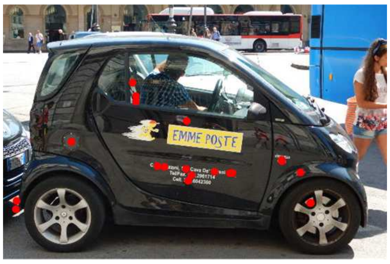
Each of the balls in this image is the maximal nerve is the 1-petal, denoted as $\mathbf{ptl}_1$. The union of all the $\mathbf{ptl}_1$ in the maximal nerve is called the $\mathbf{crl}_1$. Thus the maximal Čech nerve is equivalent to the $\mathbf{crl}_1$. All the geometric balls in the image that has a non-empty intersection with the 1-corolla($\mathbf{crl}_1$) is called the 2-petal, denoted as $\mathbf{ptl}_2$. The union of all the $\mathbf{ptl}_2$ in the image is called the 2-corolla($\mathbf{crl}_2$). The concept can be generalized to higher values of k in a similar fashion. Thus, the image shown in Fig. 9.6 also represents the 1-corolla or the $\mathbf{crl}_1$.

Let us consider the image of a car as shown in the Fig. 10.1. This image contains a black car, with many other objects such as a human, building, partial parts of a car and bus. Thus, the image contains the focal object and a lot of other objects which for the purpose of this study we consider to be noise. After the pre-processing to remove all other objects apart from the focal object we obtain the image shown in Fig. 10.2. We use this image to select keypoints(set S) which will then be used to construct a Čech complex and superimpose it on the image. The nerves in the Čech complex($\{\check{C}ech_r(s) : s \in S\}$) are color coded with respect to number of sets in them(order). This result is shown in Fig. 10.4. In contrast to the result for the image of the boat as shown in Fig. 9.4, we can notice that the collection of all the geometric balls of radius r form a single Čech complex.

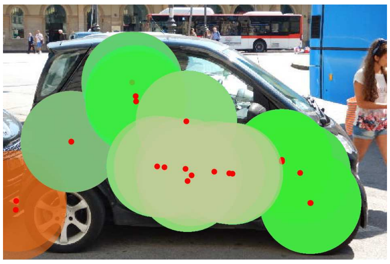
Let us now move on to the newly proposed shape signature, namely the Čech spectrum. For the approximation of the shape of the car with Čech complex, the Čech spectrum is displayed in

10.1: Original Image

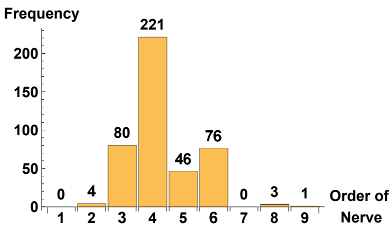
10.2: Origianl image pre-processed



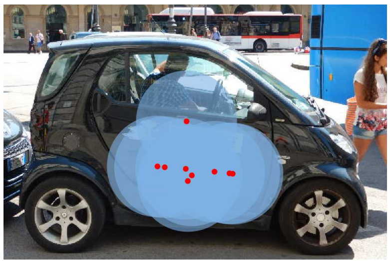
10.3: SIFT keypoints



10.4: Cech Nerves of different orders



10.5: Cech Spectrum



10.6: 1-Corolla($\mathbf{crl}_1$) of the maximal nerve

Figure 10. This figure displays the image of a car (Fig. 10.1), and the same image after preprocessing to aid in extracting topological structures(Fig. 10.2). The keypoints extracted by the SIFT algorithm are shown in Fig. 10.3.In Fig. 10.4, the Čech nerve are superimposed atop the image. Fig. 10.5 displays the Čech spectrum of the object space, denoted by $\mathscr{C}(\mathscr{O}_p^{\check{C}ech})$. Fig. 10.6 displays the 1-corolla($\mathbf{crl}_1$) of the object space or the maximal Čech nerve.

Fig. 10.5. It can also be observed that the Čech spectrum for the car is different from that of the image of the boat(Fig. 9.5). The order of the maximal Čech nerve was 11 for the boat and is 9 for the car. The number of nerves of maximal order is again one. There can be multiple maximal Čech nerves in an image. In that case we can either consider them to be multiple objects or the different(in terms of features) regions in the same image. This choice is dependent on whether we consider our image to contain a single focal object or multiple ones.

We plot the maximal Čech nerve on the image and display it in Fig. 10.6. It can be seen that the maximal nerve lies on the interior of the car on its front door. Moreover, as discussed earlier it the maximal Čech nerve is the same as the 1-corolla($\mathbf{crl}_1$) and each of the balls in it is the 1-petal($\mathbf{ptl}_1$). All the geometric balls that share a non-empty intersection with the $\mathbf{crl}_1$ are called the 2-petals($\mathbf{ptl}_2$). The union of all the 2-petals($\mathbf{ptl}_2$) is called the 2-corolla($\mathbf{crl}_2$). We can extend the concept of k-petals and k-corolla to this image as we did to the image of the boat.

4.2. *Čech complex using hole based keypoints*

After discussing the results of approximating the objects in an image using Čech complexes based on SIFT based keypoints, we develop a new type of key points. These types of keypoints are based on the notion of a hole. The notion of a hole is a vital one in topology (Alexandroff, 1965).

Let us consider an extension of this concept to a digital image. We use the notion of a **descriptive hole**, defined as below.

Definition 31. *A descriptive hole is a finite bounded sub-region of a plane with a matching description. The description is obtained by the probe function,* $\phi : 2^X \to \mathbb{R}$.

Algorithm 2: Hole based Keypoints

Input : digital image *img*, Horizontal filter radius r_x, Vertical filter radius r_y, Hole threshold t, Number of holes n_{hole}

Output: Hole locations $\mathcal{K}_{holes}$

$f_G(x,y) := \frac{1}{2\pi r_x r_y} exp(-(\frac{x^2}{2r_x^2} + \frac{y^2}{2r_y^2}))$;

$img_{filt} \leftarrow f_G(x,y) * img$;

$g := empty\ matrix$;

foreach *pixel* $\in$ *img* **do**

    $J := empty\ matrix$;

    foreach *channel* $\in$ *pixel* **do**

        $J(channel, :) := Grad(pixel)$;

    $(i, j) \leftarrow$ *location of pixel*;

    $g(i, j) \leftarrow \sqrt{\lambda_{max}(J^T J)}$;

$g :=$ *set all values of* $g < t$ *to* 1 *and rest to* 0;

$g \longmapsto$ *connected components*;

connected components $\longmapsto$ *size in terms of pixels*;

/* arrange in descending order w.r.t. size in terms of pixels */;

connected components $\longmapsto$ *arranged connected components*;

hole $\leftarrow$*first* n_{hole} *arranged connected components*;

hole $\longmapsto$ *centroids*;

$\mathcal{K}_{holes} \leftarrow$ *centroids*;

In this paper we consider the description to be the pixel intensity, which for coloured images is a vector of values in domain $\mathbb{R}^n$. Where n is the number of channels in the image. For a classical coloured image, the RGB color image, this is 3. A digital image is represented as a matrix (size $m \times n$) for computation, and for the RGB image this becomes a collection of three matrices or a multi-way array of size $m \times n \times 3$. For the purpose of detecting holes, we first convolve the image with a normalized Gaussian kernel to remove noise. The 2D normalized Gaussian is defined as:

$$f_G(x,y) = \frac{1}{2\pi r_x r_y} e^{-(\frac{x^2}{2r_x^2} + \frac{y^2}{2r_y^2})},$$

where r_x and r_y define the standard deviation of the Gaussian in the x and y direction respectively. The standard deviation dictates the radius of the smoothing filter. This smoothed image is then used to calculate the derivative for each pixel. The derivative is calculated using the a traditional derivative filter such as the sobel operator. In an image, for each pixel there are two derivative, one in the x and one in the y direction. We combine the derivatives in a Jacobian matrix(J).

To illustrate this concept, let us consider the RGB image to be a map(img) that assigns an intensity($\mathbb{R}^3$) to each pixel location($\mathbb{R}^2$). This is represented as $img : \mathbb{R}^2 \to \mathbb{R}^3$. The Jacobian matrix($J$) for each pixel would thus be a matrix of size 2×3, defined as:

$$J(img(x,y)) = \begin{bmatrix} \frac{\partial img(1)}{\partial x} & \frac{\partial img(1)}{\partial y} \\ \frac{\partial img(2)}{\partial x} & \frac{\partial img(2)}{\partial y} \\ \frac{\partial img(3)}{\partial x} & \frac{\partial img(3)}{\partial y} \end{bmatrix},$$

where $img(i)$ represents the ith channel of the image. The gradient magnitude for each pixel $g(x,y)$ is calculated as:

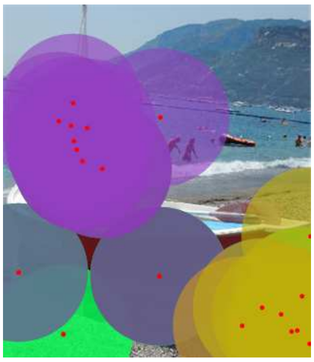
$$g(x,y) = \sqrt{\lambda_{max}(J(img(x,y))^T J(img(x,y)))},$$

where $\lambda_{max}(A)$ is the larget eigen value of matrix A.

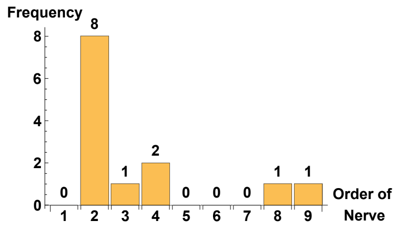
Once, we have the gradient magnitude for each pixel in the image we can then threshold to yield the location of regions with relatively constant pixel intensities. The value of the threshold decides the ammount of variation that we are willing to allow in the description of a hole. Since, we are interested in the areas with a gradient close to 0, we set all the pixels with gradient values less than the threshold to be 1 and the rest to be 0. In this fashion we mark all the regions of interest(or holes as we refer to them) as 1. We can extract these regions using connected component analysis, and then calculate the size of the region in terms of the number of pixels and its centroid. The holes are arranged in the descending order with respect to their size and the hole based keypoints($\mathscr{K}_{hole}$) are the centroids of these holes. The size of the hole is a determinet of its importance in the image. This method is summarized in Alg. 2.

Once, we have the hole-based keypoints, $\mathscr{K}_{hole}$, they are input to the Alg. 1 to build the Čech complex so that features of the objects in the image can be extracted. This process is similar to the one performed for the SIFT keypoints in § 4.1. Let us discuss the application of this new class of keypoints to the images used in § 4.1. Let us first illustrate the location of the new form of keypoints on the image of the boat, as shown in Fig. 11.1. The difference between the objects extracted from the digital images using the two differnet types of keypoints lies in the location of the keypoints. Let us compare the location of the SIFT and hole-based keypoints($\mathscr{K}_{hole}$), shown in Fig. 9.3 and Fig. 11.1 respectivly. We can see that due to the specific construction of $\mathscr{K}_{hole}$, these exist on a region with a constant intensity. It can be seen that in the SIFT based keypoints are located along edges of the main body of the boat, while one of the $\mathscr{K}_{hole}$ lies near the center of the body of the boat. The rest of the keypoints are mostly concentrated near the bottom edge of the boat on the shadow, and the life tube. These objects are very narrow, so the SIFT and the $\mathscr{K}_{hole}$ are at almost similar locations. It is easy to see that for a narrow objects the centeroids are close to the

11.1: Hole-based keypoints



11.2: Čech nerve of different order



11.3: Čech Spectrum

11.4: Maximal Čech nerve

Figure 11. Fig. 11.1 displays the hole-based keypoints on the original image of the boat. Čech nerves of different order are shown in Fig. 11.2. Fig.11.3 displays the Čech spectrum for the hole-based keypoints. The maximal Čech nerve or the 1-corolla($\mathbf{crl}_1$) is displayed in Fig. 11.4.

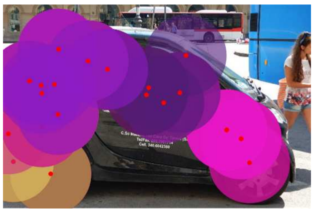
edges.

Let us now look at the Čech nerves of differnet order imposed on the image. It is shown in Fig. 11.2 and the nerves are color coded with respect order(the number of geometric balls in the nerve). The radius used to generate this result is the same as that for the SIFT based keypoints shown in Fig. 9.4. It can be seen that Čech nerve generated using $\mathscr{K}_{holes}$ covers more area of the boat, that the Čech nerve generated using the SIFT keypoints.

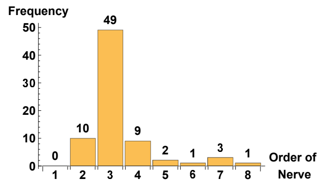
The Čech spectrum or the number of unique Čech nerves of a specific order in the Čech complex($\check{C}ech_r(K)$) is defined as Def. 20. For the Čech complex generated using the hole-based keypoints, denoted as $\check{C}ech_r(\mathscr{K}_{holes})$, the Čech spectrum($\mathscr{C}$) is displayed in Fig. 11.3. Comparing this with the result for the SIFT keypoints(Fig. 9.5), it can be seen that the Čech spectrum is quite different. It can be seen that the order of the maximal Čech nerve for $\mathscr{K}_{hole}$ is 9, while for the SIFT based case is 11.

One commonality between the Čech complexes for both the SIFT and $\mathscr{K}_{holes}$ is the location of the maximal Čech nerve or the 1-corolla($\mathbf{crl}_1$). Other topological structures can be extracted for the $\mathscr{K}_{holes}$ based $\check{C}ech_r(\mathscr{K}_{holes})$ in the same fashion as for the SIFT based case explained with

12.1: Hole-based keypoints



12.2: Čech nerve of different order



12.3: Čech Spectrum

12.4: Maximal Čech nerve

Figure 12. Fig. 12.1 displays the hole-based keypoints on the original image of the car. Čech nerves of different order are shown in Fig. 12.2. Fig.12.3 displays the Čech spectrum for the hole-based keypoints. The maximal Čech nerve or the 1-corolla($\mathbf{crl}_1$) is displayed in Fig. 12.4.

detail in § 4.1.

Let us move on to the case of extracting the shape of the car using $\mathscr{K}_{holes}$. We consider the location of the SIFT keypoints(Fig.10.3) and compare them with the location of the $\mathscr{K}_{holes}$(Fig. 12.1). For this image it can be seen that the SIFT based keypoints for the image of the car are located in the center of the object on the front door. While, the $\mathscr{K}_{holes}$ for this image are located towards the upper contour of the car. The SIFT points are located on the front door due to the text. There are lot of edges on the front door thus the SIFT based keypoints lie on these edges. Moreover, the location of $\mathscr{K}_{holes}$ are on the rims, the mirros and the bumper of the car on the back. The locations of the keypoints for the SIFT are on the on the text, bottom fender of the car on the back and on the arm of the man in the car. Due to the centrality of the text on the car the SIFT keypoints for this image are concentrated on the center of the car.

This is the main reason for the $\check{C}ech_r(K)$ generated using the SIFT keypoints(Fig. 10.4) cover the car better than the $\mathscr{K}_{holes}$(Fig. 12.2). The nerves of different order are color coded for the Čech complex generated using $\mathscr{K}_{holes}$ are displayed in Fig. 12.2. The Čech complex for the $\mathscr{K}_{holes}$ conforms very well to the top contour of the car and the tires, but the bottom part of the front door remains uncovered. The Čech spectrum of the $\check{C}ech_r(K)$ usig the $\mathscr{K}_{holes}$ is shown in Fig. 12.3. It can be seen that this Čech spectrum is different from the case of the SIFT keypoints shown in Fig. 10.5. The maximal Čech nerve for the $\mathscr{K}_{holes}$ is 8 while for the SIFT keypoints is 9. Moreover the case of the car, the location of the maximal nerver or the 1-corolla($\mathbf{crl}_1$) is also different for the SIFT keypoints and the $\mathscr{K}_{holes}$. For the SIFT the maximal nerve is on the front door while for the $\mathscr{K}_{holes}$ it is on the top of the rear end of the car. This difference is fundamentaly due to the difference in the spatial distribution of the specific types of keypoints.

4.3. Applications of the proposed framework

In this section we will present a few possible applications of the proposed framework. The idea is to formulate a proof of concept, that the current framework can be used in object extraction and recognition tasks in computer vision. We will begin with the idea of extracting the shapes from an image using the topological notion of cover.

4.3.1. Persistence of Čech Shapes

One of the major themes in this article is to develop a topological framework for covers of an object in a digital image. The idea of using simple geometrical objects to cover a topological space, so as to extract topological and geometrical information about it dates back to Poincarĕ (Poincaré, 1895). The objects used here are disks (here called Čech balls), parameterized by the location of centers and radii. An important question that arises here is related to the choice of these parameters.

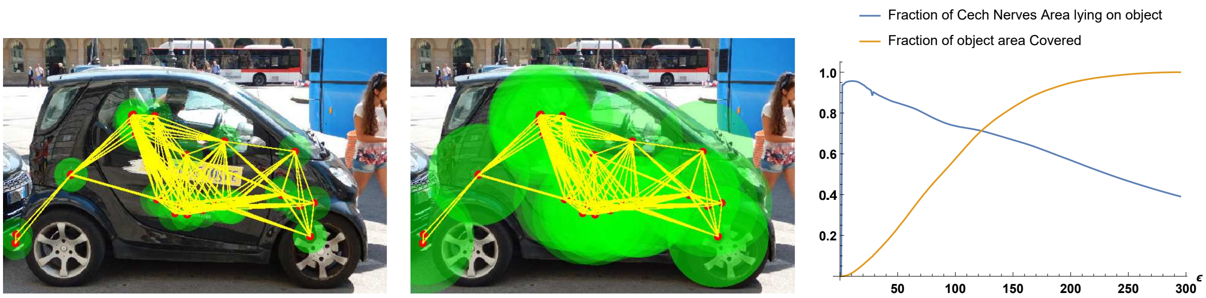
We choose the centroids to be the keypoints contributed by either SIFT or by hole based keypoints of Alg. 2. As to the choice of radii goes, we will use a recently developed technique, called persistent topology (Edelsbrunner & Harer, 2010), which is aimed at filtering out noise. The idea is that as we increase the radius of the Čech balls we get a new Čech complex which is a superset of the previous one. This can be written as: $for\, all\, r, s \in \mathbb{R}, r < s \Rightarrow cx_r \subseteq cx_s$. Here cx_r is a Čech complex yielded by Čech balls of radius r. This means that we get a filtration of Čech complexes indexed by the radius. Since, the question under investigation is the quality of a cover of the objects in digital images, we will employ appropriate measures in this regard.

The two measures that we use are the fraction of the area of the object covered by the Čech complex and the fraction of the area of Čech complex that lies on the object in the image. We want to cover the maximum area of the object while reducing the area of the background(parts of an image that are not the object), in the resulting cover. This is a trade off, which can be seen from Figs. 13.1 and 13.2. In Fig. 13.1 all of the Čech complex lies on the object, but it only covers a small area of the object. In Fig. 13.2, we are covering a significant portion of the object area, but the spillage into the background is also present. We need to strike a balance between the two parameters to attain a cover of the object that is well suited to approximating its geometrical and topological features. Hence, we plot the normalized values(on the range $[0, 1]$) of the measures for the whole filtration in the plot shown in Fig. 13.3.

From this plot we can decide upon the appropriate value of the radii for the Čech balls, which will suite our application. If the objective is to maximize the area of the object covered we would like the radius to be in the range of $170 - 220$, but the spillage of the complex would be significant. Another possible choice could be the intersection point of the two curves, which would yield a value of 120 for the radius. This perspective gives us a view of the topological and geometrical properties of the object, indexed by scale. Other possible extensions of this method could be, to consider indexing by the number of keypoints or to index by both the number of keypoints and the radius.

4.3.2. Shapes of Bird Species

Let us consider another application of the concepts developed in this article. We will consider the identification of different species of birds, based on their shape. The aim like the previous



13.1: Čech Complex with $\epsilon = 50$ 13.2: Čech Complex with $\epsilon = 150$ 13.3: Persistence of Čech shapes

Figure 13. This figure illustrates the change in the area of the object covered by the Čech complexes of varying diameters,ϵ. The plots illustrate the fraction of the object area covered by the Čech complexes and the fraction of the complex area that lies on the object of interest.

application is to demonstrate the viability of the theoretical framework. For this purpose we select the database used in (Lazebnik *et al.*, 2005). We select three images of birds belonging to three different species. On the images we perform an analysis similar to the one detailed in Sec. 4.2.

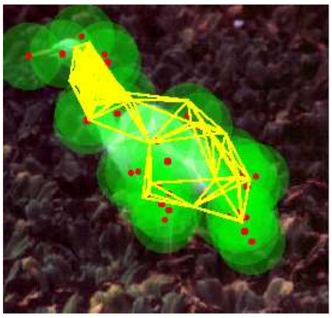
We use the holebased keypoints as they take into account the description of the image(in terms of the locations of constant pixel intensity regions). This will be a crucial feature in the classification of different patterns. The patterns inside an object and the shape of its contour are some of the most important features when it comes to classification in computer vision. This can be understood by looking at the pictures of the three birds we aim to classify, shown in Figs. 14.1,14.4 and 14.7. These differ not only in terms of the shapes of their boundary contours but the patterns on them. For each of the birds, the Čech complexes formed by considering the holebased keypoints determined using Alg. 2, are shown in Figs.14.2,14.5 and 14.8. We can see that the 1-skeleton of the Čech complex (i.e. all the 1-simplices in it) conforms to the shape of the bird. An additional point to note here is that, it represents the proximity of the centroids of the regions with different descriptions. Hence, it is a more structural representation of the bird in terms of the regions of matching description in it.

For this example we resort to looking at the Čech spectra(Def. 20) of each of the images. This is the number of Čech nerves of different order in the image. Each of the Čech Nerves is itself a hyper-connected space, as it is a intersection of varying number of Cech balls. The order of the hyper-proximity is the same as the cardinality of the nerve. Hence, the Čech spectra can be equivalently thought of as the number of hyper-connected subspaces of different order in the image. The Čech spectra for the images of the three birds are show in Figs.14.3,14.6 and 14.9. It can be seen that the Čech spectra for the three birds are significantly different, hence providing a possible feature for classification.

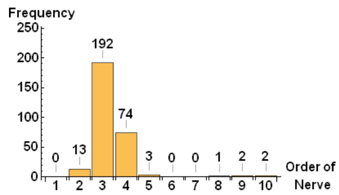
4.3.3. Shapes of Butterfly Species

We present another application of the proposed framework to object detection for classification in computer vision. The aim is to provide a proof of concept, that the theoretical framework developed in this article can be used in practical applications. For this purpose, we aim to classify

14.1: Egret

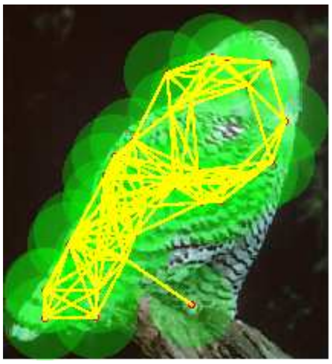


14.2: Čech Complex covering of Egret

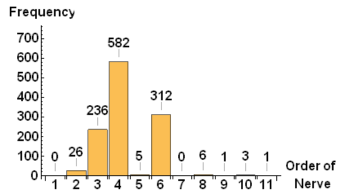


14.3: Čech Spectrum of Egret

14.4: Owl

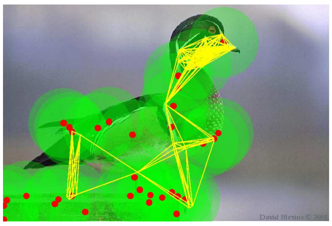


14.5: Čech Complex covering of Owl

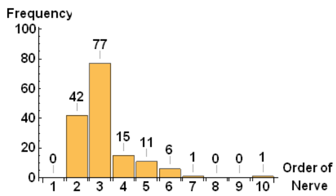


14.6: Čech Spectrum of Owl

14.7: Wood Duck



14.8: Čech Complex of Wood Duck



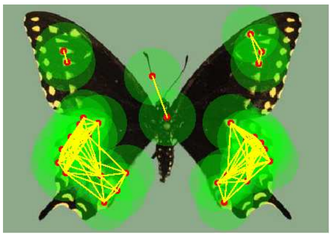
14.9: Čech Spectrum of Wood Duck

Figure 14. This figure illustrates the Čech complex coverings and the associated Cech spectra for images of three different birds taken from the database developed in (Lazebnik *et al.*, 2005).

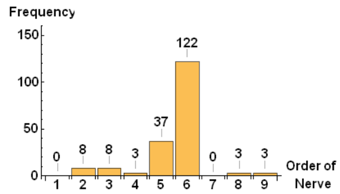
butterflies based on their shapes. The dataset used in this task is taken from Lazebnik *et al.* (2004). We take three different images of butterflies belonging to different species. These images are shown in Figs. 15.1,15.4 and 15.7. All these butterflies are different from one another based on their shape and patterns.

As we discussed in the case of classifying bird species, a classifier that incorporates the pixel intensity description would perform better than the one that considers boundary contours alone. This fact becomes even more evident when we compare Figs.15.1 and 15.4, showing Black Snowtail and Machaon butterflies respectively. Both the butterflies have almost identical boundary

15.1: Black Snowtail

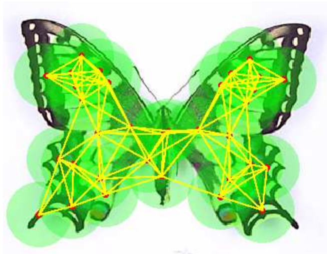


15.2: Čech Complex covering of Black Snowtail

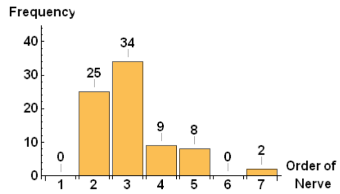


15.3: Čech Spectrum of Black Snowtail

15.4: Machaon

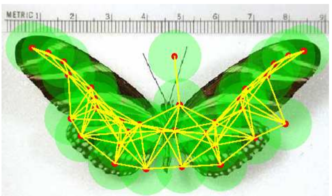


15.5: Čech Complex covering of Machaon

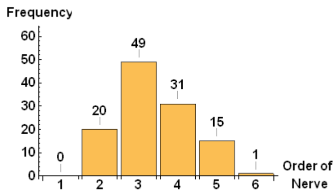


15.6: Čech Spectrum of Machaon

15.7: Zebra Butterfly



15.8: Čech Complex of Zebra Butterfly



15.9: Čech Spectrum of Zebra Butterfly

Figure 15. This figure illustrates the Čech complex coverings and the associated Cech spectra for images of three different butterflies taken from the database developed in (Lazebnik *et al.*, 2004).

contours but differ drastically in terms of patterns. Thus, we must use the holebased keypoints calculated using Alg. 2, which are centroids of regions with matching description(in this case [pixel intensities). The methodology used here is similar to the Secs. 4.2 and 4.3.2. The Čech covers of the different butterflies are shown in Figs. 15.2,15.5 and 15.8. It can be seen that the difference of patterns dictates the locations of the holebased keypoints. It leads to different Čech complexes for both the Black Snowtail(Fig. 15.1) and the Machaon(Fig. 15.4). Moreover, the 1-skeletons(all the 1-simplices) of the Čech complexes conform to the structure of the shape.

Now to exploit this difference in the Čech complexes we use Čech spectrum(Def. 20) as a measure. The relationship between the Čech spectrum and hyper-connected subspaces of the image has been detailed in Sec. 4.3.2. We can see that Čech spectra for the different butterflies as shown in Figs.15.3,15.6 and 15.9 are substantially different. This difference elucidates the possibility of classification.

5. Conclusion

This paper uses proximal Čech complexes to approximates the shape of objects in digital images. Several topological structures with closed geometric balls as the primitive are formulated to study the geometrical and topological properties of objects. Moreover, instead of considering only the boundary contours of the object we also include the interior of the shape using descriptive proximity relations. The classical notion of proximity as a relation on two subsets has been extended to functions over arbitrary number of subsets. Moreover, the usual binary proximity is also extended to a continuous valued function also yielding the extent of nearness between objects. We define the concept of a descriptive hole in an image as a finite bounded region with a matching description. These are then used to formulate an algorithm to extract keypoints from an image. The distribution of the orders f the different Čech nerves in the image is used to define a signature for the shape of an object in digital images. The results for the computational experiments along with the algorithms used have also been presented.

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On Comultisets and Factor Multigroups

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Abstract

In this paper, we vividly study the concept of comultisets of a multigroup and obtain some related results. The notion of Lagrange theorem in multigroup setting is proposed. Also, this paper proposes the notion of factor multigroups as an extension of factor groups in classical group setting and deduces some results. Subsequently, some homomorphic properties of factor multigroups are presented.

Keywords: Comultisets, Multisets, Multigroups, Normal submultigroups, Factor multigroups.
2010 MSC: 03E72, 06D72, 11E57, 19A22.

1. Introduction

Many fields of modern mathematics emerged by violating a basic principle of a known theory or concept. For example, fuzzy set theory emerged by violating the notion of definite collection of objects in cantorian set theory. Similarly, the theory of multisets (see (Knuth, 1981), (Singh *et al.*, 2007), (Syropoulos, 2001), (Wildberger, 2003) for details) has been defined by assuming that, for a given set X, an element x occurs a finite number of times. This violates the idea of distinct collection of objects.

The concept of (classical) groups is built on the foundation of cantorian (or crisp) set theory. Since group is defined over a nonempty set hence, an algebraic study of multisets is an extension of group theory. The notion of multigroup was proposed in (Nazmul *et al.*, 2013) as an algebraic structure of multiset that generalized the concept of group. The notion is consistent with other non-classical groups in (Biswas, 1989), (Rosenfeld, 1971), (Shinoj *et al.*, 2015), (Shinoj & Sunil, 2015). The term *multigroups* has been earlier mentioned in (Barlotti & Strambach, 1991), (Dresher & Ore, 1938), (Mao, 2009), (Prenowitz, 1943), (Schein, 1987), (Tella & Daniel, 2013) as an extension of group theory (with each of the authors having a divergent view). Nonetheless, the

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idea of multigroups captured in (Nazmul *et al.*, 2013) is quite acceptable because it is in consonant with other non-classical groups and defined in the light of multiset.

A complete survey on the concept of multigroups from various authors were reviewed in (Ibrahim & Ejegwa, 2016). Further studies on the concept of multigroups in the light of multisets have been carried out. See (Awolola & Ejegwa, 2017), (Awolola & Ibrahim, 2016), (Ejegwa, 2017), (Ejegwa & Ibrahim, 2017*b*), (Ejegwa & Ibrahim, 2017*c*), (Ejegwa & Ibrahim, 2017*a*), (Ibrahim & Ejegwa, 2017*a*), (Ibrahim & Ejegwa, 2017*b*) for details.

In this paper, we explore more on the concept of comultisets of a multigroup studied in (Ejegwa & Ibrahim, 2017*b*), (Nazmul *et al.*, 2013), deduce some results, and propose Lagrange theorem in multigroups context. Also, we extend the idea of factor or quotient groups to multigroups and obtain some related results. Finally, the notion of homomorphism of factor multigroups is explored.

The paper is organized as follows. Section 2 gives some basic definitions and existing results on multisets and multigroups, respectively, for the sake of reference. Section 3 presents more results on comultisets of a multigroups and proposes the analogous of Lagrange theorem in multigroup setting. Details on the notion of factor multigroups as an extension of factor groups and its homomorphic properties are explicated in Section 4. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. (Singh *et al.*, 2007) Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function,

$$C_A : X \to N = \{0, 1, ...\}$$

such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A, that is, a counting function of A (where $C_A(x) = 0$, implies $x \notin Dom(A)$).

The set X is called the ground or generic set of the class of all multisets containing objects from X. The set of all multisets over X is depicted by $MS(X)$.

Definition 2.2. (Wildberger, 2003) Let A and B be two multisets over X. Then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3. (Syropoulos, 2001) Let A and B be two multisets over X. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,
(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ represent minimum and maximum, respectively.

Definition 2.4. (Nazmul *et al.*, 2013) Let X be a group. A multiset G is called a multigroup of X if the count function of G,

$$C_G : X \to N = \{0, 1, ...\}$$

satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,
(ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$.

By implication, a multiset G is a multigroup of a group X if $\forall x, y \in X$,

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y).$$

It follows immediately that,

$$C_G(e) \geq C_G(x)\ \forall x \in X$$

where e is the identity element in X. A multigroup G is regular if

$$C_G(x) = C_G(y) \forall x, y \in X.$$

We denote the set of all multigroups of X by $MG(X)$.

The count of an element in G is the number of occurrence of the element in G, and denoted by C_G. The order of G is the sum of the count of each of the elements in G, and is given by

$$|G| = \sum_{i=1}^{n} C_G(x_i) \forall x_i \in X.$$

Remark. (Ejegwa, 2017) Every multigroup is a multiset but the converse is not necessarily true.

Theorem 2.1. (Nazmul *et al.*, 2013) *Let $A, B \in MG(X)$. Then $A \cap B \in MG(X)$.*

Definition 2.5. (Nazmul *et al.*, 2013) Let $A \in MG(X)$. Then A is said to be abelian or commutative if for all $x, y \in X$, $C_A(xy) = C_A(yx)$.

Definition 2.6. (Nazmul *et al.*, 2013) Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}.$$

Proposition 2.1. (Nazmul *et al.*, 2013) *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.7. (Ejegwa, 2017) Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

A submultigroup B of A is complete if $B_* = A_*$ and incomplete otherwise. A submultigroup B of A is also a multigroup of X itself.

Definition 2.8. (Ejegwa & Ibrahim, 2017*b*) Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$, it satisfies $C_A(xyx^{-1}) \geq C_A(y)$.

Definition 2.9. (Nazmul *et al.*, 2013) Let A and B be multigroups of X. Then A^{-1} and $A \circ B$ are defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \forall x \in X$$

and

$$C_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz}(C_A(y) \wedge C_B(z)), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise,} \end{cases}$$

respectively.

Definition 2.10. (Ejegwa & Ibrahim, 2017*b*) Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in G$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A. Similarly, for right comultiset of A. It follows that, $xA = yA$, $Ay \circ Az = Ayz$ and $yA \circ zA = yzA$ $\forall x, y, z \in X$.

Definition 2.11. (Ejegwa & Ibrahim, 2017*c*) Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies

(i) $C_A(f^{-1}(y_1y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2))$ $\forall y_1, y_2 \in Y$,
(ii) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2))$ $\forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \;\forall x \in X.$$

Definition 2.12. (Ejegwa & Ibrahim, 2017*c*) Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism from A to B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism from A to B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism from A to B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism from A to B, then A is isomorphic to B denoted by $A \cong B$.

Lemma 2.1. (Ejegwa & Ibrahim, 2017*c*) *Let $f : X \to Y$ be a homomorphism of groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively.*

(i) If f is an epimorphism, then $f(f^{-1}(B)) = B$.

(ii) If $kerf = \{e\}$, then $f^{-1}(f(A)) = A$.

The kernel of f is defined by $kerf = \{x \in X \mid C_A(x) = C_B(e'), f(e) = e'\}$, where e and e' are the identities of X and Y, respectively. The kernel of f is a normal subgroup of X, and always contains the identity element of X. It reduces to the identity element if and only if f is one to one.

Theorem 2.2. (Ejegwa & Ibrahim, 2017*c*) *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then $A \in MG(X) \Leftrightarrow f(A) \in MG(Y)$ and $B \in MG(Y) \Leftrightarrow f^{-1}(B) \in MG(X)$.*

Definition 2.13. (Ejegwa & Ibrahim, 2017*b*) Suppose $A \in MG(X)$. Then the normalizer of A is the set given by

$$N(A) = \{g \in X \mid C_A(gy) = C_A(yg) \forall y \in X\}.$$

Theorem 2.3. (Awolola & Ibrahim, 2016) *Let $A \in MG(X)$ with identity $e \in X$. Then $\forall x, y \in X$, $C_A(x) = C_A(y)$ if $C_A(xy^{-1}) = C_A(e)$.*

3. Some results on comultisets of a multigroup

We assume that if G is a multigroup of a group X, then $G_* = X$ (except otherwise stated). That is, every element of X is in G with its multiplicity or count. In this section, we present some result on comultisets of a multigroups.

Recall that, for any submultigroup A of a multigroup G of a group X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the right comultiset of A.

Example 3.1. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$$

and $G = [\rho_0^7, \rho_1^5, \rho_2^5, \rho_3^3, \rho_4^3, \rho_5^3]$ be a multigroup of X. Then $H = [\rho_0^6, \rho_1^4, \rho_2^4]$ is an incomplete submultigroup of G.

Now, we find the left comultisets of H by pre-multiplying each element of G by H.

$$\begin{aligned}
\rho_0 H &= [\rho_0^6, \rho_1^4, \rho_2^4] \\
\rho_1 H &= [\rho_2^4, \rho_0^6, \rho_1^4] \\
\rho_2 H &= [\rho_1^4, \rho_2^4, \rho_0^6] \\
\rho_3 H &= [\rho_3^0, \rho_5^0, \rho_4^0] = \emptyset \\
\rho_4 H &= [\rho_4^0, \rho_3^0, \rho_5^0] = \emptyset \\
\rho_5 H &= [\rho_5^0, \rho_4^0, \rho_3^0] = \emptyset.
\end{aligned}$$

Similarly, the right comultisets of H are thus.

$$\begin{aligned}
H\rho_0 &= [\rho_0^6, \rho_1^4, \rho_2^4] \\
H\rho_1 &= [\rho_2^4, \rho_0^6, \rho_1^4] \\
H\rho_2 &= [\rho_1^4, \rho_2^4, \rho_0^6] \\
H\rho_3 &= [\rho_3^0, \rho_4^0, \rho_5^0] = \emptyset \\
H\rho_4 &= [\rho_4^0, \rho_5^0, \rho_3^0] = \emptyset \\
H\rho_5 &= [\rho_5^0, \rho_3^0, \rho_4^0] = \emptyset.
\end{aligned}$$

Suppose $H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2]$, that is, a complete submultigroup of G. The left comultisets of H are thus listed.

$$\begin{aligned}
\rho_0 H &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\
\rho_1 H &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_5^2, \rho_3^2, \rho_4^2] \\
\rho_2 H &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_4^2, \rho_5^2, \rho_3^2] \\
\rho_3 H &= [\rho_3^2, \rho_5^2, \rho_4^2, \rho_0^6, \rho_2^3, \rho_1^3] \\
\rho_4 H &= [\rho_4^2, \rho_3^2, \rho_5^2, \rho_1^3, \rho_0^6, \rho_2^3] \\
\rho_5 H &= [\rho_5^2, \rho_4^2, \rho_3^2, \rho_2^3, \rho_1^3, \rho_0^6].
\end{aligned}$$

The right comultisets of H are below.

$$\begin{aligned}
H\rho_0 &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\
H\rho_1 &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_4^2, \rho_5^2, \rho_3^2] \\
H\rho_2 &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_5^2, \rho_3^2, \rho_4^2] \\
H\rho_3 &= [\rho_3^2, \rho_4^2, \rho_5^2, \rho_0^6, \rho_1^3, \rho_2^3] \\
H\rho_4 &= [\rho_4^2, \rho_5^2, \rho_3^2, \rho_2^3, \rho_0^6, \rho_1^3] \\
H\rho_5 &= [\rho_5^2, \rho_3^2, \rho_4^2, \rho_1^3, \rho_2^3, \rho_0^6].
\end{aligned}$$

Remark. Let H be a submultigroup of $G \in MG(X)$. We notice that

(i) H and its comultisets are equal because a multiset is an unordered collection. Consequently, $xH = yH \; \forall x, y \in X$.

(ii) the number of comultisets of H equals the cardinality of H_*, and the union and intersection of the comultisets of H are comparable to H.

(iii) there is a one-to-one correspondence between the right comultisets and the left comultisets of H.

Proposition 3.1. *Let $A, B \in MG(X)$ such that $A \subseteq B$. If $xA = yA$, then $C_A(x) = C_A(y)\ \forall x, y \in X$.*

Proof. Let X be a group and $x \in X$. Suppose $xA = yA$, then we have

$$C_{xA}(x) = C_{yA}(x) \Rightarrow C_A(x^{-1}x) = C_A(y^{-1}x) \Rightarrow C_A(e) = C_A(y^{-1}x).$$

Then, it follows that $C_A(y) = C_A(x)\ \forall x, y \in X$ by Theorem 2.3. □

Proposition 3.2. *Let $B \in MG(X)$ and A be a submultigroup of B. If $(Ay \circ Az)^{-1} = Ay \circ Az$ and $Ay \circ Az = Ayz$, then $(Ay \circ Az)^{-1} = Ayz$.*

Proof. Straightforward from Definition 2.10. □

Theorem 3.1. *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid C_A(g) = C_A(e)\}, \\ K &= \{x \in X \mid C_{Ax}(y) = C_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again $H = K$.

Proof. Let $g, h \in H$. Then

$$\begin{aligned} C_A(gh) &\geq C_A(g) \wedge C_A(h) \\ &= C_A(e) \wedge C_A(e) \\ &= C_A(e) \end{aligned}$$

$\Rightarrow C_A(gh) \geq C_A(e)$.

But, $C_A(gh) \leq C_A(e)$ from Definition 2.4. Thus, $C_A(gh) = C_A(e)$, implying that $gh \in H$. Since X is finite, it follows that H is a subgroup of X.

Now, we show that $H = K$. Let $k \in K$. Then for $y \in X$ we get

$$C_{Ak}(y) = C_{Ae}(y) \Rightarrow C_A(yk^{-1}) = C_A(y).$$

Choosing $y = e$, we obtain

$$C_A(k^{-1}) = C_A(e) \Rightarrow k^{-1} \in H,$$

and so $k \in H$ since H is a subgroup of X. Thus, $K \subseteq H$.

Again, let $h \in H$. Then $C_A(h) = C_A(e)$. Also,

$$C_{Ah}(y) = C_A(yh^{-1})\ \forall y \in X$$

and

$$C_{Ae}(y) = C_A(y)\ \forall y \in X.$$

Thus to show that $h \in K$, it suffices to prove that

$$C_A(yh^{-1}) = C_A(y)\ \forall y \in X.$$

Now,

$$\begin{aligned} C_A(yh^{-1}) &\geq C_A(y) \wedge C_A(h^{-1}) \\ &= C_A(y) \wedge C_A(h) \\ &= C_A(y) \wedge C_A(e) \\ &= C_A(y). \end{aligned}$$

Again,

$$\begin{aligned} C_A(y) &= C_A(yh^{-1}h) \\ &\geq C_A(yh^{-1}) \wedge C_A(h) \\ &= C_A(yh^{-1}) \wedge C_A(e) \\ &= C_A(yh^{-1}) \end{aligned}$$

$\Rightarrow C_A(yh^{-1}) = C_A(y)$, thus, $H \subseteq K$. Hence, $H = K$. □

Corollary 3.1. *With the same notation as in Theorem 3.1, H is a normal subgroup of X if A is a normal submultigroup of B.*

Proof. Let $y \in X$ and $x \in H$. Then

$$\begin{aligned} C_A(yxy^{-1}) &= C_A(yy^{-1}x) \text{ since } A \text{ is normal in } B \\ &= C_A(x) = C_A(e). \end{aligned}$$

Thus, $yxy^{-1} \in H$. Hence, H is normal in X. □

Theorem 3.2. *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$H = \{g \in X \mid C_A(g) = C_A(e)\}.$$

Then for $x, y \in X$, we get $Hx = Hy \Leftrightarrow Ax = Ay$. Similarly, $xH = yH \Leftrightarrow xA = yA$.

Proof. This result gives a relationship between comultisets of a submultigroup of a multigroup and the cosets of a subgroup of a given group.

By Theorem 3.1, we recall that H is a subgroup of X and

$$H = \{x \in X \mid C_{Ax}(z) = C_{Ae}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. So

$$C_{Axy^{-1}}(z) = C_{Ae}(z)$$

and

$$C_A(zyx^{-1}) = C_A(z)\ \forall z \in X.$$

Replacing z by zy^{-1}, which is also an arbitrary element of X, we get

$$C_A(zx^{-1}) = C_A(zy^{-1})\ \forall z \in X,$$

implying that $Ax = Ay$.

Conversely, suppose that $Ax = Ay$. This implies that

$$C_A(zx^{-1}) = C_A(zy^{-1})\ \forall z \in X.$$

Put $z = y$, we get

$$C_A(yx^{-1}) = C_A(e).$$

So $yx^{-1} \in H$ and therefore, $Hx = Hy$. □

Corollary 3.2. *Let X be a group. If A is a submultigroup of a multigroup B of X and $x, y \in X$. Then $xA = yA$ and $Ax = Ay \Leftrightarrow C_A(y^{-1}x) = C_A(yx^{-1}) = C_A(e)$.*

Proof. Let $x, y \in X$, and recall that $H = \{x \in X \mid C_A(x) = C_A(e)\}$. Suppose $xA = yA$ and $Ax = Ay$. Then, $y^{-1}x, yx^{-1} \in H$ as in Theorem 3.2. So, $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$.

Conversely, assume $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$. This implies that, $C_A(y^{-1}x) = C_A(x^{-1}x)$ and $C_A(yx^{-1}) = C_A(yy^{-1}) \Rightarrow C_{yA}(x) = C_{xA}(x)$ and $C_{Ax}(y) = C_{Ay}(y) \forall x, y \in X$. Hence, $xA = yA$ and $Ax = Ay$. □

Lemma 3.1. *Let X be a group. If B is a submultigroup of a finite multigroup $A \in MG(X)$, then $|B| = |xB|\ \forall x \in X$.*

Proof. Let $A \in MG(X)$. Since A is finite and $B \sqsubseteq A$, it follows that $|A| = n$ and $|B| = m$ such that $m \leq n$. Then the order of each comultisets xB of B for all $x \in X$ must be m. Hence $|B| = |xB|$ for all $x \in X$. □

Now, we state and prove the analogous of Lagrange theorem in multigroup setting.

Theorem 3.3. *Let G be a finite multigroup and let H be a complete submultigroup of G wherein the count of every element in H is a factor of the count of the corresponding element in G. Then the order of H divides the order of G.*

Proof. Let $|G| = n$ and $|H| = m$, then $m \leq n$ by Lemma 3.1. That is, since G is finite and H is a complete submultigroup of G, it follows that H is also finite and $G_* = H_*$. We prove that m is a factor of n. Because $H \sqsubseteq G$ wherein the count of every element in H is a factor of the count of the corresponding element in G, it then follows that m divides n. This completes the proof. □

Remark. Let X be a finite group and G be a regular multigroup of X, then the order of X divides the order of G.

4. Concept of factor multigroups and its homomorphic properties

In this section, we define factor multigroup as an extension of factor group and obtain some results. Unless otherwise stated, the multigroups in this section are complete.

Definition 4.1. Let A be a multigroup of a group X and B a normal submultigroup of A. Then the set of right/left comultisets of B with the property $C_{xB\circ yB}(z) = C_{xyB}(z)\ \forall\ x, y, z \in X$ form a multigroup called factor or quotient multigroup of A determined by B, denoted as A/B.

Remark. Let A be a multigroup of a group X, and C a normal submultigroup of A. Then

(i) if B is a submultigroup of A such that $C \subseteq B \subseteq A$, then B/C is a submultigroup of A/C.

(ii) every submultigroup of A/C is of the form B/C, for some submultigroup B of A such that $C \subseteq B \subseteq A$.

(iii) $|A/C| = n|C|$, where n is the number of comultisets of C in A. This is unlike in classical group where, suppose X is a group and Y a subgroup of X, then $|X/Y| = \dfrac{|X|}{|Y|}$.

Theorem 4.1. *Let A be a normal submultigroup of $B \in MG(X)$. Then A is commutative if and only if B/A is commutative.*

Proof. Let $x, y \in X$. Suppose A is commutative, then

$$C_A(xyx^{-1}y^{-1}) = C_A(e),$$

and hence,

$$C_A(xy) = C_A(yx).$$

Consequently, A is normal.
Now, since

$$C_A(xy(yx)^{-1}) = C_A(xyx^{-1}y^{-1}) = C_A(e),$$

we have

$$\begin{aligned} C_A(xy(yx)^{-1}) = C_A(e) \quad &\Rightarrow \quad C_A(xy(yx)^{-1}) = C_A(xy(xy)^{-1}) \\ &\Rightarrow \quad C_{Ayx}(xy) = C_{Axy}(xy). \end{aligned}$$

Thus, $Axy = Ayx$. It follows that, $Ax \circ Ay = Ay \circ Ax$ since $Ax \circ Ay = Axy$ and $Ay \circ Ax = Ayx$ by Definition 2.10. Hence, B/A is commutative.

Conversely, if B/A is commutative, then

$$Ax \circ Ay = Ay \circ Ax \Rightarrow Axy = Ayx.$$

Thus,

$$C_A(xy(yx)^{-1}) = C_A(e) \Rightarrow C_A(xy) = C_A(yx),$$

completes the proof. □

Theorem 4.2. *Let $A, B, C \in MG(X)$ such that A and B are normal submultigroups of C and $A \subseteq B$, then B/A is a normal submultigroup of C/A.*

Proof. Let $x \in X$. Then $C_{B/A}(x) \leq C_{C/A}(x) \forall x \in X$ since $A \subseteq B$ and A and B are normal submultigroups of C. So B/A is a submultigroup of C/A. For all $x, y \in X$,

$$C_{B/A}(yxy^{-1}) \geq C_{B/A}(x).$$

Hence, B/A is a normal submultigroup of C/A by Definition 2.8. □

Remark. Let C be a multigroup of a group X, and B a normal submultigroup of C. Then every normal submultigroup of C/A is of the form B/A, for some normal submultigroup A of C such that $A \subseteq B \subseteq C$.

Theorem 4.3. *Let $A, B \in MG(X)$ and A a normal submultigroup of B. Then $A \cap B/A_*$ is a normal submultigroup of A.*

Proof. By Proposition 2.1, A_* is a subgroup of X and $A \cap B \in MG(X)$ by Theorem 2.1. So, $A \cap B/A_*$ is a multigroup of X. Since A is a normal submultigroup of B, then $A \cap B$ is a submultigroup of B and $A \cap B/A_*$ is a submultigroup of A. We show that $A \cap B/A_*$ is a normal submultigroup of A. Let $x, y \in A_*$. Then $xyx^{-1} \in A_*$ since

$$C_A(xyx^{-1}) = C_A(y) > 0$$

by Definition of A_*. This proves that A_* is normal. Also $A \cap B$ is normal since

$$C_{A\cap B}(xyx^{-1}) = C_A(xyx^{-1}) \wedge C_B(xyx^{-1}) \geq C_A(y) \wedge C_B(y) = C_{A\cap B}(y).$$

Hence, $A \cap B/A_*$ is a normal submultigroup of A. □

Theorem 4.4. *Let $A \in MG(X)$ and $N(A)$ be a normalizer of $x \in X$. Then $N(A)$ is a subgroup of X and $A/N(A)$ is a normal submultigroup of A.*

Proof. Clearly, $e \in N(A)$. Let $x, y \in N(A)$. Then for any $z \in X$, we have

$$\begin{aligned} C_A((xy^{-1})z) &= C_A(x(y^{-1}z)) = C_A((y^{-1}z)x) \\ &= C_A(y^{-1}(zx)) = C_A(y(zx)^{-1}) \\ &= C_A(y(x^{-1}z^{-1})) = C_A(z(xy^{-1})). \end{aligned}$$

Hence, $xy^{-1} \in N(A)$. Therefore, $N(A)$ is a subgroup of X. By Definition 4.1, it follows that $A/N(A) \in MG(N(A))$ and clearly, $A/N(A)$ is a submultigroup of A. Since $C_{A/N(A)}(xyx^{-1}) = C_{A/N(A)}(y) \ \forall x, y \in X$, it implies that $A/N(A)$ is a normal submultigroup of A. □

Theorem 4.5. *Let A be a commutative multigroup of X and B a normal submultigroup of A. Then there exists a natural homomorphism $f : A \to A/B$ defined by $C_{f(A)}(y) = C_B(x^{-1}y) \ \forall \ x, y \in X$.*

Proof. Let $f : A \to A/B$ be a mapping defined by

$$C_{f(A)}(y) = C_B(x^{-1}y) \;\forall\; x, y \in X.$$

That is, $C_{f(A)}(y) = C_{xB}(y) \Rightarrow f(A) = xB$ (consequently, $f(A_*) = xB_*$). Since $f : A \to A/B$ is derived from $f : A_* \to A_*/B_*$ such that B_* is a normal subgroup of A_*, then to prove that f is a homomorphism, we show that

$$C_{xyB}(z) = C_{xB\circ yB}(z)\forall z \in X \Rightarrow f(xy) = f(x)f(y).$$

Since B is commutative, then

$$C_B(xz) = C_B(zx) \Rightarrow C_B(z^{-1}xz) = C_B(x)\forall z \in X.$$

We know that,

$$C_{xB}(z) = C_B(x^{-1}z) \text{ and } C_{yB}(z) = C_B(y^{-1}z).$$

Then

$$C_{xyB}(z) = C_B((xy)^{-1}z).$$

Now,

$$\begin{aligned} C_{xB\circ yB}(z) &= \bigvee_{z=rs}(C_{xB}(r) \wedge C_{yB}(s)) \\ &= \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

And

$$\begin{aligned} C_{xyB}(z) = C_B((xy)^{-1}z) &= C_B(y^{-1}x^{-1}z) \\ &\geq \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

Suppose by hypothesis,

$$C_B(y^{-1}x^{-1}z) = \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)),$$

then it follows that $C_{xyB}(z) = C_{xB\circ yB}(z)\,\forall z \in X$. Consequently, we have $f(xy) = f(x)f(y)\,\forall x, y \in X$. Therefore, f is a homomorphism. □

Corollary 4.1. *Let $A, B \in MG(X)$ such that $C_A(x) = C_A(y)\forall x, y \in X$ and $C_A(e) \geq C_B(x)\;\forall x \in X$. If $f : A \to A/B$ is a natural homomorphism defined by $C_{f(A)}(y) = C_B(x^{-1}y)\;\forall\; x, y \in X$, then $f^{-1}(f(B)) = A \circ B$.*

Proof. Let $x \in X$. To proof the result, we assume that $f(x) = f(y)\forall x, y \in X$. Thus,

$$\begin{aligned} C_{f^{-1}(f(B))}(x) &= \bigvee_{x\in X}(C_{f(B)}(f(x))), f(x) = f(y) \\ &= \bigvee_{x\in X}(C_B(f^{-1}(f(y))))\;\forall y \in X \\ &= C_B(y). \end{aligned}$$

Again,

$$
\begin{aligned}
C_{A\circ B}(x) &= \bigvee_{x=zy}(C_A(z) \wedge C_B(y))\ \forall y, z \in X \\
&= \bigvee_{x\in X}(C_A(xy^{-1}) \wedge C_B(y)), z = xy^{-1}\ \forall y, z \in X \\
&= \bigvee_{x\in X}(C_A(e) \wedge C_B(y))\ \forall y \in X \\
&= C_B(y).
\end{aligned}
$$

$\Rightarrow f^{-1}(f(B)) = A \circ B.$ □

Remark. We assume there is a bijective correspondence between every (normal) submultigroup of A that contains B and the (normal) submultigroups of A/B; if C is a (normal) submultigroup of A containing B, then the corresponding (normal) submultigroup of A/B is $f(C)$.

Theorem 4.6. *Let $f : X \to Y$ be an isomorphism of groups and A a normal submultigroup of $B \in MG(X)$ such that $C_B(x) = C_B(y)\ \forall x, y \in X$ with $kerf = \{e\}$. Then $B/A \cong f(B)/f(A)$.*

Proof. By Theorem 2.2 and Definition 4.1, B/A and $f(B)/f(A)$ are multigroups. Let

$$h : B/A \to f(B)/f(A)$$

be defined as

$$h(Ax) = f(A)(f(x))\forall x \in X.$$

If $Ax = Ay$, then $C_A(xy^{-1}) = C_A(e)$. Since $kerf = \{e\}$ meaning $kerf \subseteq A^*$, then $f^{-1}(f(A)) = A$ by Lemma 2.1. Thus,

$$C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e),$$

that is,

$$C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)),$$

then

$$C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(f(e)),$$

so

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)e') \text{ (where } f(e) = e').$$

Hence,

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)) \Rightarrow f(A)(f(x)) = f(A)(f(y)).$$

Hence, h is well-defined. It is also a homomorphism because

$$
\begin{aligned}
h(AxAy) = h(Axy) &= f(A)(f(xy)) \\
&= f(A)(f(x)f(y)) \\
&= f(A)(f(x))f(A)(f(y)) \\
&= h(Ax)h(Ay).
\end{aligned}
$$

Suppose f is an epimorphism, then $\exists\, x \in X$ such that $f(x) = y$. So,

$$h(Ax) = f(A)(f(x)) = f(A)(y).$$

Moreover,

$$f(A)(f(x)) = f(A)(f(y)) \Rightarrow C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(e') \Rightarrow$$
$$C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)) \Rightarrow C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e)$$

implies $C_A(xy^{-1}) = C_A(e) \Rightarrow Ax = Ay$, which proves that h is an isomorphism. Hence, the result follows. □

Corollary 4.2. *Let $f : X \to Y$ be an isomorphism of groups and B a normal submultigroup of $A \in MG(Y)$ such that $C_A(x) = C_A(y)\ \forall x, y \in Y$. Then $f(A)/f(B) \cong A/B$.*

Proof. By Theorem 2.2, $f(A), f(B) \in MG(X)$ and $f(A)/f(B)$ and A/B are multigroups by Definition 4.1. Again, since $B \in MG(Y)$, then $f(f^{-1}(B)) = B$ by Lemma 2.1. If $x \in kerf$, then $f(x) = e' = f(e)$, and so

$$C_B(f(x)) = C_B(f(e)),$$

that is,

$$C_{f^{-1}(B)}(x) = C_{f^{-1}(B)}(e).$$

Hence, $x \in f^{-1}(B)$, that is, $kerf \subseteq f^{-1}(B^*)$. The proof is completed following the same process as in Theorem 4.6. □

Theorem 4.7. *Let $A, B \in MG(X)$ and A a normal submultigroup of B. Then $B/B_* \approx B/A$.*

Proof. Let f be a natural homomorphism from B_* onto B_*/A_* defined by $f(xA_*) = xA_*\ \forall x \in B_*$. Then we have

$$C_{f(B/B_*)}(xA_*) = \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_*.$$

Since B/B_* and B are bijective correspondence to each other and $z = f^{-1}(xA_*) = xA_*$, it follows that

$$\begin{aligned} C_{f(B/B_*)}(xA_*) &= \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_* \\ &= \vee(C_B(y)), \forall y \in xA_* \\ &= C_{B/A}(xA_*)\forall x \in B_*, \end{aligned}$$

because B/A and B are bijective correspondence to each other. Therefore, $B/B_* \approx B/A$. □

Lemma 4.1. *If $f : X \to Y$ and $A \in MG(X)$, then $(f(A))_* = f(A_*)$.*

Proof. Straightforward. □

Theorem 4.8. *Let $B \in MG(X)$. Suppose Y is a group and $C \in MG(Y)$ such that $B \approx C$. Then there exists a normal submultigroup A of B such that $B/A \cong C/C_*$.*

Proof. Since $B \approx C$, $\exists$ an epimorphism f of X onto Y such that $f(B) = C$. Define $A \in MG(X)$ as follows: $\forall\, x \in X$,

$$C_A(x) = \begin{cases} C_B(x) & \text{if } x \in kerf \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $A \subseteq B$. If $x \in kerf$, then $yxy^{-1} \in kerf\ \forall y \in X$, and so

$$C_A(yxy^{-1}) = C_B(yxy^{-1}) \geq C_B(x) = C_A(x)\ \forall y \in X.$$

If $x \notin kerf$, then $C_A(x) = 0$ and so

$$C_A(yxy^{-1}) \geq C_A(x) = 0\ \forall y \in X.$$

Hence, A is a normal submultigroup of B. Also, $B \approx C \Rightarrow f(B) = C$ which further implies $(f(B))_* = C_*$ and $f(B_*) = C_*$ by Lemma 4.1. Let $f = g$. Then g is a homomorphism of B_* onto C_* and $kerg = A_*$. Thus, there exists an isomorphism h of B_*/A_* onto C_* such that $h(xA_*) = g(x) = f(x)\ \forall x \in B_*$. For such an h, we have

$$\begin{aligned} C_{h(B/A)}(z) &= \vee(C_{B/A}(xA_*)), \forall x \in B_*, h(xA_*) = z \\ &= \vee(\vee[C_B(y)], \forall y \in xA_*), \forall x \in B_*, g(x) = z \\ &= \vee(C_B(y)), \forall y \in B_*, g(y) = z \\ &= \vee(C_B(y)), \forall y \in X, f(y) = z \\ &= C_B(f^{-1}(z)) = C_{f(B)}(z) = C_C(z), \forall z \in C_*. \end{aligned}$$

Therefore, $B/A \cong C/C_*$. □

Theorem 4.9. *Let $B \in MG(X)$ and A be a normal submultigroup of B. Then $B/A \cap B \simeq A \circ B/A$.*

Proof. From Proposition 2.1, we infer that A_* is also a normal subgroup of X. By the Second Isomorphism Theorem for groups, we deduce

$$B_*/A_* \cap B_* \cong A_*B_*/A_*.$$

We know that

$$(A \cap B)_* = A_* \cap B_*,$$
$$(A \circ B)_* = A_*B_*.$$

Consequently, we have

$$B_*/(A \cap B)_* \cong (A \circ B)_*/A_*,$$

where f is given by

$$f(x(A \cap B)_*) = xA_* \forall x \in B_*.$$

Thus,

$$\begin{aligned} C_{f(B/A\cap B)}(yA_*) &= C_{B/A\cap B}(y(A \cap B)_*) \\ &= \vee(C_B(z)), \forall z \in y(A \cap B)_* \\ &\leq \vee(C_{A\circ B}(z)), \forall z \in y(A_* \cap B_*) \\ &\leq \vee(C_{A\circ B}(z)), \forall z \in yA_* \\ &= C_{A\circ B/A}(yA_*),\ \forall y \in B_*. \end{aligned}$$

Hence, $f(B/A \cap B) \subseteq A \circ B/A$. Therefore, $B/A \cap B \simeq A \circ B/A$. □

Theorem 4.10. *Let $A, B, C \in MG(X)$ such that $A \subseteq B$, and A and B are normal submultigroups of C. Then $(C/A)/(B/A) \cong C/B$.*

Proof. If $A, B \in MG(X)$ and A is a normal submultigroup of B, obviously, A_* is a normal subgroup of B_* and both A_* and B_* are normal submultigroups of C_*. From the principle of Third Isomorphism Theorem for groups, we have

$$(C_*/A_*)/(B_*/A_*) \cong C_*/B_*,$$

where f is given by

$$f(xA_*(B_*/A_*)) = xB_* \ \forall x \in C_*.$$

Then

$$\begin{aligned}
C_{f((C/A)/(B/A))}(xB_*) &= C_{(C/A)/(B/A)}(xA_*(B_*/A_*)) \\
&= \vee(C_{C/A}(yA_*)), \forall y \in C_*, yA_* \in xA_*(B_*/A_*) \\
&= \vee(\vee[C_C(z)], \forall z \in yA_*), \forall y \in C_*, yA_* \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in C_*, zA_* \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in C_*, f(z) \in xB_* \\
&= \vee(C_C(z)), \forall z \in C_*, f(z) = z \\
&= C_{C/B}(xB_*)
\end{aligned}$$

$\forall x \in C_*$, where the equalities hold since f is one-to-one. Hence, the result follows. □

5. Conclusion

An indepth work on comultisets had been carried out and some results were deduced. We have extended the notion of factor groups to multigroups and explicated some properties of factor multigroups. Finally, we explored some homomorphic properties of factor multigroups. Nonetheless, more results on comultisets and factor multigroups could be exploited.

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