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A Common Fixed Point Theorem in Cone Metric Spaces over Banach Algebras

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Abstract

In this paper, a common fixed point theorem for four mappings in cone metric spaces over Banach algebras is proved without assuming the normality of underlying cone. The results of this paper unify, generalize and extend some known results in cone metric spaces over Banach algebras. An example is presented which shows the significance of the result proved herein.

Keywords: Cone metric space, coincidence point, common fixed point.
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1. Introduction

The study of K-metric and K-normed spaces were introduced in the mid-20th century (Aliprantis & Tourky, 2007; Kantorovich, 1957; Vandergraft, 1967; Zabreĭko, 1997). In these papers, the set of real numbers was replaced by an ordered Banach space, as the codomain for a metric. In 2007, such spaces are reintroduced by (Huang & Zhang, 2007) under the name of cone metric spaces. (Huang & Zhang, 2007) defined convergent and Cauchy sequences in cone metric spaces in terms of interior points of underlying cone. Some basic versions of the fixed point theorems in cone metric spaces can be found in (Huang & Zhang, 2007). (Abbas & G.Jungck, 2008) proved some common fixed point results in these spaces. (Radenović, 2009) obtained a coincidence point theorem for two mappings in this new setting, which satisfy a new type of contractive condition. The result of (Radenović, 2009) was extended by (Rangamma & Prudhvi, 2012) for three mappings which satisfy a generalized contractive condition without exploiting the notion of continuity.

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In the papers (Radenović, 2009) and (Rangamma & Prudhvi, 2012) the contractive conditions were generalized by using the norm function. Notice that, the norm function is defined from the Banach space into the set of real numbers, hence, the results of (Huang & Zhang, 2007) can not be obtained by the results of (Rangamma & Prudhvi, 2012). Inspired by this fact, (Malhotra *et al.*, 2012) used a more competent function ϕ instead of the norm function $\|\cdot\|$. The benefit of using the function ϕ was that, the new results generalize and unify the results of (Rangamma & Prudhvi, 2012) as well as (Huang & Zhang, 2007) and (Vetro, 2007). (Malhotra *et al.*, 2012) defined a function ϕ from a normal cone into another normal cone, and so, their results cannot be applied if the cone is non-normal.

Some recent studies (see, (Çakallı *et al.*, 2012; Du, 2010; Feng & Mao, 2010; Kadelburg *et al.*, 2011)) show that the fixed point results proved in cone metric space are direct consequences of their usual metric versions. To overcome this drawback, recently, (Liu & Xu, 2013) improved the concept of cone metric spaces by defining the cone metric with values in a Banach algebra, instead of a Banach space, so that, the contractive conditions can involve the vector constants. The fixed point results thus obtained cannot be derived by their usual metric versions which was shown by an example in (Liu & Xu, 2013).

Inspired by the results of (Liu & Xu, 2013), in this paper, we prove some coincident and common fixed point results for four mappings in cone metric spaces over Banach algebras with solid cone which are not necessarily normal. We improve the definition of function ϕ used by (Malhotra *et al.*, 2012), by removing the normality condition from the domain and codomain cones of ϕ, as well as, we use a vector constant, instead the scalar in the contractive condition involving the function ϕ. Our result generalizes and unifies the results of (Liu & Xu, 2013), (Radenović, 2009) and (Rangamma & Prudhvi, 2012) and several other results, in cone metric spaces over Banach algebras.

2. Preliminaries

We first state some known definitions and facts which will be used throughout the paper.

Let E be a real Banach algebra with a unit e_E and a zero element 0_E. A nonempty closed subset P of E is called a cone if the following conditions hold:

(1) $\{0_E, e_E\} \subset P$;

(2) if $\alpha, \beta \in [0, \infty)$, then $\alpha P + \beta P \subseteq P$;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{0_E\}$.

A cone P is called a solid cone if P° is nonempty, where P° stands for the interior of P.

We can always define a partial ordering $\preceq_P$ with respect to P by $x \preceq_P y$ if and only if $y - x \in P$. We shall write $x \ll_P y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on E. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0_E \preceq_P x \preceq_P y$ implies $\|x\| \leq K\|y\|$.

Throughout the paper, we consider the real Banach algebras.

Definition 2.1. (Liu & Xu, 2013) Let X be a nonempty set and E be a Banach algebra. A mapping $d\colon X \times X \to E$ is called a cone metric if it satisfies:

(i) $0_E \preceq_P d(x,y)$, for all $x, y \in X$, $d(x,y) = 0_E$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \preceq_P d(x,z) + d(z,y)$ for all $x, y \in X$.

In this case, the pair (X,d) is called a cone metric space over Banach algebra E. If the cone P is normal then (X,d) is called a normal cone metric space.

Definition 2.2. (Dordević *et al.*, 2011) A sequence $\{u_n\}$ in a P is said to be a c-sequence in P if for each $c \gg_P 0_E$ (i.e., $0_E \ll_P c$), there exists $N \in \mathbb{N}$ such that $u_n \ll_P c$ for all $n > N$.

Definition 2.3. (Huang *et al.*, 2017) Let (X,d) be a cone metric space over Banach algebra E and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n,x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n,x_m)\}$ is a c-sequence for n,m, i.e., for each $c \gg_P 0_E$, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m) \ll_P c$ for all $n,m > N$.

(iii) (X,d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X,d) over Banach algebra E is unique.

Lemma 2.1. *(Janković* et al.*, 2011) Let E be a Banach algebra and $u, v, w \in E$. Then*
(1) $u \ll_P w$ if $u \preceq_P v \ll_P w$ or $u \ll_P v \preceq_P w$;
(2) $u = 0_E$ if $0_E \preceq_P u \ll_P c$ for each $c \gg_P 0_E$.

The following results are well known and will be used in the sequel.

Lemma 2.2. *Let E be a Banach algebra and $u \in E$. Then the spectral radius of u is equal to* $\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.

Lemma 2.3. *Let E be a Banach algebra and $k \in E$. If $\rho(k) < \lambda$, for some $\lambda > 0$ then $\lambda e_E - k$ is invertible in E, moreover,* $(\lambda e_E - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$.

Lemma 2.4. *(Huang* et al.*, 2016) Let P be a cone in a Banach algebra E, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in P, and $\alpha, \beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in P.*

Lemma 2.5. *(Huang & Radenović, 2015) Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence in P.*

Let X be a nonempty set and f, g be two self-maps on X and $x, z \in X$. Then x is called coincidence point of pair (f,g) if $fx = gx$, and z is called point of coincidence of pair (f,g) if $fx = gx = z$. The pair (f,g) is called weakly compatible if f and g commutes at their coincidence point, i.e. $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

For results on weakly compatible mappings in cone metric spaces, see (Janković *et al.*, 2010; Jungck *et al.*, 2009).

Now we can state our main results.

3. Main Results

Let E, B be two real Banach algebras, P and C be solid cones in E and B respectively. Let "$\preceq_P$" and "$\preceq_C$" be the partial orderings induced by P and C in E and B respectively, 0_E and 0_B are the zero vectors of E and B respectively; and e_E and e_B are the units of E and B respectively.

The following definition of function ϕ is an improved version of the definition used by (Malhotra *et al.*, 2012) (see, also, (Khan *et al.*, 2015)). Let $\phi : P \to C$ be a function satisfying:

(i) if $a, b \in P$ with $a \preceq_P b$ then $\phi[a] \preceq_C k\phi[b]$, for some positive real k;
(ii) $\phi[a + b] \preceq_C \phi[a] + \phi[b]$ for all $a, b \in P$;
(iii) the sequence $\{\phi[a_n]\}$ is c-sequence in C if and only if the sequence $\{a_n\}$ is a c-sequence in P.

We denote the set of all such functions by $\Phi(P, C)$, i.e., $\phi \in \Phi(P, C)$ if ϕ satisfies all above properties. It is clear that $\phi[a] = 0_B$ if and only if $a = 0_E$.

Let (X, d) be a cone metric space with solid cone P and $\phi \in \Phi(P, C)$. Then $d(x, y) \preceq_P d(x, z) + d(z, y)$ for all $x, y, z \in X$, therefore

$$\phi[d(x, y)] \preceq_C k\phi[d(x, z)] + k\phi[d(z, y)]. \tag{3.1}$$

Example 3.1. Let $E = L[0, 1]$ be the real Banach algebra of integrable functions $f(x)$ such that $\int_0^1 |f(x)|dx < \infty$ with norm $\|f\| = \int_0^1 |f(x)|dx$, the point-wise multiplication and the unit 1. Let $P = \{f \in E\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in E. Let $B = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, point-wise multiplication and unit 1. Let $C = \{f \in B\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in B. Define $\phi\colon P \to B$ by $\phi[f] = \int_0^t f(x)dx$, $t \in [0, 1]$ for all $f \in P$. Then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.2. Let E be a Banach algebra with solid cone P. Define $\phi\colon P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Example 3.3. Let $E = \mathbb{R}^2, P = \{(a, b) : a, b \in \mathbb{R}$ and $a, b \geq 0\}$ and $B = \mathbb{R}^3, C = \{(a, b, c) : a, b, c \in \mathbb{R}$ and $a, b, c \geq 0\}$, with coordinatewise multiplication and units $(1, 1)$ and $(1, 1, 1)$ respectively. Then P and C are solid cones. Define $\phi\colon P \to C$ by $\phi[(x, y)] = (x, y, ax + by)$, where a, b are positive constants, then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.4. Let E be any real Banach algebra with normal cone P and normal constant K. Define $\phi : P \to [0, \infty)$ by $\phi[a] = \|a\|$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $B = \mathbb{R}, C = [0, \infty)$ and $k = K$.

Example 3.5. Let E be the real vector space defined by

$$E = \{ax + b\colon a, b \in \mathbb{R}, x \in [1/2, 1]\}$$

with supremum norm and $P = \{ax + b \in E\colon a \leq 0, b \geq 0\}$. Then E is a real Banach algebra with point-wise multiplication and unit $e_E = 1$ and P is a normal cone with normal constant $K > 2$ (see (Rezapour & Hamlbarani, 2008)). Let $B = \mathbb{R}^2$ be with Euclidean norm, coordinate-wise

multiplication, unit $e_B = (1, 1)$ and $C = \{(a, b) : a \geq 0, b \geq 0\}$. Then C is a normal cone with normal constant $K = 1$. Define $\phi : P \to C$ by

$$\phi[ax + b] = (-a, b) \text{ for all } ax + b \in P.$$

Then, $\phi \in \Phi(P, C)$ with $k = 1$.

The following theorem is an improved version of the main results of (Radenović, 2009), (Rangamma & Prudhvi, 2012) and (Liu & Xu, 2013), and unifies and generalizes these results.

Theorem 3.1. *Let (X, d) be a complete cone metric space over Banach algebra E and P be a solid cone. Suppose that f, g, h, l be four self-maps of X, $f(X) \subset l(X), g(X) \subset h(X)$ and the following condition is satisfied: there exist $\phi \in \Phi(P, C)$ and $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$\phi[d(fx, gy)] \preceq_C \alpha\phi[d(hx, ly)] \text{ for all } x, y \in X. \tag{3.2}$$

If $h(X)$, $l(x)$ are closed subsets of X, and the pairs (f, h), (g, l) are weakly compatible, then the mappings f, g, h and l have a unique common fixed point.

Proof. Suppose, x_0 be any arbitrary point of X. Since $f(X) \subset l(X)$, there exists $x_1 \in X$ such that $fx_0 = lx_1$. Again, as $g(X) \subset h(X)$, there exists $x_2 \in X$ such that $gx_1 = hx_2$. Continuing in this manner, we obtain a sequence $\{z_n\}$ such that

$$\begin{aligned} z_{2n} &= fx_{2n} = lx_{2n+1}, \\ z_{2n+1} &= gx_{2n+1} = hx_{2n+2} \text{ for all } n \geq 0. \end{aligned}$$

We shall prove that $\{z_n\}$ is a Cauchy sequence in X.

Note that, if there exists $n \in \mathbb{N}$ such that $z_n = z_{n+1}$, e.g., suppose, $z_{2n_0} = z_{2n_0+1}$, then it follows from (3.2) that

$$\begin{aligned} \phi[d(z_{2n_0+2}, z_{2n_0+1})] &= \phi[d(fx_{2n_0+2}, gx_{2n_0+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n_0+2}, lx_{2n_0+1})] \\ &= \alpha\phi[d(z_{2n_0+1}, z_{2n_0})]. \end{aligned}$$

As, $z_{2n_0} = z_{2n_0+1}$ the above inequality yields

$$\phi[d(z_{2n_0+2}, z_{2n_0+1})] = 0_B.$$

As, $\phi \in \Phi(P, C)$ therefore the above equality implies that $d(z_{2n_0+2}, z_{2n_0+1}) = 0_E$, i.e., $z_{2n_0+2} = z_{2n_0+1}$. Similarly, we obtain that

$$z_{2n_0} = z_{2n_0+1} = z_{2n_0+2} = z_{2n_0+3} = \cdots .$$

Therefore, $\{z_n\}$ is a Cauchy sequence.

Now, suppose that z_n and z_{n+1} are distinct for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n}, z_{2n+1})] &= \phi[d(fx_{2n}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n-1}, z_{2n})]. \end{aligned}$$

Writing $d_n = \phi[d(z_n, z_{n+1})]$, we obtain

$$d_{2n} \preceq_C \alpha d_{2n-1} \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Again, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n+2}, z_{2n+1})] &= \phi[d(fx_{2n+2}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n+2}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n+1}, z_{2n})]. \end{aligned}$$

It follows from the above inequality that

$$d_{2n+1} \preceq_C \alpha d_{2n} \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$d_n \preceq_C \alpha d_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

Repeated use of (3.5) that

$$d_n \preceq_C \alpha d_{n-1} \preceq_C \alpha^2 d_{n-2} \preceq_C \cdots \preceq_C \alpha^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Let $n, m \in \mathbb{N}$ and $m > n$, then by (3.1) and (3.6) we obtain

$$\begin{aligned} \phi[d(z_n, z_m)] &\preceq_C k\phi[d(z_n, z_{n+1})] + k\phi[d(z_{n+1}, z_{n+2})] + \cdots + k\phi[d(z_{m-1}, z_m)] \\ &= k[d_n + d_{n+1} + \cdots + d_{m-1}] \\ &\preceq_C k[\alpha^n d_0 + \alpha^{n+1} d_0 + \cdots + \alpha^{m-1} d_0] \\ &= k\alpha^n [e_B + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}] d_0 \\ &\preceq_C k\alpha^n \left[\sum_{i=0}^{\infty} \alpha^i\right] d_0 \\ &= k\alpha^n (e_B - \alpha)^{-1} d_0. \end{aligned}$$

Since $\rho(\alpha) < 1$, therefore, by Lemma 2.4 and Lemma 2.5, the sequence $\{k\alpha^n (e_B - \alpha)^{-1} d_0\}$ is a c-sequence in C. Hence, by Lemma 2.1 and the fact that $\phi \in \Phi(P, C)$, we have, the sequence $\{z_n\} = \{hx_{n+1}\}$ is a Cauchy sequence.

Since X is complete, there exists $w \in X$ such that $z_n \to w$ as $n \to \infty$. Since, $z_{2n} = lx_{2n+1} \in l(X)$, $z_{2n+1} = hx_{2n+2} \in h(X)$ for all $n \in \mathbb{N}$ and $l(X)$, $h(X)$ are closed subsets of X, there exist $u, v \in X$ such that

$$w = lu = hv.$$

Therefore, from by (3.1) and (3.2) we obtain

$$\begin{aligned} \phi[d(fv, w)] &\preceq_C k\phi[d(fv, gx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &\preceq_C k\alpha\phi[d(hv, lx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &= k\alpha\phi[d(w, z_{2n})] + k\phi[d(z_{2n+1}, w)]. \end{aligned}$$

Since, $z_n \to w$ as $n \to \infty$, the sequences $\{d(w, z_{2n})\}$ and $\{d(z_{2n+1}, z)\}$ are c-sequences in P. As, $\phi \in \Phi(P, C)$, the sequences $\phi[\{d(w, z_{2n})\}]$ and $\phi[\{d(z_{2n+1}, z)\}]$ are c-sequences in C. By Lemma 2.1, Lemma 2.4 and the above inequality, the sequence $\{\phi[d(fv, w)]\}$ is a c-sequence in C. This shows that the sequence $\{d(fv, w)\}$ is a c-sequence in P, and so, we must have $d(fv, w) = 0_E$, i.e., $fv = w$. Therefore

$$\begin{aligned} \phi[d(w, gu)] &= \phi[d(fv, gu)] \\ &\preceq_C \alpha\phi[f(hv, lu)] \\ &= \alpha\phi[f(w, w)] \\ &= \theta_B. \end{aligned}$$

Hence, $d(w, gu) = 0_E$, i.e., $w = gu$. Thus

$$w = gu = lu = hv = fv. \tag{3.7}$$

As, the pairs (f, h), (g, l) are weakly compatible it follows from (3.7) that $fw = fhv = hfv = hw$ and $gw = glu = lgu = lw$. Hence

$$gw = lw = hw = fw. \tag{3.8}$$

Using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} \phi[d(w, gw)] &= \phi[d(fv, gw)] \\ &\preceq_C \alpha\phi[d(hv, lw)] \\ &= \alpha\phi[d(w, gw)]. \end{aligned}$$

Thus, $\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)]$. Successive use of this inequality yields

$$\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)] \preceq_C \alpha^2\phi[d(w, gw)] \preceq_C \cdots \preceq_C \alpha^n\phi[d(w, gw)].$$

As, $\rho(\alpha) < 1$, the sequence $\{\alpha^n\}$ is a c-sequence in C, and by Lemma 2.1, Lemma 2.4 we obtain that the sequence $\{\phi[d(w, gw)]\}$ is a c-sequence in C. By definition, $d(w, gw)$ is a c-sequence in P, and so, $d(w, gw) = 0_E$, i.e., $gw = w$. Hence, we obtain from (3.8) that

$$w = gw = lw = hw = fw. \tag{3.9}$$

Thus, w is a common fixed point of the mappings f, g, h and l.

For uniqueness of fixed point, suppose w' is a common fixed point of the mappings f, g, h and l and w and w' are distinct. Then, we have

$$w' = gw' = lw' = hw' = fw'.$$

Using (3.2) we obtain

$$\begin{aligned} \phi[d(w, w')] &= \phi[d(fw, gw')] \\ &\preceq_C \alpha\phi[d(hw, lw')] \\ &= \alpha\phi[d(w, w')]. \end{aligned}$$

Again, since $\rho(\alpha) < 1$, the above inequality yields $\phi[d(w, w')] = 0_B$. This shows that $d(w, w') = 0_E$, i.e., $w = w'$. This contradiction proves the uniqueness of common fixed point. □

The following corollary is a generalized version of Theorems 2.1 of (Rangamma & Prudhvi, 2012) and Theorem 2.1 of (Radenović, 2009).

Corollary 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists a number $a \in [0, 1)$ such that*

$$\|d(fx, gy)\| \leq a\|d(hx, hy)\| \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $B = \mathbb{R}, C = [0, \infty)$, $lx = hx$ for all $x \in X$, and $\phi[a] = \|a\|$ for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = K$ =normal constant of P and the result follows from Theorem 3.1. □

The following corollary is an improved and generalized version of (Huang & Zhang, 2007) and (Liu & Xu, 2013).

Corollary 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$d(fx, gy) \preceq \alpha d(hx, hy) \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $E = B, P = C$ and define $\phi : P \to P$ by $\phi[a] = a$, for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = 1$ and the result follows from Theorem 3.1. □

Example 3.6. Let $E = \mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ and the multiplication defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. The unit of E is $e_E = (1, 0)$. Let $P = \{(x_1, x_2) : x_1, x_2 \geq 0\}$. Then, P is a solid cone. Define $\phi : P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Let $X = \mathbb{R}^2$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|).$$

Then, (X, d) is a complete cone metric space over Banach algebra E. Define the mappings $f, g, h : X \to X$ by

$$f(x_1, x_2) = g(x_1, x_2) = (\ln(1 + |x_1|), \arctan(2 + |x_2|) + 2ax_1), \; g(x_1, x_2) = h(x_1, x_2) = (2x_1, x_2)$$

for all $(x_1, x_2) \in X$, where $a > 0$. Then, it is easy to see that the condition (3.2) is satisfied with $\alpha = \left(\frac{1}{2}, a\right)$. Obviously, $h(X)$ is a closed subset of X and $f(X) \subset h(X)$. It is easy to see that f and h are weakly compatible. All the conditions of Theorem 3.1 are satisfied, hence f and h has a unique common fixed point.

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Introduction to Non-Diophantine Number Theory

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Abstract

In the 19^{th} century, non-Euclidean geometries were discovered and studied. In the 20th century, non-Diophantine arithmetics were discovered and studied. Construction of non-Diophantine arithmetics is based on more general mathematical structures, which are called abstract prearithmetics, as well as on the projectivity relation between abstract prearithmetics. In a similar way, as set theory gives a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of the Diophantine and non-Diophantine arithmetics. In this paper, we use abstract prearithmetics for developing fundamentals of non-Diophantine number theory, which can be also called non-Diophantine higher arithmetic as the conventional number theory is called higher arithmetic. In particular, we prove the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics.

Keywords: number, arithmetic, number theory, divisibility, prime number, factorization, addition, difference, multiplication

1. Introduction

The arithmetic $\mathbb{N}$ of all natural numbers is one of the most basic objects in mathematics. People in general and mathematicians in particular believe that the laws of this arithmetic are universal and unique. The equality $2 + 2 = 4$ is treated as an eternal absolute truth. However, the best thinkers started questioning universality of $\mathbb{N}$ long ago suggesting various examples when the rules of this arithmetic, which is called the Diophantine arithmetic, are not true (cf., for example, ((Helmholtz, 1887), (Kline, 1982), (Kline, 1985), (Davis, 1972) (Davis & Hersh, 1998), (Burgin, 1997), (Burgin, 2001), (Gardner, 2005), (Cleveland, 2008)).

Here we give only three of such examples although it is possible to find much more.

1. One raindrop added to another raindrop does not make two raindrops (Helmholtz, 1887). Mathematically, it is described by the equality $1 + 1 = 1$.
2. If one puts a lion and a rabbit in a cage, one will not find two animals in the cage later on (cf. (Kline, 1985)). In terms of numbers, it will mean $1 + 1 = 1$.

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3. When a cup of milk is added to a cup of popcorn then only one cup of mixture will result because the cup of popcorn will very nearly absorb a whole cup of milk without spillage (Davis & Hersh, 1998). So, in this case we also have 1 + 1 = 1.

Besides, recently the expression 1 + 1 = 3 has become a popular metaphor for synergy in a variety of areas: in business and industry (cf., for example, (Beechler, 2013), (Gottlieb, 2013), (Grant & Johnston, 2013), (Marks & Mirvis, 2010)), in economics and finance (cf., for example, (Burgin & Meissner, 2017)), in psychology and sociology (cf., for example, (Brodsky *et al.*, 2004), (Bussmann, 2013), (Enge, 2017), (Frame & Meredith, 2008), (Klees, 2006), (Mane, 1952), (Trott, 2015)), library studies (cf., for example, (Marie, 2007)), biochemistry and bioinformatics (cf., for example, (Kroiss *et al.*, 2009)), computer science (cf., for example, (Derboven, 2011), (Glyn, 2017), (Lea, 2016)), physics (cf., for example, (Lang, 2014)), medicine (cf., for example, (Lawrence, 2011), (Phillips, 2016), (Trabacca *et al.*, 2012)) and pedagogy (cf., for example, (Nieuwmeijer, 2013)).

All these situations indicated existence of other non-Diophantine arithmetics in which it would be possible to describe all these situations in a rigorous mathematical way. Thus, the first class of non-Diophantine arithmetics was discovered and explored in 1975 while the first publication appeared in 1977 (Burgin, 1977). Later other classes were introduced (Burgin, 2010). Recently non-Diophantine arithmetics found applications in physics (Czachor, 2016), (Czachor, 2017*a*), (Czachor & Posiewnik, 2016) and psychology (Czachor, 2017*b*). Following the classical understanding of arithmetic, here non-Diophantine arithmetics are considered as arithmetics of natural numbers.

Construction of non-Diophantine arithmetics is based on more general mathematical structures, which are called abstract prearithmetics, as well as on projectivity between abstract prearithmetics. (Burgin, 2010). In a similar way, as set theory is a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of non-Diophantine arithmetics and the Diophantine arithmetic. In addition, the theory of abstract prearithmetics includes theories of various conventional mathematical structures, such as rings, fields, ordered rings, ordered fields, lattices and Boolean algebras, as its subtheories. This allows using constructions from the theory of abstract prearithmetics for its subtheories of conventional mathematical structures. Abstract prearithmetics also provide a unified algebraic context for some traditional mathematical constructions, such as logarithmic scales, modular arithmetics and computer arithmetics, which are used in many applications in mathematics, science and technology.

In essence, an abstract prearithmetic is a universal algebra (algebraic system) with two operations and a partial order. Operations are called addition and multiplication but in a general case, there are no restrictions on these operations. Some of abstract prearithmetics are numerical, that is, their elements are numbers, e.g., natural numbers or real numbers. A numerical prearithmetic that satisfies additional conditions, in particular, containing all natural numbers and no other elements is called an arithmetic of natural numbers. A numerical prearithmetic that satisfies additional conditions, in particular, contains all whole numbers and no other elements is called an arithmetic of whole numbers. Everybody knows the conventional Diophantine arithmetic $\mathbb{N}$. However, there are also non-Diophantine arithmetics of natural numbers introduced and studied in (Burgin, 1977), (Burgin, 1997), (Burgin, 2001), (Burgin, 2007), (Burgin, 2010).

In this paper, we use abstract prearithmetics for developing fundamentals of non-Diophantine number theory, which can be also called non-Diophantine higher arithmetic as the conventional number theory is called higher arithmetic (cf., for example, (Broadbent, 1971), (Davenport, 1999), (Hayes, 2009). Number theory has three basic goals:

- Exploration of properties of and relations between natural numbers
- Classification of natural numbers and formation of important and interesting classes of natural numbers
- Exploration of properties of and relations between classes of natural numbers.

In this paper, we pursue these goals in the context of abstract prearithmetics. We pay the main attention to the problems of divisibility and primality. In particular, we prove the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics.

In what follows, we use the following notation:

- N is the set of all natural numbers,
- $\mathbb{N}$ is the conventional (Diophantine) arithmetic of all natural numbers,
- W is the set of all whole numbers
- $\mathbb{W}$ is the conventional (Diophantine) arithmetic of all natural numbers,
- R is the set of all real numbers,
- $\mathbb{R}$ is the conventional (Diophantine) arithmetic of all real numbers.

2. Abstract prearithmetics

An *abstract prearithmetic* is a set (often a set of numbers) A with a partial order $\leq$ and two binary operations + (addition) and $\cdot$ (multiplication), which are defined for all its elements. It is denoted by $\mathbb{A} = (A; +, \cdot, \leq)$. The set A is called *the set of the elements or numbers* or *the carrier* of the prearithmetic $\mathbb{A}$. As always, if $x \leq y$ and $x \neq y$, then we denote this relation by $x < y$. Operation + is called *addition* and operation $\cdot$ is called *multiplication* in the abstract prearithmetic $\mathbb{A}$. Note that an abstract prearithmetic can have more than two operations and more than one order relations.

Example 2.1. Naturally, the conventional Diophantine arithmetic $\mathbb{N}$ of all natural numbers, the conventional arithmetic $\mathbb{R}$ of all real numbers and the conventional Diophantine arithmetic $\mathbb{W}$ of all whole numbers are an abstract prearithmetics.

Example 2.2. Another example of abstract prearithmetics is a *modular arithmetic*, which is sometimes known as *residue arithmetic* or *clock arithmetic* (Kurosh, 1963)). It is studied in mathematics and used in physics and computing. In modular arithmetic, operations of addition and multiplication are defined but in contrast to the conventional arithmetic, its numbers form a cycle upon reaching a certain value, which is called *the modulus*. The rigorous approach to the theory of modular arithmetic was worked out by Carl Friedrich Gauss.

All these examples show that conventional arithmetics are abstract prearithmetics. However, there are many unusual abstract prearithmetics.

Example 2.3. Let us consider the set N of all natural numbers with the standard order $\leq$ and introduce the following operations:

$$a \oplus b = a \cdot b$$

$$a \otimes b = a^b$$

Then the system $\mathbb{A} = (\mathbb{N}; \oplus, \otimes, \leq)$ is an abstract prearithmetic with addition $\oplus$ and multiplication $\otimes$.

Example 2.4. Let us consider the set $\mathbb{R}^{++}$ of all positive real numbers is with the standard order $\leq$ and introduce the following operations:

$$a \boxplus b = a + b$$

$$a \circledast b = a \div b$$

Then the system $\mathbb{B} = (\mathbb{R}^{++}; \boxplus, \circledast, \leq)$ is an abstract prearithmetic with addition $\boxplus$ and multiplication $\circledast$.

Example 2.5. Many algebraic structures studied in algebra are abstract prearithmetics with a trivial order, i.e., any ring, lattice, Boolean algebra, linear algebra, field, Ω-group, Ω-ring, Ω-algebra (Kurosh, 1963), (Baranovich & Burgin, 1975), topological ring, topological field, normed ring, normed algebra, normed field, and in essence, any universal algebra with two operations is an abstract prearithmetic with a trivial order. The same structures with nontrivial order are also abstract prearithmetics. Examples are given by ordered rings, ordered linear algebras and ordered fields. Besides, it is possible to treat universal algebras with one operation as abstract prearithmetics with a trivial order and trivial multiplication.

Elements 0 and 1 have very special properties in the conventional Diophantine arithmetic. We explore these properties in the general setting of abstract prearithmetics.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$.

Definition 2.1. a) An element z, which is usually denoted by 0 or 0_A , is called an *additive zero* of $\mathbb{A}$ if $a + z = z + a = a$ for any element a from A.

b) An element z, which is usually denoted by 0 or 0_{mA} , is called a *multiplicative zero* of $\mathbb{A}$ if $a \cdot z = z \cdot a = z$ for any element a from A.

c) An element b, which is usually denoted by 1 or 1_A , is called a *multiplicative one* of $\mathbb{A}$ if $a \cdot b = b \cdot a = a$ for any element z from A.

Lemma 2.1. *An additive zero is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two additive zeros 0_1 and 0_2 . Then we have

$$0_1 = 0_1 + 0_2 = 0_2$$

Lemma 2.2. *A multiplicative zero is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two multiplicative zeros 0_1 and 0_2 . Then we have

$$0_1 = 0_1 \cdot 0_2 = 0_2$$

Lemma 2.3. *A multiplicative* 1 *is unique.*

Indeed, suppose an abstract prearithmetic $\mathbb{A}$ has two additive zeros 1_1 and 1_2 . Then we have

$$1_1 = 1_1 \cdot 1_2 = 1_2$$

The number 0 in the conventional Diophantine arithmetic is both additive and multiplicative zero while the number 1 is the multiplicative one. However, in a general case of abstract prearithmetics, additive and multiplicative zeros do not coincide as the following examples demonstrate.

Example 2.6. Let us define an abstract prearithmetic $\mathbb{A} = (N; \oplus, \otimes, \leq)$ where N is the set of all natural numbers by the following rules:

$$m \oplus n = m + n$$

$$m \otimes n = m \cdot n + 3$$

where $m + n$ are arbitrary natural numbers, while + is conventional addition and $\cdot$ is conventional multiplication of natural numbers.

We can see that 0 is the additive zero but not the multiplicative zero in $\mathbb{A}$.

Example 2.7. Let us define an abstract prearithmetic $\mathbb{A} = (Z; \oplus, \otimes, \leq)$ where Z is the set of all integer numbers by the following rules:

$$m \oplus n = m + n + 2$$

$$m \otimes n = m \cdot n$$

where $m + n$ are arbitrary integer numbers, while + is conventional addition and $\cdot$ is conventional multiplication of integer numbers.

We can see that 0 is the multiplicative zero but not the multiplicative zero in $\mathbb{A}$. At the same time, -2 is the additive zero in $\mathbb{A}$.

However, in some cases, additive and multiplicative zeros coincide.

Proposition 1. *If an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *has the additive zero* 0*, contains an opposite element* $-x$ *for each element* x*, multiplication is distributive with respect to addition and preserves opposite elements, i.e.,* $z \cdot (-x) = -(z \cdot x)$ *for any elements* z *and* x *from* A*, then* 0 *is also the multiplicative zero.*

Proof. Let us take an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ that satisfies all initial conditions of the Proposition. We remind that if x is an element from A, then an element y is called opposite to x and denoted by x when $x + y = 0$. Taking arbitrary elements x and z from A, we have

$$0 \cdot x = (z + (-z)) \cdot x = z \cdot x + (-z) \cdot x = z \cdot x + (-(z \cdot x)) = 0$$

The identity $x \cdot 0 = 0$ is proved in a similar way. Consequently, 0 is also the multiplicative zero. □

Proposition is proved.

As a corollary, we obtain a well-known result from the theory of rings (cf. (Kurosh, 1963)).

Corollary 2.1. *In a ring, additive and multiplicative zeros coincide.*

An important property of the Diophantine arithmetic $\mathbb{N}$ is existence of the successor Sx and the predecessor Px for any number x from $\mathbb{N}$. The successor Sx is defined by the following conditions

$$x < S\,x \text{ and if } x \leq z \leq S\,x, \text{ then } z \text{ is equal either to } x \text{ or to } S\,x$$

The predecessor Px is defined by the following conditions

$$Px < x \text{ and if } Px \leq z \leq x, \text{ then } z \text{ is equal either to } x \text{ or to} Px$$

In what follows we assume that all considered abstract prearithmetics have this property, i.e., any element x has the successor Sx and the predecessor Px.

It is possible to extend addition and multiplication in abstract prearithmetics to n-ary addition and multiplication by the following formulas using induction on n.

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

If $\sum_{i=1}^{n-1} a_i$ is defined, then

$$\sum_{i=1}^{n} a_i = (\sum_{i=1}^{n-1} a_i) + a_n$$

In the same way, we have

$$\prod_{i=1}^{1} a_i = a_1$$

$$\prod_{i=1}^{2} a_i = a_1 \cdot a_2$$

If $\prod_{i=1}^{n-1} a_i$ is defined, then

$$\prod_{i=1}^{n} a_i = (\prod_{i=1}^{n-1} a_i) \cdot a_n$$

Note that while in the conventional Diophantine arithmetic, addition and multiplication are commutative and associative, for arbitrary abstract prearithmetic, this is not always true and it is possible to define other n-ary operations.

When all a_i are equal to the same element, say a, we use the following notation

$$\sum_{i=1}^{n} a_i = n[a]$$

$$\prod_{i=1}^{n} a_i = [a]^n$$

Definitions imply the following result.

Lemma 2.4. *For any natural number n and any element a, we have $(n + 1)[a] = n[a] + a$ and $[a]^{n+1} = [a]^n \cdot a$. When addition $+$ is associative, it is possible to remove parentheses and we have*

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

$$n[a] = na.$$

and when multiplication $\cdot$ is associative, it is also possible to remove parentheses and we have

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

$$[a]^n = a^n$$

The Diophantine (conventional) arithmetic of natural numbers has the, so-called, Archimedean property, which is named after the great ancient Greek mathematician Archimedes of Syracuse and is important for proofs of many results in arithmetic and number theory. For instance, the Archimedean property, which is often called the Archimedean axiom, is important for proving that the set of all natural numbers and the set of all prime numbers are infinite. This property (axiom) is also very important for axiomatics in geometry (cf. (Veronese, 1889), (Hilbert, 1899)).

The Archimedean property (axiom) states that if we take any two natural numbers m and n, in spite that n may be enormously larger than m, it is always possible to add m enough times to itself, i.e., to take a sum $m + m + \ldots + m$, so that the result will be larger than n.

In contrast to the Diophantine arithmetic $\mathbb{N}$, the Archimedean property is invalid in many Diophantine arithmetics, such as $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$, and many non-Diophantine arithmetics. Other examples of non-Archimedean arithmetics are: the arithmetic of cardinal numbers (cf., for example,

(Fraenkel *et al.*, 1973)), the nonstandard arithmetic of hyperreal numbers (Robinson, 1966) , and the arithmetic of real hypernumbers (Burgin, 2012). However, many non-Diophantine arithmetics have the Archimedean property and we study it because it is important for number theory. In abstract prearithmetics, there are three principal structures - one is relational and two are operational, that is why it is natural to consider four Archimedean properties, which do not coincide in the general case.

Definition 2.2. a) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Successively Archimedean Property,* or is a *successively Archimedean prearithmetic,* if the inequality $a < b$ for $a,\ b \in A$ implies existence of a natural number n such that $S^n a$ is larger than or equal to b.

b) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Additively Archimedean Property*, or is *additively Archimedean prearithmetic,* if the inequality $a < b$ for $a,\ b \in A$ implies existence of a natural number n such that

$$b \leq n[a]. \tag{2.1}$$

c) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ satisfies the *Multiplicatively Archimedean Property*, or is *multiplicatively Archimedean prearithmetic,* if for any elements a and b from A, the inequality $a < b$ implies existence of a natural number n such that

$$b \leq [a]^n. \tag{2.2}$$

d) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive 0 satisfies the *left Binary Archimedean Property* for addition, or is a *binary for addition Archimedean prearithmetic* from the left, if for any elements a and b from A, the inequality $0 < a < b$ implies that there is an element q less than b such that

$$b \leq q + a \tag{2.3}$$

and satisfies the *right Binary Archimedean Property* for addition, or is a *binary for addition Archimedean prearithmetic* from the right, if for any elements a and b from A, the inequality $0 < a < b$ implies that there is an element q less than b such that

$$b \leq a + q \tag{2.4}$$

When addition + is commutative, then the right Binary Archimedean Property for addition coincides with the left Binary Archimedean Property for addition. When an abstract prearithmetic has both the right and left Binary Archimedean Properties for addition, then it has the *Binary Archimedean Property for addition.*

Example 2.8. The conventional arithmetic $2\mathbb{N}$ of all even numbers and conventional arithmetic $3\mathbb{N}$ of all natural numbers divisible by 3 have all four properties - the Successively Archimedean Property, Binary Archimedean Property for addition, Additively Archimedean Property and Multiplicatively Archimedean Property.

However, in general, these properties are independent because there are prearithmetics and arithmetics, which have only one part of the Archimedean Properties. For instance, the Diophantine arithmetic $\mathbb{N}$ does not have the Multiplicatively Archimedean Property but has other Archimedean Properties. That is why when the Archimedean Property is defined for multiplicative groups or semigroups, its validity is assumed for all elements but the unit element e (Fuchs, 1963).

Example 2.9. The conventional Diophantine arithmetic $\mathbb{W}$ of all whole numbers has the Successively Archimedean Property and Binary Archimedean Property for addition but does not have the Multiplicatively Archimedean Property and Additively Archimedean Property because these properties do not hold for the number 0.

That is why when the Archimedean Property is defined for additive groups or semigroups, its validity is assumed for all elements but the zero 0 (Fuchs, 1963).

Example 2.10. The conventional arithmetic $\mathbb{R}_1$ of all larger than 1 real numbers does not have the Successively Archimedean Property because real numbers do not have successors but has the Binary Archimedean Property for addition, Multiplicatively Archimedean Property, and Additively Archimedean Property.

There are also prearithmetics and arithmetics, which do not have any of the Archimedean Properties. Examples are: the arithmetic ***Ord*** of all ordinal numbers, the arithmetic $\mathbb{NW}$ of all nonstandard whole numbers (Robinson, 1966) and the arithmetic NH of all whole hypernumbers (Burgin, 2012).

Lemma 2.5. *An additively Archimedean prearithmetic cannot have the additive zero* 0.

Indeed, if $0 < b$, then any element $n[0]$ is equal to 0 and still less than b.

Lemma 2.6. *A multiplicatively Archimedean prearithmetic cannot have the multiplicative one* 1 *or the multiplicative zero* 0.

Indeed, if $1 < b$, then any element $[1]^n$ is equal to 1 and still less than b. In a similar way, if $0 < b$, then any element $[0]^n$ is equal to 0 and is still less than b.

Proposition 2. *Any additively Archimedean prearithmetic is a binary for addition Archimedean prearithmetic from the left.*

Proof. Let us consider an additively Archimedean abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ and its elements $a < b$. By Definition 2.2, A satisfies condition (2.3), i.e., for some natural number n, we have $b \leq n[a] = (\ldots(((a + a) + a) + a)\ldots) + a$. As $a < b$ and $1 \leq n$, we can take the least n such that $b \leq n[a]$. It means that $q = (n - 1)[a] < b$ and $b \leq (n - 1)[a] + a = q + a$. □

Proposition is proved.

Corollary 2.2. *Any additively Archimedean prearithmetic with associative addition is a binary for addition Archimedean prearithmetic from the left and from the right.*

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$ and in which addition + preserves the order $\leq$.

Lemma 2.7. *If $Sb \leq b + a$ for any elements a and b from A, then for any natural number n, we have $S^n a \leq (n + 1)[a]$.*

Proof. We use induction on n to show that $S^n a \leq (n + 1)[a]$. For $n = 1$, taking a as b, we have

$$Sa \leq a + a = 2[a]$$

For $n = 2$, taking Sa as b, we have

$$S^2 a = S(Sa) \leq Sa + a \leq (a + a) + a = 3[a]$$

as addition + preserves the order $\leq$. Let us assume that our statement is true for $n - 1$, i.e.,

$$Sn - 1a \leq n[a]$$

Then we have

$$S^n a = S(S^{n-1} a) \leq S^{n-1} a + a \leq n[a] + a = (n + 1)[a]$$

as addition + preserves the order $\leq$. The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 3. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have $Sb \leq b + a$ for any elements a and b from A and addition + preserves the order $\leq$, then $\mathbb{A}$ is an additively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is a successively Archimedean prearithmetic, there is a natural number n such that $S^n a$ is larger than or equal to b, i.e., $b \leq S^n a$. Then by Lemma 2.7, we have

$$b \leq S^n a \leq (n + 1)[a]$$

It means that $\mathbb{A}$ is an additively Archimedean prearithmetic. □

Proposition is proved.

Proposition 4. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive zero 0, addition + preserves the order $\leq$ and 0 is the least element in $\mathbb{A}$, then $\mathbb{A}_P = (A \setminus \{0\}; +, \cdot, \leq)$ is an additively Archimedean prearithmetic.*

Proof. As $0 < a$ for any element a from $\mathbb{A}_P$, we have $b = b + 0 \leq b + a$. By Definition 2.2, we have

$$b < Sb \leq b + a$$

Thus, by Proposition 2, $\mathbb{A}_P$ is an additively Archimedean prearithmetic. □

Proposition is proved.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$ and multiplication $\cdot$ preserves the order $\leq$.

Lemma 2.8. *If $Sb \leq b \cdot a$ for any elements a and b from A, then for any natural number n, we have $S^n a \leq [a]^{(n+1)}$.*

Proof. We use induction on n to show that $S^n a \leq [a]^{(n+1)}$. For $n = 1$, taking a as b, we have

$$Sa \leq a \cdot a = [a]^2$$

For $n = 2$, taking Sa as b, we have

$$S^2 a = S(Sa) \leq Sa \cdot a \leq (a \cdot a) \cdot a = [a]^3$$

as multiplication $\cdot$ preserves the order $\leq$. Let us assume that our statement is true for $n - 1$, i.e.,

$$S^{n-1} a \leq [a]^n$$

Then we have

$$S^n a = S(S^{n-1} a) \leq S^{n-1} a \cdot a \leq [a]^n \cdot a = [a]^{(n+1)}$$

as multiplication $\cdot$ preserves the order $\leq$. The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 5. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$, we have $Sb \leq b \cdot a$ for any elements a and b from A and multiplication $+$ preserves the order $\leq$, then $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is a successively Archimedean prearithmetic, there is a natural number n such that $S^n a$ is larger than or equal to b, i.e., $b \leq S^n a$. Then by Lemma 2.8, we have

$$b \leq S^n a \leq [a]^{(n+1)}$$

It means that $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic. □

Proposition is proved.

Corollary 2.3. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with a discrete order $\leq$, we have $Sb \leq b \cdot a$ for any elements a and b from A and multiplication $\cdot$ preserves the order $\leq$, then $\mathbb{A}$ is a binary Archimedean prearithmetic.*

Proposition 6. *If in a successively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative 1, multiplication $\cdot$ preserves the order $\leq$ and 1 is the smallest element in $\mathbb{A}$, then $\mathbb{A}_C = (A \backslash \{1\}; +, \cdot, \leq)$ is a multiplicatively Archimedean prearithmetic.*

Proof. As $1 < a$ for any element a from $\mathbb{A}_C$, we have $b = b \cdot 1 \leq b \cdot a$. By Definition 2.2, we have

$$b < S\,a \leq b \cdot a$$

Thus, by Proposition 5, $\mathbb{A}_C$ is an multiplicatively Archimedean prearithmetic. □

Proposition is proved.

Let us study relations between multiplication and addition.

Lemma 2.9. *If $b + a \leq b \cdot a$ for any elements a and b from A, then for any natural number n, we have $n[a] \leq [a]^n$.*

Proof. We use induction on n to prove the lemma. For $n = 2$, taking a as b, we have

$$2[a] = a + a \leq a \cdot a = [a]^2.$$

Let us assume that our statement is true for $n - 1$, i.e.,

$$n[a] \leq [a]^n$$

Then we have

$$(n + 1)[a] = n[a] + a \leq [a]^n \cdot a = [a]^{(n+1)}$$

The principle (axiom) of the mathematical induction gives us the necessary result. □

Lemma is proved.

Proposition 7. *If in an additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ we have $b + a \leq b \cdot a$ for any elements a and b from A, then $\mathbb{A}$ is a multiplicatively Archimedean prearithmetic.*

Proof. Let us consider elements a and b from A such that $a < b$. As $\mathbb{A}$ is an additively Archimedean prearithmetic, there is a natural number n such that $n[a]$ is larger than or equal to b, i.e., $b \leq n[a]$. Then by Lemma 2.9, we have

$$b \leq n[a] \leq [a]^n$$

□

Proposition is proved.

Corollary 2.4. *If in a additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ we have $b + a \leq b \cdot a$ for any elements a and b from A, then $\mathbb{A}$ is a binary Archimedean prearithmetic.*

Lemma 2.10. *In a additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves order, we have $a < a + a$ for any element a from A, which is not maximal.*

Proof. Let us assume that $a + a \leq a$ for some element a from A. Then

$$(a + a) + a \leq a + a \leq a$$

By induction we can prove that $n[a] \leq a$ for any natural number n.
As the element a is not maximal, $a < b$ for some element b from A. At the same time, we have

$$n[a] \leq a < b$$

This means that $\mathbb{A}$ is not an additively Archimedean prearithmetic. Thus, in an additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves order, we have $a < a + a$ for any element a from A, which is not maximal. □

Lemma is proved.

Corollary 2.5. *In a totally ordered additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition preserves strict order, the inequality $m < n$ implies the inequality $m[a] < n[a]$ for any element a from A, which is not maximal.*

When addition is associative and commutative, we have a stronger result.

Lemma 2.11. *In a totally ordered additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ where addition strictly preserves order and is associative and commutative, we have $a < a + b$ for any elements a and b from A, which are not maximal.*

Proof. If $a \leq b$, then the statement of the lemma follows from Lemma 2.10 because

$$a < a + a \leq a + b$$

as addition strictly preserves order and order relation is transitive.
Let us consider the case when $b < a$ and assume $a + b \leq a$. Because $\mathbb{A}$ is additively Archimedean prearithmetic, we have $a \leq n[b]$ for some natural number n. As addition strictly preserves order and order relation is transitive, we have

$$b + b < a + b \leq a$$

Adding b to both sides of the inequality $b + b \leq a + b$, we obtain

$$b + b + b < a + b + b \leq a + b \leq a$$

Continuing this process, we obtain

$$n[b] < a + b \leq a$$

This contradicts the equality $a \leq n[b]$ and by the Principle of excluded middle completes the proof. □

Lemma 2.12. *If $b \cdot a \leq b + a$ for any elements a and b from A, then for any natural number n, we have $[a]^n \leq n[a]$.*

Proof is similar to the proof of Lemma 2.1.16.

Proposition 8. *If in a multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$, *we have* $b \cdot a \leq b + a$ *for any elements* a *and* b *from* A, *then* $\mathbb{A}$ *is an additively Archimedean prearithmetic.*

Proof is similar to the proof of Proposition 7.

Lemma 2.13. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication preserves order, we have* $a < a \cdot a$ *for any element* a *from* A, *which is not maximal.*

Proof is similar to the proof of Lemma 2.10.

Corollary 2.6. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication preserves strict order, the inequality* $m < n$ *implies the inequality* $[a]^m < [a]^n$ *for any element* a *from* A, *which is not maximal.*

When addition is associative and commutative, we have a stronger result.

Lemma 2.14. *In a totally ordered multiplicatively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *where multiplication strictly preserves order and is associative and commutative, we have* $a < a \cdot b$ *for any elements* a *and* b *from* A, *which are not maximal.*

Proof is similar to the proof of Lemma 2.11.
We also introduce and study exact Archimedean properties.

Definition 2.3. a) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Successively Archimedean* if for any elements a and b from A, the inequality $a < b$ implies that there is natural number n such that $S^n a$ is equal to b.

b) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Additively Archimedean* if there is an element d from A, which is called the *additive generator* of $\mathbb{A}$, such that for any elements a and b from A, the inequality $a < b$ implies that there is a natural number n such that

$$a + n[d] = b \tag{2.5}$$

c) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *Exactly Multiplicatively Archimedean* if there is an element d from A, which is called the *multiplicative generator* of $\mathbb{A}$, such that for any elements a and b from A, the inequality $a < b$ implies that there is a natural number n such that

$$a \cdot [d]^n = b \tag{2.6}$$

d) An abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative 1 satisfies the *left Exactly Binary Archimedean Property* for addition, or is *left exactly additive prearithmetic* from the left, if for any elements a and b from A, the inequality $1 < a < b$ implies that there is an element q less than b such that

$$b = q + a \tag{2.7}$$

and satisfies the *right Exactly Binary Archimedean Property*, or is *exactly additive prearithmetic from the right*, if for any elements a and b from A, the inequality $1 < a < b$ implies that there is an element q less than b such that

$$b = a + q \tag{2.8}$$

The Diophantine arithmetic $\mathbb{N}$ is Exactly Successively Archimedean and Exactly Additively Archimedean because 1 is the additive generator of $\mathbb{N}$. The conventional arithmetic $\mathbb{W}$ of all whole numbers is also Exactly Successively Archimedean and Exactly Additively Archimedean because 1 is the additive generator of $\mathbb{W}$.

Remark. Exact Archimedean properties are intrinsically related to the concept of the natural order in partially ordered groupoids, groups and semigroups. We remind that if $H$is a partially ordered groupoid (semigroup, group), then its order is natural if $a < b$ implies $ax = ya = b$ for some elements a and b from H (Fuchs, 1963). Thus, the order in an Exact Archimedean partially ordered groupoid (semigroup, group) is natural.

Lemma 2.15. *An abstract prearithmetic with the linear order is Exactly Successively Archimedean if and only it is Successively Archimedean.*

Proof. Necessity. Any Exactly Successively Archimedean abstract prearithmetic is Successively Archimedean because for any element a in a partially ordered set, we have $a \leq a$, i.e., $S^n a = b$ implies $b \leq S^n a$.

Sufficiency. Let us assume that an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is Successively Archimedean and for some elements a and b from A, we have $a < b$. Then by definition, there is the least natural number n such that $b \leq S^n a$. As the order $\leq$ is linear, it means that we have the inequalities

$$S^{n-1}a \leq b \leq S^n a$$

Because $S^n a = S(S^{n-1}a)$ and by definition, there are no elements that larger than $S^{n-1}a$ and smaller than $S(S^{n-1}a)$, we have either $b = S^{n-1}a$ or $b = S^n a$. Consequently, the abstract prearithmetic $\mathbb{A}$ is Exactly Successively Archimedean. □

Lemma is proved.

Remark. For Exactly Additively Archimedean prearithmetics and Exactly Multiplicatively Archimedean prearithmetics a similar statement is not always true.

Lemma 2.16. *In an Exactly Additively Archimedean prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *with linear order and associative commutative addition, which strictly preserves order, either* 0 *or the additive generator* d *of* $\mathbb{A}$*, which is not maximal, is the least element.*

Proof. At first, we show that any element $b = n[d]$ if $d \leq b$. As $\mathbb{A}$ is Exactly Additively Archimedean, in this case, $b = d + n[d]$. As addition is commutative, $b = n[d] + d = (n + 1)[d]$.

At the same time, if there is an element b with $d < b$. Then assuming $d + d \leq d$, we obtain

$$(d + d) + d \leq d + d \leq d$$

By induction we can prove that $n[d] \leq d < b$ for any natural number n and the equality $b = n[d]$ becomes impossible. Consequently, we have

$$d < d + d = 2[d] < \ldots < n[d] < (n + 1)[d] < \ldots$$

as addition strictly preserves order and order relation is transitive.

Now let us suppose there is an element a that is less than d. As $\mathbb{A}$ is Exactly Additively Archimedean and d is the additive generator of $\mathbb{A}$, in this case, $d = a + n[d]$. As we demonstrated, we have $d < n[d]$. Then by the same token, if $d < a + d$, then $d < a + n[d]$ for any natural number n. If $d > a + d$, then $d > a + n[d]$ for any natural number n. Thus, we come to conclusion that $d = a + d$.

Applying mathematical induction, we see that $a + n[d] = n[d]$ for any natural number n. Thus, for any element b larger than d, we have $b = a + b$, i.e., a is the additive zero for all elements $b \geq d$.

Let us suppose there is an element c that is less than a. Then $a = c + n[d]$ and $d = c + m[d]$ because $c < a < d$ and d is the additive generator of $\mathbb{A}$.

At the same time, $c + d < c + n[d] < d$. Thus,

$$d > c + d > c + d + d = 2[d] > \ldots > m[d]$$

This contradict the equality $d = c + m[d]$ demonstrating that a is the least element and the additive zero in $\mathbb{A}$. □

Lemma is proved.

The Exactly Additively Archimedean Property allows representing successors Sa using addition as it is done in the conventional Diophantine arithmetic $\mathbb{N}$ of natural numbers where $Sn = n+1$. The same is true for many abstract prearithmetics.

Proposition 9. *If $\mathbb{A} = (A; +, \cdot, \leq)$ is an Exactly Additively Archimedean prearithmetic with linear order, the successor function S, the additive generator d and strictly monotone associative addition, then $Sa = a + d$ for any element a from A.*

Proof. Let us take an Exactly Additively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the successor function S, an the additive generator d and monotone addition. As by definition $a < Sa$ and the prearithmetic $\mathbb{A}$ is Exactly Additively Archimedean, we have

$$Sa = a + n[d] \tag{2.9}$$

As the order in A is linear, we have three options: $a > a + d, a = a + d$, or $a < a + d$.

If we have the first option, i.e., $a + d < a$, then

$$a + 2[d] = a + d + d < a + d < a,$$

because addition is strictly monotone and associative. By induction, for any natural number n, we have

$$a + n[d] < a$$

This contradicts equality (2.9) and shows that the first option is impossible.

If we have the second option, i.e., $a + d = a$, then

$$a + 2[d] = a + d + d = a + d = a,$$

because addition is associative. By induction, for any natural number n, we have

$$a + n[d] = a$$

This contradicts equality (2.9) and shows that the second option is impossible.

If we have the third option, i.e., $a + d > a$, then

$$a + 2[d] = a + d + d > a + d > a,$$

because addition is strictly monotone and associative.

By the definition of the successor Sa, if $a \leq z \leq Sa$, then z is equal either to a or to Sa. Because $a + 2[d] > a + d > a$, equality (2.9) implies $n = 1$ in (2.5) and $Sa = a + d$. □

Proposition is proved.

Applying Proposition 9 several times, we obtain the following result.

Proposition 10. *If $\mathbb{A} = (A; +, \cdot, \leq)$ is an Exactly Additively Archimedean prearithmetic (ESAPA) with linear order, the successor function S, an additive generator d and strictly monotone associative addition, then for any element a from A, $S^n a = a + n[a]$.*

Proof is left as an exercise.

Proposition 11. *Any Exactly Additively Archimedean prearithmetic is an exactly additive prearithmetic from the right.*

Proof. Let us consider a additively Archimedean abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ and its elements $a < b$. By Definition 2.3, it satisfies condition (2.5), i.e., for some natural number n, we have $b = a + n[d] = a + (\ldots(((d + d) + d) + d)\ldots) + d)$ where d is an additive generator. It means that we can take $q = n[d]$ and $b = a + n[d] = a + q$. □

Proposition is proved.

Corollary 2.7. *Any exactly additively Archimedean prearithmetic with commutative addition is an exactly additively Archimedean prearithmetic from the left and from the right.*

3. Elements of non-Diophantine number theory

Obtained properties of abstract prearithmetics allow building non-Diophantine number theory, which is also called non-Diophantine higher arithmetic. Here we develop only the fundamentals of this theory starting with such properties as subtractability and divisibility.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$.

Definition 3.1. a) An element b from $\mathbb{A}$ is subtractable *from the right (from the left)* by an element a from $\mathbb{A}$ if $b = d + a$ (correspondingly, $b = a + d$) for some element d from $\mathbb{A}$, which is called the *difference from the right* (correspondingly, *from the left*) of b and a. We call a and d *additive factors* of b and denote *subtractability from the right* by $b\lceil a$ (*from the left* by $a\rceil b$) and the *difference from the right* (correspondingly, *from the left*) by $d = b \rightharpoonup a$ (correspondingly, by $d = b \leftharpoonup a$).

b) An element a from $\mathbb{A}$ is *subtractable* by an element b if it is subtractable by b from the right and from the left with the same difference, i.e., $b = d + a = a + d$. We denote *subtractability* by $b(a$ and the difference of b and a by $d = b - a$.

For instance, in the conventional Diophantine arithmetic $\mathbb{N}$, any number is subtractable from the right and from the left by any smaller number because $n = 1 + (n - 1) = (n - 1) + 1$ for any natural number $n > 1$. However, this is not true for many abstract prearithmetics and non-Diophantine arithmetics.

Example 3.1. Let us consider the set N of all natural numbers with the standard order $\leq$ and introduce the following operations:

$$a \oplus b = a \cdot b$$

$$a \otimes b = a^b$$

Then the system $\mathbb{A} = (N; \oplus, \otimes, \leq)$ is an abstract prearithmetic. Taking numbers 5_A and 3_A from this prearithmetic, we see that there is no number n_A in $\mathbb{A}$ such that $3_A \oplus b = 5_A$. It means that 5_A is not subtractable by 3_A . Moreover, we can see that in this prearithmetic, subtractability means divisibility.

This example shows that subtractability is an additive counterpart of divisibility.

Lemma 3.1. *If addition + is commutative in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, then for any elements a and b from $\mathbb{A}$, a is subtractable by b if and only if it is subtractable by b from the right or from the left.*

Proof is left as an exercise.

Proposition 12. *If addition + is associative in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a\lceil b$ and $b\lceil c$ imply $a\lceil c$.*

(2) *$b\rceil a$ and $c\rceil b$ imply $c\rceil a$.*

(3) *$a(b$ and $b(c$ imply $a(c$.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. If $b\lceil c$, then $b = e + c$ for some element e from $\mathbb{A}$. Consequently,

$$a = d + b = d + e + c = (d + e) + c$$

It means that $a\lceil c$. Statements (2) and (3) are proved in a similar way. □

Proposition is proved.

Proposition 13. *If addition + is associative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$*, then for any elements* a*,* b *and* c *from* $\mathbb{A}$*, we have:*

(1) $a\lceil b$ *imply* $a + c\lceil b + c$.

(2) $b\rceil a$ *imply* $c + b\rceil c + a$.

(3) $a(b$ *imply* $a + c(b + c$ *when addition + is also commutative.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. Consequently,

$$a + c = d + b + c = d + (b + c)$$

It means that $a + c\lceil b + c$.
(2) If $b\rceil a$, then $a = b + d$ for some element $\mathbb{A}$. Consequently,

$$c + a = c + b + d = (c + b) + d$$

It means that $c + b\rceil c + a$. By Lemma 3.1, the statement (3) follows from statements (1) and (2) when addition + is commutative. □

Proposition is proved.

Proposition 14. *If addition + is associative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$*, then for any elements* a*,* b *and* c *from* $\mathbb{A}$*, we have:*

(1) $a\lceil b$ *imply* $c + a\lceil b$.

(2) $b\rceil a$ *imply* $b\rceil a + c$.

(3) $a(b$ *imply* $a + c(b$ *when addition + is also commutative.*

Proof. (1) If $a\lceil b$, then $a = d + b$ for some element d from $\mathbb{A}$. Consequently,

$$c + a = c + d + b = (c + d) + b$$

It means that $c + a\lceil b$. (2) If $b\rceil a$, then $a = b + d$ for some element d from $\mathbb{A}$. Consequently,

$$a + c = b + d + c = (c + b) + d + c$$

It means that $a + c\rceil b$. By Lemma 3.1, the statement (3) follows from statements (1) and (2) when addition + is commutative. □

Proposition is proved.

Proposition 15. *In an Exactly Additively Archimedean abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$*, for any elements* a*,* b *and* c *from* $\mathbb{A}$*, if* $a < b$*, then* b *is subtractable from the left by* a.

Proof is left as an exercise.

An important property of numbers is the cancelation law. For instance, if $n + 5 = m + 5$, then $n = m$ and if $n + 5 > m + 5$, then $n > m$ for any whole numbers n and m. Here we consider nine forms of the cancelation law and study them for abstract prearithmetics.

Definition 3.2. a) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation* if $a + c \leq b + c$ implies $a \leq b$.

b) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation* if $a + c = b + c$ implies $a = b$.

c) $\mathbb{A}$ is an abstract *prearithmetic with strict additive additive cancelation from the right* if $a + c < b + c$ implies $a < b$.

d) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation from the left* if $c + a \leq c + b$ implies $a \leq b$.

e) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation from the left* if $c + a = c + b$ implies $a = b$.

f) $\mathbb{A}$ is an abstract *prearithmetic with strict additive cancelation from the left* if $c + a < c + b$ implies $a < b$.

g) $\mathbb{A}$ is an abstract *prearithmetic with ordered additive cancelation* if it is an abstract prearithmetic with ordered additive cancelation from the left and from the right.

h) $\mathbb{A}$ is an abstract *prearithmetic with additive cancelation* if it is an abstract prearithmetic with additive cancelation from the left and from the right.

j) $\mathbb{A}$ is an abstract *prearithmetic with strict additive cancelation* if it is an abstract prearithmetic with strict additive cancelation from the left and from the right.

Let us consider some examples.

Example 3.2. The arithmetic $\mathbb{N}$ of all natural numbers is an abstract prearithmetic with additive cancelation, with ordered additive cancelation and with strict additive cancelation.

Example 3.3. However, the Diophantine arithmetic $\mathbb{W}$ of all whole numbers is an abstract prearithmetic with ordered additive cancelation but does not have additive cancelation or strict additive cancelation because any number multiplied by 0 is equal to 0.

Lemma 3.2. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with additive cancelation has the additive zero 0_A , then for any element a from $\mathbb{A}$, $a - a$ is defined and equal to 0_A.*

Indeed, by Definition 3.1, $a + 0_A = a$. Consequently, $a - a = 0_A$ because if $a + c = a$, then $c = 0_A$.

Proposition 16. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with additive cancelation has the additive zero 0_A, then for any element a from $\mathbb{A}$, $a - 0_A$ is defined and equal to a.*

Proof is left as an exercise.

Let us find conditions for ordered additive cancelation.

Lemma 3.3. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation if in* $\mathbb{A}$*, the order is linear (total) and addition preserves the strict order.*

Indeed, let us assume that $a + c \leq b + c$. If it is not true $a \leq b$, then $b < a$ because the order is total. However, as addition preserves the strict order, it would be $b + c < a + c$. As this contradicts our assumption, we conclude that $a \leq b$.

Lemma 3.4. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation (from the right or from the left) if it is with additive cancelation (from the right or from the left) and strict additive cancelation(from the right or from the left).*

Proof is left as an exercise.

Lemma 3.5. $\mathbb{A} = (A; +, \cdot, \leq)$ *is an abstract prearithmetic with ordered additive cancelation (from the right or from the left), then it is an abstract prearithmetic with additive cancelation (from the right or from the left).*

Indeed, $a + c = b + c$ implies $a + c \leq b + c$ and $b + c \leq a + c$. As $\mathbb{A}$ is an abstract prearithmetic with ordered additive cancelation, this implies $a \leq b$ and $b \leq a$. Consequently, $a = b$, which means that $\mathbb{A}$ is an abstract prearithmetic with additive cancelation.

Cancelation property allows strengthening of results in Lemma 3.5.

Lemma 3.6. *For any elements a and b from an abstract prearithmetic* $\mathbb{A}$ *with additive cancelation, we have:*

a) $a\lceil b$ *if and only if subtraction from the left* $a \leftharpoonup b$ *is defined.*

b) $b\rceil a$ *if and only if subtraction from the right* $a \rightharpoonup b$ *is defined.*

c) $a(b$ *if and only if full subtraction* $a - b$ *is defined.*

Proof is left as an exercise.

Lemma 3.7. *For any elements a and b from an abstract prearithmetic* $\mathbb{A}$ *with additive cancelation, we have:*

a) If subtraction from the left $\leftharpoonup$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation from the left.*

b) If subtraction from the right $\rightharpoonup$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation from the right.*

c) If full subtraction $a-$ *is defined in an abstract prearithmetic* $\mathbb{A}$*, then* $\mathbb{A}$ *is with additive cancelation.*

Proof is left as an exercise.

In some cases, additive and multiplicative zeros coincide.

Proposition 17. *If an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *with additive cancelation has the additive zero* 0_A *and multiplication is distributive over addition, then* 0_A *is also the multiplicative zero.*

Proof. Let us take an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ that satisfies all initial conditions. Taking arbitrary element x from $\mathbb{A}$, we have

$$0_A \cdot x = 0_A \cdot x + 0_A = (0_A + 0_A) \cdot x = 0_A \cdot x + 0_A \cdot x \tag{3.1}$$

As we have additive cancelation in $\mathbb{A}$, it is possible to cancel $0_A \cdot x$ in (3.1). This gives us $0_A \cdot x = 0_A$. The identity $x \cdot 0_A = 0_A$ is proved in a similar way. Consequently, 0_A is also the multiplicative zero. □

Proposition is proved.

Corollary 3.1. *In a semiring with additive cancelation, additive and multiplicative zeros coincide.*

An important property of numbers is divisibility. Here we study it for abstract prearithmetics.

Definition 3.3. a) An element a from $\mathbb{A}$ is *divisible* from the right (from the left) by an element b from $\mathbb{A}$ if $a = d \cdot b$ ($a = b \cdot d$) for some element d from $\mathbb{A}$. We call b and d *multiplicative factors* or *divisors* of a, the element a is called a *multiple* of b from the left (from the right), and *divisibility from the right* is denoted by $a \rfloor b$ (*from the left* by $b \lfloor a$).

b) An element a from $\mathbb{A}$ is *divisible* by an element b if it is divisible by b from the right and from the left. We denote this by $a|b$ and a is called a *multiple* of b.

Remark. If a is divisible by b, it is also denoted by $b|a$ in some publications.

Lemma 3.8. *If multiplication* $\cdot$ *is commutative in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *with the multiplicative one* 1*, then in it, any element a is divisible by b if and only if it is divisible by b from the right or from the left.*

Proof is left as an exercise.

Proposition 18. *If multiplication* $\cdot$ *in an abstract prearithmetic* $\mathbb{A} = (A; +, \cdot, \leq)$ *is associative, then for any elements a, b and c from* $\mathbb{A}$*, we have:*

(1) $a \rfloor b$ *and* $b \rfloor c$ *imply* $a \rfloor c$.

(2) $a \lfloor b$ *and* $b \lfloor c$ *imply* $a \lfloor c$.

(3) $a|b$ *and* $b|c$ *imply* $a|c$.

Proof. (1) If $a \lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. If $b \lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently,

$$a = d \cdot b = d \cdot e \cdot c = (d \cdot e) \cdot c$$

It means that $a \lfloor c$. Statements (2) and (3) are proved in a similar way. □

Proposition is proved.

Corollary 3.2. ((Landau *et al.*, 1999): Theorem 2). *If a, b and c are integer numbers, then $a|b$ and $b|c$ imply $a|c$.*

In other words, Proposition 18 and Corollary 3.2 mean that a divisor of a divisor of an element is a divisor of this element. It is also possible to say that a multiple of a multiple of an element is a multiple of this element.

Proposition 19. *If multiplication $\cdot$ in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a \lfloor b$ imply $a \cdot c \lfloor b \cdot c$.*

(2) *$b \rfloor a$ imply $c \cdot b \rfloor c \cdot a$.*

(3) *$a|b$ imply $a \cdot c|b \cdot c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a \lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. Consequently,

$$a \cdot c = d \cdot b \cdot c = d \cdot (b \cdot c)$$

It means that $a \cdot c \lfloor b \cdot c$.
(2) If $b \rfloor a$, then $a = b \cdot d$ for some element d from $\mathbb{A}$. Consequently,

$$c \cdot a = c \cdot b \cdot d = (c \cdot b) \cdot d$$

It means that $c \cdot b \rfloor c \cdot a$. By Lemma 3.8, the statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.3. ((Landau *et al.*, 1999): Theorem 3b). *If a, b and c are integer numbers, then $a|b$ imply $ac|bc$.*

Proposition 20. *If multiplication $\cdot$ in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then for any elements a, b and c from $\mathbb{A}$, we have:*

(1) *$a \lfloor b$ imply $c \cdot a \lfloor b$.*

(2) *$b \rfloor a$ imply $b \rfloor a \cdot c$.*

(3) *$a|b$ imply $a \cdot c|b$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a \lfloor b$, then $a = d \cdot b$ for some element d from $\mathbb{A}$. Consequently,

$$c \cdot a = c \cdot d \cdot b = (c \cdot d) \cdot b$$

It means that $c \cdot a \lfloor b$

(2) If $b \rfloor a$, then $a = b \cdot d$ for some element d from $\mathbb{A}$. Consequently,

$$a \cdot c = b \cdot d \cdot c = b \cdot (d \cdot c)$$

It means that $b \rfloor a \cdot c$. By Lemma 3.8, the statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.4. ((Landau *et al.*, 1999): Theorem 4). *If a, b and c are integer numbers, then $a|b$ imply $ac|b$.*

In other words, Proposition 20 and Corollary 3.4 mean that a divisor of an element is also a divisor of any multiple of this element. It is also possible to say that a multiple of a multiple of an element is a multiple of this element.

Proposition 21. *For any elements a, b and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a \lfloor c$ and $b \lfloor c$ imply $a + b \lfloor c$ when multiplication $\cdot$ is distributive from the right over addition $+$.*

(2) *$c \rfloor a$ and $c \rfloor b$ imply $c \rfloor a + b$ when multiplication $\cdot$ is distributive from the left over addition $+$.*

(3) *$a|c$ and $b|c$ imply $a + b|c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) If $a \lfloor c$, then $a = d \cdot c$ for some element d from $\mathbb{A}$. If $b \lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the right, we have

$$a + b = d \cdot c + e \cdot c = (e + d) \cdot c$$

It means that $a + b \lfloor c$.

(2) If $c \rfloor a$, then $a = c \cdot d$ for some element d from $\mathbb{A}$. If $c \rfloor b$, then $b = c \cdot e$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the left, we have

$$a + b = c \cdot d + c \cdot e = c \cdot (e + d)$$

It means that $c \rfloor a + b$. The statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative. □

Proposition is proved.

Corollary 3.5. ((Landau *et al.*, 1999): Theorem 5). *If a, b and c are integer numbers, then $a|b$ and $b|c$ imply $(a + b)|c$.*

In other words, Proposition 21 and Corollary 3.5 mean that that a common divisor of two elements is also a divisor of the sum of these elements.

Corollary 3.6. *For any elements a_i $(i = 1, 2, 3, \dots, n)$ from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a_i|c$ for all $i = 1, 2, 3, \dots, n$ imply $\sum_{i=1}^{n} a_i|c$ when multiplication $\cdot$ is associative and distributive from the right over addition $+$.*

(2) *$c|a_i$ for all $i = 1, 2, 3, \dots, n$ imply $c|\sum_{i=1}^{n} a_i$ when multiplication $\cdot$ is associative and distributive from the left over addition $+$.*

(3) *$a_i|c$ for all $i = 1, 2, 3, \dots, n$ imply $\sum_{i=1}^{n} a_i|c$ when multiplication $\cdot$ is also commutative.*

Proposition 22. *For any elements a, b, k, h and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a\lfloor c$ and $b\lfloor c$ imply $(a \cdot k + b \cdot h)\lfloor c$ when multiplication $\cdot$ is associative and distributive from the right over addition $+$.*

(2) *$c\rfloor a$ and $c\rfloor b$ imply $c\rfloor(a \cdot k + b \cdot h)$ when multiplication $\cdot$ is associative and distributive from the left over addition $+$.*

(3) *$a|c$ and $b|c$ imply $(a \cdot k + b \cdot h)|c$ when multiplication $\cdot$ is also commutative.*

Proof. (1) By Proposition 20, for any elements a, b, k, h and c from $\mathbb{A}$, we have:

$$(a \cdot k)\lfloor c \text{ and } (b \cdot h)\lfloor c$$

Thus, By Proposition 20,

$$(a \cdot k + b \cdot h)\lfloor c$$

Statements (2) and (3) are proved in the same way based on Proposition 20 and 21. □

Proposition is proved.

Corollary 3.7. ((Landau *et al.*, 1999): Theorem 6). *If a, b, k, h and c are integer numbers, then $a|c$ and $b|c$ imply $(a \cdot k + b \cdot h)|c$.*

Proposition 23. *For any elements a, b and c from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, we have:*

(1) *$a\lfloor c$ and $b\lfloor c$ imply $a \rightharpoonup b\lfloor c$ when the right difference $a \rightharpoonup b$ of a and b exists and multiplication $\cdot$ is distributive from the right over the right difference.*

(2) *$c\rfloor a$ and $c\rfloor b$ imply $c\rfloor a \leftharpoonup b$ when the left difference $a \leftharpoonup b$ of a and b exists and multiplication $\cdot$ is distributive from the left over the left difference.*

(3) *$a|c$ and $b|c$ imply $a - b|c$ when the difference $a - b$ of a and b exists and multiplication $\cdot$ is commutative and distributive over the difference.*

Proof. (1) If $a\lfloor c$, then $a = d \cdot c$ for some element d from $\mathbb{A}$. If $b\lfloor c$, then $b = e \cdot c$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the right, we have

$$a \rightharpoonup b = d \cdot c \rightharpoonup e \cdot c = (e \rightharpoonup d) \cdot c$$

It means that $a \rightharpoonup b\lfloor c$.
(2) If $c\rfloor a$, then $a = c \cdot d$ for some element d from $\mathbb{A}$. If $c\rfloor b$, then $b = c \cdot e$ for some element e from $\mathbb{A}$. Consequently, by distributivity from the left, we have

$$a \leftharpoonup b = c \cdot d \leftharpoonup c \cdot e = c \cdot (e \leftharpoonup d)$$

It means that $c\rfloor a \leftharpoonup b$. The statement (3) follows from statements (1) and (2) when multiplication $\cdot$ is commutative and distributive over the difference. □

Proposition is proved.

Corollary 3.8. ((Landau *et al.*, 1999): Theorem 5). *If a, b and c are integer numbers, then $a|c$ imply $b|c$ imply $(a - b)|c$.*

In other words, Proposition 23 and Corollary 3.8 mean that that a common divisor of two elements is also a divisor of the difference of these elements.

Lemma 3.9. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the multiplicative zero 0^m, then 0^m is divisible by any element from $\mathbb{A}$.*

Indeed, we have $0^m = a \cdot 0^m = 0^m \cdot a$ for any element a from $\mathbb{A}$.

Existence of the additive zero impacts subtractability.

Lemma 3.10. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the additive zero 0, then in it, any element is subtractable by itself and by 0.*

Indeed, we have $a = a + 0 = 0 + a$ for any element a from $\mathbb{A}$.

Lemma 3.11. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative one 1_A , then in it, any element is divisible by itself and by 1_A.*

Indeed, we have $a = a \cdot 1_A = 1_A \cdot a$ for any element a from $\mathbb{A}$.

Number theory begins with classification of numbers and studying their properties. An important class of numbers in the Diophantine arithmetic $\mathbb{N}$ consists of prime numbers, which are extensively studied in number theory (cf., for example, (Davenport, 1999), (Landau *et al.*, 1999)).

In abstract prearithmetics in general and in non-Diophantine arithmetics in particular, there are two classes of prime numbers - additively prime numbers and multiplicatively prime numbers. There are also additively composite numbers and multiplicatively composite numbers. They are counterparts of the well-known concepts of prime and composite numbers in the Diophantine arithmetic $\mathbb{N}$. Here we define these classes in abstract prearithmetics.

Definition 3.4. a) An element p from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the additive zero 0_A is *additively prime* in $\mathbb{A}$ if there are no elements $a, b \neq 0_A$ in $\mathbb{A}$ such that $p = a + b$.

b) An element p from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ with the multiplicative one 1_A is *multiplicatively prime* in $\mathbb{A}$ if $p \neq 0_A$, $p \neq 1_A$ and there are no elements $a, b \neq 1_A$ in $\mathbb{A}$ such that $p = a \cdot b$.

c) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively composite* in $\mathbb{A}$ if it is not additively prime.

d) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively composite* in $\mathbb{A}$ if it is not multiplicatively prime.

Example 3.4. In the Diophantine arithmetic $\mathbb{N}$, there is only one additively prime number 1 and infinitely many multiplicatively prime numbers. That is why in the conventional (Diophantine) number theory, additively prime numbers are not even introduced but multiplicatively prime numbers, which are simply called prime numbers, are studied with great interest by many mathematicians.

Remark. It is interesting that the great Greek philosopher Aristotle defined additively prime numbers and found only two additively prime numbers 2 and 3 because at that time, Greek mathematicians did not consider 1 as a number (Aristotle, 1984).

Note that there are many abstract prearithmetics that do not have additively prime numbers, i.e., all numbers are composite.

Example 3.5. In any modular arithmetic $\mathbb{Z}_n$, there are no additively prime numbers because any number in $\mathbb{Z}_n$ is a sum of two non-zero numbers.

This is a particular case of the following result.

Lemma 3.12. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is a group with respect to addition, then it does not have additively prime elements.*

Indeed, any element a in $\mathbb{A}$ has the opposite element $-a$ and if $a \neq b$ in $\mathbb{A}$, then $b = (b+(-a))+a$ where $b + (-a) \neq 0_A$ i.e., b is an additively composite element.

At the same time, there are many abstract prearithmetics that have infinitely many additively prime numbers as the following example demonstrates.

Example 3.6. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which operations are defined by the following formula

$$n \oplus m = (n + m)^2$$

$$n \circ m = (n \cdot m)^2$$

where $+$ is the standard addition and $\cdot$ is the standard multiplication of natural numbers.

Then all natural numbers that are not squares in the Diophantine arithmetic $\mathbb{N}$ will be additively prime numbers in this prearithmetic $\mathbb{A}$.

Proposition 24. *There is continuum of abstract prearithmetics that have infinitely many additively prime numbers.*

Proof. Abstract prearithmetics are different when they have different multiplication. Because multiplication in abstract prearithmetics is defined independently from addition, we can take the abstract prearithmetic A_f, in which addition is defined as in the abstract prearithmetic $\mathbb{A}$ from Example 3.6 while multiplication is defined by an arbitrary function f from $N \times N$ into N. By construction, all natural numbers that are not squares in $\mathbb{N}$ will be additively prime numbers in this prearithmetic A_f. As there is continuum of such functions f (Fraenkel *et al.*, 1973), there is also continuum of abstract prearithmetics that have infinitely many additively prime numbers. □

Proposition is proved.

However, the following result shows that for an arbitrary abstract prearithmetic, the situation can be essentially different.

Theorem 3.1. *There are infinite abstract prearithmetics, in which for any natural number $n > 1$, there are exactly n additively prime elements.*

Proof. Let us consider the set W of all whole numbers, take a natural number n and define the following functions

$$g(m) = \begin{cases} 0 & \text{when } m = 0 \\ n + m & \text{when } m > 0 \end{cases}$$

and

$$h(q) = \begin{cases} q & \text{when } 0 \leq q < n + 1 \\ q - n & \text{when } q > n \end{cases}$$

We can build an abstract prearithmetic $\mathbb{A}_n = (W; \oplus, \otimes, \leq)$ with addition defined for whole numbers m and n larger than 0 by the following formula

$$m \oplus k = h(g(m) + g(k)) = (m + n) + (k + n) - n = m + k + n$$

Besides,

$$0 \oplus m = m \oplus 0 = h(g(m) + 0) = (m + n) - n = m$$

for all $m > 0$.

Then the least additively composite number is $n + 2 = 1 \oplus 1$. At the same time, any larger number $r = 2 + n + k$ is also additively composite because $2 \oplus k = h(g(2) + g(k)) = 2 + k + n = r$. Consequently, there are exactly $n + 1$ additively prime elements in $\mathbb{A}_n$ and it is possible to build such a prearithmetic $\mathbb{A}_n$ for all $n = 1, 2, 3, \ldots$ □

Theorem is proved.

Considering multiplicatively prime numbers, we see that there are also many abstract prearithmetics that do not have multiplicatively prime numbers.

Example 3.7. In the modular arithmetic $\mathbb{Z}_p$ where p is a prime number, there are no multiplicatively prime numbers because any non-zero number in $\mathbb{Z}_p$ is a product of two non-zero numbers.

This is a particular case of the following result.

Lemma 3.13. *If an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is a group with respect to multiplication, then it does not have multiplicatively prime elements because any non-zero element a in $\mathbb{A}$ has the inverse element a^{-1}.*

Indeed, any element a in $\mathbb{A}$ has the inverse element a^{-1} and if $a \neq b$ in $\mathbb{A}$, then $b = (b \cdot a^{-1}) \cdot a$ where $b \cdot a^{-1} \neq 1_A$.

At the same time, there are many abstract prearithmetics that have infinitely many multiplicatively prime numbers.

Proposition 25. *There is continuum of abstract prearithmetics that have infinitely many multiplicatively prime numbers.*

Proof. Abstract prearithmetics are different when they have different addition. Because multiplication in abstract prearithmetics is defined independently from addition, we can take the abstract prearithmetic $\mathbb{N}_f$, in which multiplication is defined as in the Diophantine arithmetic $\mathbb{N}$ while addition is defined by an arbitrary function f from $N \times N$ into N. As it is proved (cf., for example, (Davenport, 1999)) that $\mathbb{N}$ has infinitely many multiplicatively prime numbers. Thus, there are infinitely many multiplicatively prime elements in this prearithmetic N_f . As there is continuum of such functions f (Fraenkel *et al.*, 1973), there is also continuum of abstract prearithmetics that have infinitely many multiplicatively prime numbers. □

Proposition is proved.

However, the following result shows that for an arbitrary abstract prearithmetic, the situation can be essentially different.

Theorem 3.2. *There are infinite abstract prearithmetics, in which for any natural number $n > 3$, there are exactly n multiplicatively prime elements.*

Proof. Let us consider the set N of all natural numbers, take a natural number n and define the following functions

$$g(m) = \begin{cases} 1 & \text{when } m = 1 \\ 2^{n+m} & \text{when } m > 1 \end{cases}$$

and

$$h(q) = \begin{cases} q & \text{when } 1 \leq q < n+1 \\ |\log_2 q| - n & \text{when } q > 2^n \end{cases}$$

Note that if $q = 2^m$, then $h(q) = m - n$. We can build an abstract prearithmetic $\mathbb{B}_n = (N; \oplus, \otimes, \leq)$ with multiplication defined for whole numbers m and n larger than 1 by the following formula

$$m \otimes k = h(g(m) + g(k)) = h(2^{n+m} \cdot 2^{n+k}) = h(2^{2n+m+k}) =$$

$$|\log_2 2^{2n+m+k}| - n = \log_2 2^{2n+m+k} - n = m + k + n$$

Besides,

$$1 \otimes m = m \otimes 1 = h(g(m) \cdot 1) = \log_2 2^{n+m} - n = (m + n) - n = m$$

for all $m > 0$.

Then the least composite number is $n + 4 = 2 \otimes 2$. At the same time, any larger number $r = n + 4 + k$ with $k = 1, 2, 3, \ldots$ is also composite as it is divisible by 2 because $2 \otimes (2 + k) = h(g(2) + g(k)) = 2 + k + 2 + n = r$. Consequently, there are exactly $n + 3$ additively prime elements in $\mathbb{B}_n$ and it is possible to build such a prearithmetic $\mathbb{B}_n$ for all $n = 1, 2, 3, \ldots$ □

Theorem is proved.

Let us also consider other traditional classes of natural numbers, for example, even and odd numbers. It is also natural to define even and odd elements in abstract prearithmetics.

Definition 3.5. a) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively even* in $\mathbb{A}$ (with respect to an element b) if there is an element c in $\mathbb{A}$ such that $a = 2 + c$ $(a = b + c)$.

b) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively even* in $\mathbb{A}$ (with respect to an element b) if there is an element c in $\mathbb{A}$ such that $a = 2 \cdot c$ $(a = b \cdot c)$.

c) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *additively odd (with respect to* an element b in $\mathbb{A}$) if it is not additively even (with respect to the element b)

d) An element a from an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is *multiplicatively odd (with respect to* an element b in $\mathbb{A}$) if it is not multiplicatively even (with respect to the element b)

Note that when an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ has the number 2, then multiplicatively even in $\mathbb{A}$ with respect to 2 elements are simply *even elements (numbers)* although they might be essentially different from even numbers in the Diophantine arithmetic $\mathbb{N}$. Naturally, all elements, which are not multiplicatively even in $\mathbb{A}$ with respect to 2, are *odd elements (numbers).*

Example 3.8. In the arithmetic $2\mathbb{N}$, of all even numbers with conventional addition and multiplication, all numbers larger than 2 are additively and multiplicatively even.

However, there are many abstract prearithmetics that have only one additively (multiplicatively) even number as the following example demonstrates

Example 3.9. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which addition is defined by the following formulas

$$n \oplus 2 = 2 \oplus n = 2$$

$$n \oplus m = n + m \text{ if } n, m \neq 2$$

where + is the standard addition of natural numbers. We see that in $\mathbb{A}$, only 2 is an additively even number.

Example 3.10. Let us consider an abstract prearithmetic $\mathbb{A} = (N; \oplus, \circ, \leq)$ which contains the set all natural numbers N and in which multiplication is defined by the following formulas

$$n \circ 2 = 2 \circ n = 2$$

$$n \circ m = n \cdot m \text{ if } n, m \neq 2$$

where $\cdot$ is the standard addition of natural numbers. We see that in $\mathbb{A}$, only 2 is an multiplicatively even number.

In some abstract prearithmetics, even numbers have usual properties.

Proposition 26. *If addition in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then the sum of any additively even in $\mathbb{A}$ element with respect to an element b with any element c is additively even in $\mathbb{A}$ with respect to the element b.*

Indeed, if a is an additively even in $\mathbb{A}$ element with respect to an element b, then $a = b + d$. Consequently, as addition in $\mathbb{A}$ is associative, we have

$$a + c = (b + d) + c = b + (d + c)$$

It means that $a + c$ is additively even in $\mathbb{A}$ with respect to the element b.

Proposition 27. *If multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is distributive with respect to addition, then the sum of two multiplicatively even with respect to an element b in $\mathbb{A}$ elements is multiplicatively even with respect to the element b.*

Indeed, if a and d are multiplicatively even in $\mathbb{A}$ elements with respect to an element b, then $a = b \cdot u$ and $d = b \cdot w$. Consequently, as multiplication in $\mathbb{A}$ is distributive with respect to addition, we have

$$a + d = (b \cdot u) + (b \cdot w) = b \cdot (u + w)$$

It means that $a + d$ is multiplicatively even in $\mathbb{A}$ with respect to an element b.

Proposition 28. *If multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is associative, then the product of any multiplicatively even in $\mathbb{A}$ element with respect to an element b with any element c is additively even in $\mathbb{A}$ with respect to the element b.*

Indeed, if a is an additively even in $\mathbb{A}$ element with respect to an element b, then $a = b \cdot d$. Consequently, as multiplication in $\mathbb{A}$ is associative, we have

$$a \cdot c = (b \cdot d) \cdot c = b \cdot (d \cdot c)$$

It means that $a \cdot c$ is multiplicatively even in $\mathbb{A}$ with respect to the element b.

One of the basic results of the conventional number theory is the prime decomposition theorem, proofs of which it is possible to find in many books (cf., for example, (Landau *et al.*, 1999), (Davenport, 1999)).

Theorem 3.3. ***Prime Decomposition Theorem.*** *For any natural number larger than 1 in the conventional Diophantine arithmetic* $\mathbb{N}$*, there is a unique up to the order of factors decomposition (factoring) of this number into the product of prime numbers.*

It is also called the Fundamental Theorem of Arithmetic. According to (Davenport, 1999) Davenport, 1992), the first clear statement and proof of this theorem seem to have been given by Gauss in 1801. An equivalent form of the Fundamental Theorem of Arithmetic states that any factoring of a natural number can be extended to a unique up to the order of factors prime factorization. It is interesting that the same result is evidently true for additively prime numbers. Namely, we have the following result.

Proposition 29. *For any natural number in the conventional Diophantine arithmetic* $\mathbb{N}$*, there is a unique up to the order of factors decomposition of this number into the sum of additively prime numbers.*

Indeed, 1 is an additively prime number and any natural number is the sum of some number of $1s$.

Note that in the Diophantine arithmetics $\mathbb{N}$ and $\mathbb{W}$, there is only one additively prime number. At the same time, as it is demonstrated in Proposition 24, there are prearithmetics that have an infinite set of additively prime numbers. An equivalent form of Proposition 29 states that any decomposition (factoring) of a natural number into a sum can be extended to a unique up to the order of factors decomposition (factorization) of this number into the sum of additively prime numbers.

These results bring us to the following concepts.

Definition 3.6. a) An abstract prearithmetic $\mathbb{A}$ has the *additive factoring property* if for any of its non-zero elements, any factorization (additive decomposition) of this element into a sum can be extended to a factorization (additive decomposition) of this number into the sum of additively prime elements.

b) An abstract prearithmetic $\mathbb{A}$ has the *strong additive factoring property* if for any of its non-zero elements, any factorization (additive decomposition) of this element into a sum can be extended to a unique up to the order of factors factorization (additive decomposition) of this number into the sum of additively prime elements.

c) An abstract prearithmetic $\mathbb{A}$ has the *multiplicative factoring property* if for any of its elements but zero 0 and the multiplicative one 1, any factorization (multiplicative decomposition) of this element into a product can be extended to a factorization (multiplicative decomposition) of this number into the product of multiplicatively prime elements.

d) An abstract prearithmetic $\mathbb{A}$ has the *strong multiplicative factoring property* if for any of its elements but zero 0 and the multiplicative one 1, any factorization (multiplicative decomposition) of this element into a product can be extended to a unique up to the order of factors factorization (multiplicative decomposition) of this number into the product of multiplicatively prime elements.

For abstract prearithmetics and even for whole-number and natural number prearithmetics (cf. Section 2.7), the additive factoring property is not true in a general case as the following example demonstrates.

Example 3.11. Let us consider the arithmetic $\mathbb{R}^{++} = (\mathbb{R}^{++}; +, \cdot, \leq)$ of all positive real numbers with standard addition, multiplication and order. The prearithmetic $\mathbb{R}^{++}$ does not have additively prime numbers because any positive real number a is equal to $a/2$ plus $a/2$. Consequently, this prearithmetic does not have the additive factoring property and Proposition 29 is not true for this prearithmetic.

Lemma 3.14. *The strong additive factoring property implies the additive factoring property.*

Proof is left as an exercise.

The inverse implication is not true as the following example demonstrates.

Example 3.12. Let us consider the set $F = \{0, 1, \frac{1}{3}, \frac{1}{2}\}$ and the set P of all expressions of the form $a_1 + a_2(\frac{1}{2}) + a_3(\frac{1}{3})$ where a_1, a_2 and a_3 are natural numbers. We see that the sum of these expressions has the same form. The multiplication is defined by the following formula

$$(a_1 + a_2(\frac{1}{2}) + a_3(\frac{1}{3})) \cdot (b_1 + b_2(\frac{1}{2}) + b_3(\frac{1}{3})) = (a_1 \cdot b_1) + (a_2 \cdot b_2)(\frac{1}{2}) + (a_3 \cdot b_3)(\frac{1}{3}))$$

Now we can define the set A_2 of numbers that can be represented as expressions from P. Naturally, some of these polynomials define the same number. For instance, $1 = 2(\frac{1}{2})$ or $(\frac{1}{2}) \cdot (\frac{1}{3}) = 0$.

This gives us the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, in which order is the same as in the arithmetic $\mathbb{R}$ of all real numbers, while addition and multiplication are defined above. In it, $\frac{1}{2}$ and $\frac{1}{3}$ are additively prime elements, while number 1 has two additive prime decompositions (factorizations)

$$1 = 2(\frac{1}{2}) = 1(\frac{1}{2}) + 1(\frac{1}{2})$$

and

$$1 = 3(\frac{1}{3}) = 1(\frac{1}{3}) + 2(\frac{1}{3})$$

Similar to the additive factoring property, for abstract prearithmetics and even for whole-number and natural number prearithmetics, the multiplicative factoring property is not true in a general case as the following example demonstrates.

Example 3.13. Let us consider the arithmetic $\mathbb{R}^{++} = (\mathbb{R}^{++}; +, \cdot, \leq)$ of all positive real numbers with standard addition, multiplication and order. The prearithmetic $\mathbb{R}^{++}$ does not have multiplicatively prime numbers because any positive real number a is equal to $a^{\frac{1}{2}}$ times $a^{\frac{1}{2}}$. Consequently, the Fundamental Theorem of Arithmetic is not true for this prearithmetic and it does not have the multiplicative factoring property.

There are also modular arithmetics, which do not have multiplicatively prime numbers.

Example 3.14. Let us consider the modular arithmetic $\mathbb{Z}_5$. It has five elements 0, 1, 2, 3 and 4. There are following multiplicative decompositions in $\mathbb{Z}_5$:

$$2 \cdot 3 = 1; \quad 3 \cdot 4 = 2; \quad 2 \cdot 4 = 3; \quad \textit{and} \quad 2 \cdot 2 = 4$$

This shows that all numbers in $\mathbb{Z}_5$ are multiplicatively composite.
As a result, we can build multiplicative decompositions of an arbitrary length. For instance, we have

$$2 = 3 \cdot 4 = (2 \cdot 4) \cdot 4 = ((3 \cdot 4) \cdot 4) \cdot 4 = \dots$$

At the same time, some modular arithmetics have multiplicatively prime numbers. For instance, 3 is a multiplicatively prime number in $\mathbb{Z}_4$.

Lemma 3.15. *The strong multiplicative factoring property implies the multiplicative factoring property.*

Proof is left as an exercise.
The inverse implication is not true as the following example demonstrates.

Example 3.15. Let us consider the set $F = \{1, 2, 2^{\frac{1}{3}}, 2^{\frac{1}{2}}\}$ and the set P of all expressions of the form $2^{a_1+a_2(\frac{1}{2})+a_3(\frac{1}{3})}$ where a_1, a_2 and a_3 are natural numbers. We see that the products of these expressions has the same form. Now we can define the set A_2 of numbers that can be represented as expressions from P and their arbitrary sums. Naturally, some of these polynomials define the same number. For instance, $2 = 2^{2(\frac{1}{2})} = 2^{3(\frac{1}{3})}$.

This gives us the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, in which order is the same as in the arithmetic $\mathbb{R}$ of all real numbers, while addition and multiplication are defined above. In it, $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ are multiplicatively prime elements, while as we have demonstrated, number 2 has two multiplicative prime decompositions (factorizations).

Theorem 3.4. *An infinite abstract well-ordered additively Archimedean prearithmetic* $\mathbb{A}=(A;+,\cdot,\leq)$*, with associative and commutative addition ,which strictly preserves the order, has the additive factoring property.*

Proof. As the abstract prearithmetic $\mathbb{A}$ is infinite and well-ordered, it does not have maximal elements. Indeed, if a is a maximal element in a well-ordered set, then there is only a finite number of elements less than a. At the same time, as well-ordering is also a total ordering (Fraenkel *et al.*, 1973), only one maximal element can exist and thus, the set has to be finite. Consequently, as the prearithmetic $\mathbb{A}$ is infinite, it does not have maximal elements.

Then by Lemma 2.11, the sum $a + b$ is larger than both its factors a and b. In other words, an additive factor of an element is less than this element. Consequently, any element a from $\mathbb{A}$ has only a finite numbers of additive factors by the properties of well-ordered sets (Fraenkel *et al.*, 1973).

Let us consider an element a from $\mathbb{A}$ that is not equal to the additive zero 0. If it is additively prime, then the statement of the theorem is valid for a. If a is composite, then for $n > 1$, there is a factoring

$$a = a_1 + a_2 + \ldots + a_n$$

.

If one of the elements a_i is not additively prime, then we will have more additive factors of a. As a has only a finite numbers of additive factors, at some step of the decomposition of the element a into additive factors, we will have only additively prime factors. □

Theorem is proved. Let us consider a natural example of an arithmetic with the additive factoring property.

Example 3.16. Let us take the arithmetic $\mathbb{N}_1$ of all whole numbers larger than 1 with conventional addition, multiplication and order. In this arithmetic, there are two additively prime numbers 2 and 3. Then it is possible to represent any even number from $\mathbb{N}_1$ as the sum of numbers all of which are equal to 2. It is also possible to represent any odd number from $\mathbb{N}_1$ as the sum of numbers some of which are equal to 2 while others are equal to 3. It will give an additive factorization of any number into the sum of additively prime numbers. However, this factorization is not unique. For instance, we have $6 = 2 + 2 + 2 = 3 + 3$.

Let us consider an infinite abstract well-ordered multiplicatively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with associative and commutative multiplication, which strictly preserves the order.

Lemma 3.16. *The least element b of the set B of elements from A that are not equal to the multiplicative one* 1 *is multiplicatively prime.*

Indeed, as the set B is well-ordered, it has the least element b (cf.,(Fraenkel *et al.*, 1973)). If b is not prime, then $b = a \cdot d$ where by Lemma 2.14, both factors a and d are less than b and are not equal to the multiplicative one 1. As b is the least element of B, this is impossible and thus, b is multiplicatively prime.

Theorem 3.5. *An infinite abstract well-ordered multiplicatively Archimedean prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with associative and commutative multiplication, which strictly preserves the order, has the multiplicative factoring property.*

Proof. At first, let us show that the prearithmetic $\mathbb{A}$ has the multiplicative factoring property. As the abstract prearithmetic $\mathbb{A}$ is infinite and well-ordered, it does not have maximal elements. Indeed, if a is a maximal element in a well-ordered set, then there is only a finite number of elements less than a. At the same time, as well-ordering is also a total ordering (cf.,(Fraenkel *et al.*, 1973)), only one maximal element can exist and thus, the set has to be finite. Consequently, as the prearithmetic $\mathbb{A}$ is infinite, it does not have maximal elements.

Then by Lemma 2.14, the product $a \cdot b$ is larger than both its factors a and b. In other words, a divisor of an element is less than this element. Consequently, any element a from A has only a finite numbers of divisors by the properties of well-ordered sets (Fraenkel *et al.*, 1973).

Let us consider an element a from $\mathbb{A}$ that is not equal to the multiplicative one 1. If we have a factoring

$$a = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

and one of the elements a_i is not multiplicatively prime, then we will have more divisors of a. As a has only a finite numbers of divisors, at some step of the decomposition of the element a into multiplicative factors, we will have only multiplicatively prime factors. □

Theorem is proved.

Multiplication in an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is distributive from the left with respect to difference if for any elements a, b, c and d from $\mathbb{A}$, the equality $c \cdot a + d = c \cdot b$ implies the equality $d = c \cdot e$ where the element e is the difference of b and a. For instance, for integer numbers, it means that if $k = n - m$, $m = uw$ and $n = uv$, then

$$k = n - m = uv - uw = u(v - w)$$

Proposition 30. *The prime factorization obtained in Theorem 3.5 is unique if the abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$ is, in addition, countable, exactly additive with multiplicative cancellation and in which multiplication is distributive with respect to addition and difference.*

Proof. To prove uniqueness up to the order of factors of prime factoring, it is possible to use mathematical induction because the prearithmetic $\mathbb{A}$ is countable and totally ordered.

By Lemma 3.16, the least element b of the set B of elements from $\mathbb{A}$ that are not equal to the multiplicative one 1 is prime. Consequently, it has the unique prime factorization.

Let us assume that a least element in $\mathbb{A}$, which has two or more different prime factorizations

$$a = p_1 \cdot p_2 \cdot \ldots \cdot p_n = q_1 \cdot q_2 \cdot \ldots \cdot q_m \tag{3.2}$$

As multiplication is commutative and is a total order, it is possible to assume

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

and

$$q_1 \leq q_2 \leq \ldots \leq q_m$$

Suppose that two elements, say p_1 and q_1, coincide. Then we have

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n = p_1 \cdot q_2 \cdot \ldots \cdot q_m$$

We can cancel p_1 from both sides of this equality. As a result, we obtain different prime factorizations of a divisor of a, which is less than a by Lemma 2.14. This contradicts minimality of a and shows that any equality $p_i = q_j$ is impossible. Let us take the least elements p_1 and q_1 from both decompositions. As $p_1 \leq q_1$, it is possible to suppose that $p_1 < q_1$. As multiplication strictly preserves the order, we have

$$p_1 \cdot q_1 < q_1 \cdot q_1 \leq q_1 \cdot q_2 \leq q_1 \cdot q_2 \cdot q_3 \cdot \ldots \cdot q_m = a$$

Then

$$a = p_1 \cdot q_1 + c$$

because the prearithmetic $\mathbb{A}$ is exactly additive.

Element c is the difference of a and $p_1 \cdot q_1$. Multiplication in $\mathbb{A}$ is distributive with respect to difference and a is divisible by p_1 and q_1 . Consequently, c is also divisible by p_1 and q_1. Because c is less than a, it has unique prime factorization of the form

$$c = p_1 \cdot q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t$$

Multiplication in $\mathbb{A}$ is distributive with respect to addition. It gives us

$$a = p_1 \cdot p_2 \cdot \ldots \cdot p_n = p_1 \cdot q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + p_1 \cdot q_1 = p_1 \cdot (q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + q_1)$$

Cancelling p_1, we obtain

$$p_2 \cdot \ldots \cdot p_n = q_1 \cdot r_3 \cdot r_4 \cdot \ldots \cdot r_t + q_1$$

As multiplication in A is distributive with respect to addition, we have

$$p_2 \cdot \ldots \cdot p_n = q_1 \cdot (r_3 \cdot r_4 \cdot \ldots \cdot r_t + 1)$$

Thus, q_1 is a divisor of $p_2 \cdot \ldots \cdot p_n$. As q_1 is multiplicatively prime and all $p_2, \ldots, p_n$ are multiplicatively prime, q_1 has to be equal to one of the elements $p_2 \cdot \ldots \cdot p_n$ because the prime factorization $p_2 \cdot \ldots \cdot p_n$ is unique up to the order of factors. However, before we found that such an equality is impossible.

This implies that a has the unique prime factorization. By the principle of mathematical induction, this is true for any element from $\mathbb{A}$. □

Proposition is proved.

Thus, we proved the Fundamental Theorem of Arithmetic for a wide range of abstract prearithmetics because Theorem 3.5 and Proposition 30 imply the following result.

Theorem 3.6. ***Generalized Fundamental Theorem of Arithmetic.*** *An infinitely countable abstract well-ordered multiplicatively Archimedean exactly additive prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, with multiplicative cancellation, distributive with respect to addition and difference, associative and commutative multiplication, which strictly preserves the order, has the multiplicative factoring property.*

Corollary 3.9. *(Fundamental Theorem of Arithmetic). For any natural number larger than* 1 *in the conventional Diophantine arithmetic $\mathbb{N}$, there is a unique up to the order of factors decomposition (factoring) of this number into the product of prime numbers.*

Note that there are non-Diophantine arithmetics in which not all natural numbers have prime factorization (Burgin, 1997).

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order.

Theorem 3.7. *a) If for some elements a and b from $\mathbb{A}$, we have $a < b$, then for some natural number n either*

$$b = n[a] \tag{3.3}$$

or

$$b = n[a] + r \tag{3.4}$$

where $r < a$.

b) If in addition, the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left, then the representation (3.3) or (3.4) is unique.

Proof. a) As the abstract prearithmetic $\mathbb{A}$ is additively Archimedean, for some natural number n, we have

$$b \leq (n+1)[a] \tag{3.5}$$

It is possible to assume that n is the least number for which the inequality (3.5) is true. If in the inequality (3.5), we have equality, then b satisfies formula (3.3) and the statement (a) is proved.

If the inequality (3.5) is strict and n is the least natural number for which the inequality (3.5) is valid, then we obtain

$$n[a] \leq b \leq (n+1)[a] = n[a] + a \tag{3.6}$$

As the abstract prearithmetic $\mathbb{A}$ is exactly additive, then for some natural number k, we have $b = n[a] + r$.

By construction, $r < a$. Indeed, if this is not true, then $a \leq r$ because the order $\leq$ is total. As addition preserves the order, we have

$$n[a] + a = (n+1)[a] \leq n[a] + r$$

.

Because it is assumed $b < (n+1)[a]$, we come to a contradiction, which by the principle of excluded middle, concludes the proof of the part (a).

b) By construction the part $n[a]$ in the representation (3.4) is unique because n is the largest natural number for which $n[a] \leq b$. Now let us suppose

$$b = n[a] + r = n[a] + q$$

. Because the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left, $r = q$.

□

Theorem is proved.

When the abstract prearithmetic $\mathbb{A}$ has the additive zero 0 and is additively Archimedean for all non-zero elements, then it is possible to reduce formulas (3.3) and (3.4) to one formula. Namely, we have the following result.

Corollary 3.10. *a) If for some elements a and b from $\mathbb{A}$, we have $a < b$, then for some natural number n, we have*

$$b = n[a] + r$$

where $0 \leq r < a$.

b) If in addition, the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left for all non-zero elements, then this representation is unique.

Theorem 3.7 also implies a well-known important result from number theory.

Corollary 3.11. ((Landau *et al.*, 1999): Theorem 7). *If for natural numbers a and b, we have $a < b$, then there is a natural number n such that*

$$b = n[a] + r$$

where $0 \leq r < a$.

Note that Theorem 3.7 implies that decompositions (3.3) and (3.4) are true not only for arithmetics of natural or whole numbers but also for the arithmetic of all positive rational numbers, arithmetic of all positive real numbers, arithmetic $2\mathbb{N}$ of all even numbers as well as for many non-Diophantine arithmetics (Burgin, 1997).

Corollary 3.12. *a) In a non-Diophantine arithmetic $\mathbb{A} = (N; +, \cdot, \leq)$, which is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order, the inequality $a < b$, implies either*

$$b = n[a] \tag{3.7}$$

or

$$b = n[a] + r \tag{3.8}$$

for some natural number n and $r < a$.

b) If in addition, the non-Diophantine arithmetic $\mathbb{A}$ is with additive cancellation from the left, then the representation (3.7) or (3.8) is unique.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, multiplicatively Archimedean and exactly multiplicative and in which multiplication preserves the order.

Theorem 3.8. *If for some elements a and b from $\mathbb{A}$, we have $a < b$, then*

$$b = [a]^n$$

or

$$b = [a]^n \cdot r$$

where $r < a$.

Proof is similar to the proof of Theorem 3.7.

This result is a multiplicative counterpart of Theorem 3.7. It is not valid for the Diophantine arithmetic $\mathbb{N}$ but there are abstract prearithmetics and arithmetics that have this property. For instance, let us consider the arithmetic $\mathbb{A}_{pow} = (A; +, \cdot, \leq)$, in which A consists of powers of some natural number m, i.e., $A = \{m^n;\ n = 1, 2, 3, \ldots\}$, multiplication is the same as the conventional multiplication of natural numbers and addition is trivial, i.e., the sum of any two numbers from A is equal to m. In this arithmetic, Theorem 3.8 is valid.

Corollary 3.13. *If for natural numbers a and b from $\mathbb{A}_{pow}$, we have $a < b$, then there is a natural number n such that*

$$b = a^n \cdot r$$

where $1 \leq r < a$.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which is totally ordered, additively and multiplicatively Archimedean and exactly additive and in which addition is associative and preserves the order.

Theorem 3.9. *a) For any elements a and b from $\mathbb{A}$, the following property is valid*

$$b = k_n[a]^n + k_{n-1}[a]^{n-1} + \ldots + k_1[a] + k_0[r] \tag{3.9}$$

where $r < a$, the element k_0 is either 1 *or the symbol $\oslash$, and for $i = 1, 2, 3, \ldots, n$, the element k_i is either a natural number or the symbol $\oslash$, which means that the corresponding element $[a]^i$ is absent in the left part of (3.9).*

b) If the abstract prearithmetic $\mathbb{A}$ is with additive cancellation from the left, then the representation (3.9) is unique.

Proof. a) To prove existence, we use mathematical induction on n.

Given two elements a and b from $\mathbb{A}$, we have either $a > b$ or $a = b$ or $a < b$ because the order in $\mathbb{A}$ is total. In the first two cases, the statement (a) is evident. Indeed, if $a > b$, we can take

$$b = k_0[r] = r$$

If $a = b$, we can take

$$b = 1[a] = a$$

In the case when $a < b$, we suppose that for all elements d from $\mathbb{A}$, such that $a \leq d < b$ the statement (a) is true and prove the equality (3.9). As the abstract prearithmetic $\mathbb{A}$ is multiplicatively Archimedean, there is a natural number n for which

$$b < [a]^{n+1} \tag{3.10}$$

because by Corollary 2.6, $[a]^n < [a]^{n+1}$ for all $n = 1, 2, 3, \ldots$. Taking the least n for which the inequality is valid, we obtain

$$[a]^n \leq b < [a]^{n+1}$$

where by our supposition, $n > 1$. If $b = [a]^n$, then the statement (a) is proved because

$$b = [a]^n = 1[a]^n,$$

If $b > [a]n$, then by Theorem 3.7, we have

$$b = k_n[a]^n + c \tag{3.11}$$

where $c < [a]^n$. Then by our supposition, formula (3.9) is true for c, i.e., we obtain the following equality

$$c = k_m[a]^m + k_{m-1}[a]^{m-1} + \ldots + k_1[a] + k_0[r] \tag{3.12}$$

with $m < n$ and $r < a$. If we substitute c in the equality (3.11) by the right side of the equality (3.12) and add the necessary number of expressions when $m < n - 1$, we obtain the equality (3.9) as addition is associative. The principle of mathematical induction implies that the statement (a) is true for all elements a and b from $\mathbb{A}$.

b) By Theorem 3.7, decomposition (3.11) is unique, while uniqueness of decomposition (3.12) is assumed according to the proof by induction. Uniqueness of decompositions (3.11) and (3.12) implies uniqueness of decomposition (3.9) for the chosen element b. Then the principle of mathematical induction allows us to conclude that the decomposition (3.9) is also unique for any element from $\mathbb{A}$, which is larger than a where a is an arbitrary element from $\mathbb{A}$.

□

Theorem is proved.

Corollary 3.14. ((Landau *et al.*, 1999): Theorem 8). *If a number a is larger than* 1, *then a natural number b can be expressed in one and only one way in the form*

$$b = k_n a^n + k_{n-1}a^{n-1} + \ldots + k_1 a + k_0$$

where $n > 0$, $k_0 > 0$ and $0 \leq k_j < a$ for all $j = 1, 2, \ldots, n$.

Let us consider an abstract prearithmetic $\mathbb{A} = (A; +, \cdot, \leq)$, which has multiplicative 1, is totally ordered, additively Archimedean and exactly additive and in which addition preserves the order and multiplication is associative and distributive from the left with respect to difference and addition.

Theorem 3.10. *If m is the smallest common multiple of elements a and b from $\mathbb{A}$, then any common multiple u of elements a and b is divisible by m.*

Proof. As m is the smallest common multiple of elements a and b, we have $m < u$. Then by Theorem 3.7, $u = n[m]$ or $u = n[m] + r$ and $r < m$.

In the first case, $u = m \cdot n[1]$ because multiplication is distributive from the left with respect to addition and $m \cdot 1 = m$. It means that the statement of Theorem 3.7 is true.

As it was demonstrated, $n[m]$ is divisible by m and thus, by Proposition 19, it is divisible by a and b. As u is also divisible by a and b, the element r is divisible by a and b. It means that in the second case, r is a common multiple of elements a and b. However, this contradicts to the condition that m is the smallest common multiple of elements a and b. Consequently, only the first case is possible. □

Theorem is proved.

Corollary 3.15. ((Landau *et al.*, 1999): Theorem 9). *If m is the smallest common multiple of natural numbers k and h, then any common multiple n of numbers k and h is divisible by m.*

The proved results show that it is possible to develop number theory in abstract prearithmetics similar to the conventional number theory when abstract prearithmetics satisfy relevant conditions such as having an Archimedean property or associative multiplication.

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On Orthogonal Polynomials

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Abstract

The aim of this article is to present some general properties of orthogonal sequence of functions, respectively orthogonal polynomials. We obtain some connections between the classical orthogonal polynomials: Legendre, Chebyshev, Laguerre and Hermite.

Keywords: orthogonal polynomials, orthogonal basis, special functions.
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1. Introduction

The (classical) orthogonal polynomials represent one of the most important problems, extensively investigated in the literature from different points of view (see (Carlitz, 1968), (Chihara, 2011), (Dominici & Maier, 2008), (Grinshpun, 2004), (Mason & Handscomb, 2002) and references therein).

In (McCarthy *et al.*, 1993) the authors study the Dirichlet polynomials as a generalization of the Legendre polynomials and prove a formula of Rodrigues type and some results for the zeros of the generalized Legendre polynomials and an interesting approach is made in (Sun, 2014).

Relations for products of Chebyshev polynomials and the related generating functions are shown in (Cesarano, 2012) and in (Siyi, 2015) some connections between the Chebyshev polynomials, Fibonacci numbers and Lucas numbers are emphasized.

The Laguerre polynomials have been studied in mathematical physics, combinatorics and special functions (see (Koeph, 1997), (Micu & Papp, 2005)) and T. Kim, D. S. Kim, K. W. Hwang and J. J. Seo ((Kim *et al.*, 2016)) derive a family of ordinary differential equations from the generating function of the Laguerre polynomials and prove new identities for those polynomials.

Y. He and F. Yang ((He & Yang, 2018)) obtain recurrence formulas for the Hermite polynomials using generating function methods and Padé approximation techniques and in (Kim & Kim, 2013) a formula for a product of two Hermite polynomials is given. Also, recently, the differential equations associated with squared Hermite polynomials are treated in (Kim *et al.*, 2017).

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In this paper we treat the orthogonal sequence of functions and the (classical) orthogonal polynomials. Some relations for the classical orthogonal polynomials (Legendre, Chebyshev, Laguerre, Hermite) are proved.

2. The main results

Definition 2.1. We consider $\rho : (a,b) \to \mathbb{R}$ a nonnegative function on (a,b) and $\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b)$. We assume that there exist $\int_a^b f_n^2(x)\rho(x)\,dx$. The mapping $\langle\cdot,\cdot\rangle : L^2_{\mathbb{R}}(a,b) \times L^2_{\mathbb{R}}(a,b) \to \mathbb{R}$,

$$\langle f_m, f_n\rangle = \int_a^b f_m(x)f_n(x)\rho(x)\,dx$$

is *inner product* on $L^2_{\mathbb{R}}(a,b)$ and the function ρ is called *the weight function* of the inner product.

Proposition 1. *The inner product from Definition 2.1 satisfies the following properties:*

(*i*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}}, \{f_p\}_{p\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_m + f_n, f_p\rangle = \langle f_m, f_p\rangle + \langle f_n, f_p\rangle$;

(*ii*) $\forall\alpha \in \mathbb{R},\ \forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle \alpha f_m, f_n\rangle = \alpha\langle f_m, f_n\rangle$;

(*iii*) $\forall\{f_m\}_{m\in\mathbb{N}}, \{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_m, f_n\rangle = \langle f_n, f_m\rangle$;

(*iv*) $\forall\{f_n\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(a,b) : \langle f_n, f_n\rangle \geq 0$.

Definition 2.2. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is called *orthogonal* if

$$\langle f_m, f_n\rangle = 0,\ \forall m \neq n.$$

Proposition 2. *Any orthogonal sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on (a,b) is a linearly independent system.*

Theorem 2.1. *Any linearly independent sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ can be orthogonalized.*

Definition 2.3. The orthogonal polynomials $\{p_n\}_{n\in\mathbb{N}}$ on (a,b) are called *classical orthogonal polynomials* relative to the weight function ρ if the following differential equation is checked:

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x),$$

where τ is a polynomial of degree 1 and σ is given by:

$$\sigma(x) = \begin{cases} (x-a)(b-x), & \text{if } a,b \in \mathbb{R} \\ x-a, & \text{if } a \in \mathbb{R}, b = \infty \\ b-x, & \text{if } a = -\infty, b \in \mathbb{R} \\ 1, & \text{if } a = -\infty, b = \infty \end{cases}$$

and

$$\lim_{x\to a} x^n\sigma(x)\rho(x) = \lim_{x\to b} x^n\sigma(x)\rho(x) = 0,\ \forall n \in \mathbb{N}.$$

Remark 1. From Definition 2.3, we obtain the expression of the function ρ:

$$\rho(x) = \begin{cases} (x-a)^{\alpha}(b-x)^{\beta}, & \alpha = \frac{\tau(a)}{b-a} - 1, \beta = -\frac{\tau(b)}{b-a} - 1 \ (a, b \in \mathbb{R}) \\ (x-a)^{\alpha} e^{x\tau'(x)}, & \alpha = \tau(a) - 1 \ (a \in \mathbb{R}, b = \infty) \\ (b-x)^{\beta} e^{-x\tau'(x)}, & \beta = -\tau(b) - 1 \ (a = -\infty, b \in \mathbb{R}) \\ e^{\int \tau(x)dx}, & a = -\infty, b = \infty \end{cases}$$

Remark 2. In particular, considering in Definition 2.3

- $(a, b) = (-1, 1)$ and $\rho(x) = 1$, $\sigma(x) = 1 - x^2$, $\tau(x) = -2x$, $\forall x \in (-1, 1)$, we obtain *the Legendre polynomials* denoted P_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \dfrac{1}{\sqrt{1-x^2}}$, $\sigma(x) = 1 - x^2$, $\tau(x) = -x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the first kind* denoted T_n;
- $(a, b) = (-1, 1)$ and $\rho(x) = \sqrt{1-x^2}$, $\sigma(x) = 1 - x^2$, $\tau(x) = -3x$, $\forall x \in (-1, 1)$, we obtain *the Chebyshev polynomials of the second kind* denoted U_n;
- $(a, b) = (0, \infty)$ and $\rho(x) = x^{\alpha} e^{-x}$, $\sigma(x) = x$, $\tau(x) = -x + \alpha + 1$, $\forall x \in (0, \infty)$ (where $\alpha > -1$), we obtain *the Laguerre polynomials* denoted L_n^{α};
- $(a, b) = (-\infty, \infty)$ and $\rho(x) = e^{-x^2}$, $\sigma(x) = 1$, $\tau(x) = -2x$, $\forall x \in (-\infty, \infty)$, we obtain *the Hermite polynomials* denoted H_n.

Remark 3. Also, the expression of the Chebyshev polynomials of the first, respectively second kind, can be given by:

$$T_n(x) = \cos(n \arccos x), \quad \text{respectively} \quad U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N}.$$

Now we emphasize some properties of the Legendre, Chebyshev, Laguerre and Hermite polynomials (see (Bochner, 1929), (Szego, 1939)).

Proposition 3. *The Legendre polynomials satisfy the following:*

$$(i) \ P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k C_{n-k}^k \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-2k-1)}{(n-k)! 2^k} x^{n-2k}, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N};$$

$$(ii) \ P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N};$$

$$(iii) \ (1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N};$$

$$(iv) \ (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \ \forall x \in (-1, 1), \ \forall n \in \mathbb{N}^*;$$

$$(v) \ \int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n. \end{cases}$$

Proposition 4. *The Chebyshev polynomials of the first kind verify the following:*

(*i*) $T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k C_n^{2k} x^{n-2k}(1-x^2)^k$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*ii*) $T_n(x) = \frac{(-1)^n\sqrt{1-x^2}}{(2n-1)!!} \cdot \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*iii*) $(1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*iv*) $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$.

(*v*) $\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n \neq 0 \\ \pi, & \text{if } m = n = 0. \end{cases}$

Proposition 5. *The Chebyshev polynomials of the second kind verify the following:*

(*i*) $U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k C_{n+1}^{2k+1} x^{n-2k}(1-x^2)^k$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*ii*) $U_n(x) = \frac{(-1)^n(n+1)}{(2n+1)!!\sqrt{1-x^2}} \cdot \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*iii*) $(1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}$;

(*iv*) $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0$, $\forall x \in (-1,1)$, $\forall n \in \mathbb{N}^*$.

(*v*) $\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases}$

Proposition 6. *The Laguerre polynomials verify the following:*

(*i*) $L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} C_{n+\alpha}^{n-k} x^k$, $\forall x \in (0,\infty), \alpha > -1$, $\forall n \in \mathbb{N}$;

(*ii*) $L_n^\alpha(x) = \frac{1}{n!}x^{-\alpha}e^x \cdot \frac{d^n}{dx^n}(x^{n+\alpha}e^{-x})$, $\forall x \in (0,\infty), \alpha > -1$, $\forall n \in \mathbb{N}$;

(*iii*) $x(L_n^\alpha(x))'' + (1+\alpha-x)(L_n^\alpha(x))' + nL_n^\alpha(x) = 0$, $\forall x \in (0,\infty), \alpha > -1$, $\forall n \in \mathbb{N}$;

(*iv*) $(n+1)L_{n+1}^\alpha(x) + (x-2n-1-\alpha)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0$, $\forall x \in (0,\infty), \alpha > -1$, $\forall n \in \mathbb{N}^*$.

(*v*) $\int_0^{\infty} L_m^\alpha(x)L_n^\alpha(x)x^\alpha e^{-x}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{1}{n!}\Gamma(n+1+\alpha), & \text{if } m = n. \end{cases}$

Proposition 7. *The Hermite polynomials satisfy the following:*

(*i*) $H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k C_n^{2k} \cdot (k+1) \cdot (k+2) \cdot \ldots \cdot (2k) \cdot (2x)^{n-2k}$, $\forall x \in (-\infty,\infty)$, $\forall n \in \mathbb{N}$;

(*ii*) $H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n}(e^{-x^2})$, $\forall x \in (-\infty,\infty)$, $\forall n \in \mathbb{N}$;

(*iii*) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N};$

(*iv*) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*;$

(*v*) $\int\limits_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 2^n n!\,\sqrt{\pi}, & \text{if } m = n. \end{cases}$

Further, we show other connections between the classical orthogonal polynomials indicated in Remark 2.

Proposition 8. *For the Legendre polynomials, the following relations hold:*

(*i*) $xP_{n+1}'(x) - P_n'(x) = (n+1)P_{n+1}(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*;$

(*ii*) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$

Proof. Using Proposition 3, we deduce

$$P_n'(x) = \frac{nP_{n-1}(x) - nxP_n(x)}{1 - x^2},\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

Thus, making the computations, we obtain:
(*i*)

$$xP_{n+1}'(x) - P_n'(x) =$$
$$= x \cdot \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1 - x^2} - \frac{nP_{n-1}(x) - nxP_n(x)}{1 - x^2} =$$
$$= \frac{(2n+1)xP_n(x) - nP_{n-1}(x) - nx^2P_{n+1}(x) - x^2P_{n+1}(x)}{1 - x^2} =$$
$$= \frac{(n+1)P_{n+1}(x) - x^2(n+1)P_{n+1}(x)}{1 - x^2} = (n+1)P_{n+1}(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*;$$

(*ii*)

$$P_{n+1}'(x) - xP_n'(x) = \frac{(n+1)P_n(x) - (n+1)xP_{n+1}(x)}{1 - x^2} - x \cdot \frac{nP_{n-1}(x) - nxP_n(x)}{1 - x^2} =$$
$$= \frac{(n+1)P_n(x) - nxP_{n+1}(x) + nx^2P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x)}{1 - x^2} =$$
$$= \frac{(n+1)P_n(x) - xP_{n+1}(x) - nxP_{n-1}(x) + xP_{n+1}(x) - nx^2P_n(x) - x^2P_n(x) + nxP_{n-1}(x)}{1 - x^2} =$$
$$= \frac{(n+1)P_n(x) - x^2(n+1)P_n(x)}{1 - x^2} = (n+1)P_n(x),\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$$

□

Proposition 9. *The Chebyshev polynomials of the first, respectively second kind, verify:*

(*i*) $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*;$

(*ii*) $U_n(x) - 2T_n(x) - U_{n-2}(x) = 0,\ \forall x \in (-1, 1),\ \forall n \in \mathbb{N}^*.$

Proof. From Remark 3, after some computations we obtain:

$$U_n(x) = \frac{T'_{n+1}(x)}{n+1}, \forall x \in (-1,1),\ \forall n \in \mathbb{N}.$$

(i) We have:

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - 2x\frac{T'_n(x)}{n} + \frac{T'_{n-1}(x)}{n-1} =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) + \sin((n-1)\arccos x) - 2x\sin(n\arccos x)] =$$

$$= \frac{1}{\sqrt{1-x^2}}[2x\sin(n\arccos x) - 2x\sin(n\arccos x)] = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(ii) Similarly with (i), we deduce:

$$U_n(x) - 2T_n(x) - U_{n-2}(x) = \frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} - 2T_n(x) =$$

$$= \frac{1}{\sqrt{1-x^2}}[\sin((n+1)\arccos x) - \sin((n-1)\arccos x)] - 2\cos(n\arccos x) =$$

$$= \frac{1}{\sqrt{1-x^2}}2\sin(\arccos x)\cos(n\arccos x) - 2\cos(n\arccos x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

□

Proposition 10. *For the Hermite polynomials, the following relations are checked:*

(i) $H'_{n+1}(x) + H'_{n-1}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x),\ \forall x \in (-\infty,\infty),\ \forall n \geq 2;$

(ii) $H'_{n+1}(x) - 2xH'_n(x) = 2H_n(x) - 4n(n-1)H_{n-2}(x),\ \forall x \in (-\infty,\infty),\ \forall n \geq 2.$

Proof. From Proposition 7, $\forall x \in (-\infty,\infty)$, $\forall n \in \mathbb{N}^*$ it results

$$H'_n(x) = 2nH_{n-1}(x)$$

and then

(i)

$$H'_{n+1}(x) + H'_{n-1}(x) = 2(n+1)H_n(x) + 2(n-1)H_{n-2}(x) =$$

$$= (2n+1)H_n(x) + H_n(x) + 2(n-1)H_{n-2}(x) = (2n+1)H_n(x) + 2xH_{n-1}(x);$$

(ii)

$$H'_{n+1}(x) - 2xH'_n(x) = 2(n+1)H_n(x) - 4nxH_{n-1}(x) =$$

$$= 2H_n(x) + 2n(H_n(x) - 2xH_{n-1}(x)) = 2H_n(x) - 4n(n-1)H_{n-2}(x).$$

□

Proposition 11. *The classical orthogonal polynomials from Remark 2 satisfy:*

(i) $(n+1)P_{n+1}(x) + (1-x^2)P'_n(x) - x(n+1)P_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*;$

(ii) $nT_{n+1}(x) + (1-x^2)T'_n(x) - nxT_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*;$

(*iii*) $nU_{n+1}(x) + (1-x^2)U_n'(x) - nxU_n(x) - U_{n-1}(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*$;

(*iv*) $(n+1)L_{n+1}^{\alpha+1}(x) - (L_n^{\alpha}(x))' - (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n+\alpha)L_{n-1}^{\alpha+1}(x) = 0,\ \forall x \in (0,\infty), \alpha > -1,\ \forall n \in \mathbb{N}^*$;

(*v*) $H_{n+1}(x) + H_n'(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty,\infty),\ \forall n \in \mathbb{N}^*$.

Proof. (*i*) From Proposition 3, (*ii*), respectively (*iv*), we obtain

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x),$$

respectively

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

and then

$$(n+1)P_{n+1}(x) + (1-x^2)P_n'(x) + (nx - 2n - 1)P_n(x) =$$

$$= (2n+1)xP_n(x) - nP_{n-1}(x) + nP_{n-1}(x) - nxP_n(x) - x(n+1)P_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(*ii*) From Proposition 4, (*ii*), respectively (*iv*), we deduce

$$(1-x^2)T_n'(x) = -nxT_n(x) + nT_{n-1}(x),$$

respectively

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

and then

$$nT_{n+1}(x) + (1-x^2)T_n'(x) - nxT_n(x) =$$

$$= 2nxT_n(x) - nT_{n-1}(x) - nxT_n(x) + nT_{n-1}(x) - nxT_n(x) = 0,\ \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(*iii*) From Proposition 5, (*ii*), respectively (*iv*), we observe

$$(1-x^2)U_n'(x) = -nxU_n(x) + (n+1)U_{n-1}(x),$$

respectively

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

and then

$$nU_{n+1}(x) + (1-x^2)U_n'(x) - nxU_n(x) - U_{n-1}(x) =$$

$$= 2nxU_n(x) - nU_{n-1}(x) - nxU_n(x) + (n+1)U_{n-1}(x) - nxU_n(x) - U_{n-1}(x) = 0, \forall x \in (-1,1),\ \forall n \in \mathbb{N}^*.$$

(*iv*) From Proposition 6, (*ii*), respectively (*iv*), it results

$$(L_n^{\alpha})'(x) = -L_{n-1}^{\alpha+1}(x),$$

respectively

$$(n+1)L_{n+1}^{\alpha} = (\alpha - x + 2n + 1)L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x)$$

and then

$$(n+1)L_{n+1}^{\alpha+1}(x) - (L_n^{\alpha}(x))' - (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n+\alpha)L_{n-1}^{\alpha+1}(x) =$$

$$= (\alpha - x + 2n + 2)L_n^{\alpha+1}(x) - (n + \alpha + 1)L_{n-1}^{\alpha+1}(x) + L_{n-1}^{\alpha+1}(x) -$$

$$-(\alpha - x + 2n + 2)L_n^{\alpha+1}(x) + (n+\alpha)L_{n-1}^{\alpha+1}(x) = 0,\ \forall x \in (0,\infty),\ \alpha > -1,\ \forall n \in \mathbb{N}^*.$$

(v) From Proposition 7, (ii), respectively (iv), it follows

$$H_n'(x) = 2nH_{n-1}(x),$$

respectively

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

and then

$$H_{n+1}(x) + H_n'(x) - 2xH_n(x) = 2xH_n(x) - 2nH_{n-1}(x) + 2nH_{n-1}(x) - 2xH_n(x) = 0,\ \forall x \in (-\infty, \infty),\ \forall n \in \mathbb{N}^*.$$

□

Open problem: An important open problem is to define other orthogonal polynomials that fulfill similar conditions with those emphasized for the classical orthogonal polynomials in this article.

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