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Advancing Research, Inspiring Discovery

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Journal's Scope

The journal **Theory and Applications of Mathematics & Computer Science** focuses on Applied Mathematics & Computation. It publishes, free of charge, original papers of high scientific value in all areas of applied mathematics and computer science, but giving a preference to those in the areas represented by the editorial board. In addition, the improved analysis, including the effectiveness and applicability, of existing methods and algorithms, is of importance.

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Foreword

It is with great pride and anticipation that we present this new issue of the journal ***Theory and Applications of Mathematics*** & ***Computer Science***, marking the long-awaited return of our publication after several years of pause. During this time of silence, the world has changed in many ways, yet the pursuit of knowledge and academic inquiry remains as vital as ever. The resumption of this journal is not only a renewal of our commitment to scholarly engagement, but also a celebration of the remarkable individuals who have shaped the landscape of our discipline.

In this spirit, this issue is dedicated to one such individual, whose contributions have left an indelible mark on both our field and our academic community: *Professor Emeritus Mihail Megan*. His work has influenced generations of scholars and practitioners, shaping both theoretical frameworks and practical approaches across *Mathematics* & *Computation*. Whether through his groundbreaking research, his exceptional teaching, or his mentorship of countless students and colleagues, the legacy of *Professor Megan* is one of profound impact.

This special issue serves as both a tribute to his extraordinary body of work and a testament to the lasting influence he continues to have on our academic community.

As we look forward to the future of ***Theory and Applications of Mathematics*** & ***Computer Science***, this special issue also represents a renewal of our mission: to foster thoughtful dialogue, to challenge the boundaries of knowledge.

Thank you for joining us in this celebration, and we hope that the contributions in this issue will inspire new generations of thinkers and researchers, just as *Professor Megan* has inspired so many of us.

Sincerely,
Sorin NĂDĂBAN
Editor-in-Chief
Arad, December 2025

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Sendov's Conjecture and the Geometry of Cubic Polynomials

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Abstract

Sendov's conjecture proposes a tight upper bound for the distance from a zero of a polynomial having roots in the unit disk to the closest critical point. In the particular case of cubic polynomials, the Siebeck-Marden theorem provides a geometric relation between roots and critical points. Based on this, geometric arguments are employed to prove Sendov's conjecture for cubic polynomials and explore its sharpness.

Keywords: Sendov's conjecture, geometry, cubic polynomial.

2020 MSC: 30C15, 52A40.

Every polynomial is characterized by its complex roots, up to the leading coefficient. Moreover, since the complex numbers have a well established geometric structure, it is natural to investigate geometric aspects related to polynomial roots. In the following we will often identify a point in the plane with the associated complex number. Given a non-constant polynomial P of degree at least equal to two, consider the derivative P' and its roots, called critical points pf P. The well known Gauss-Lucas theorem says that the critical points lie in the convex hull of the roots of P. Various works in the literature search for relations between roots and critical points. Among these, there is the following famous conjecture by Sendov Marden (1983), solved for $\deg P \leq 8$ in Brown & Xiang (1999) and for all sufficiently large degrees in Tao (2020).

Conjecture 1. *Suppose the roots of P lie in the unit disk. Then if* $\mathbf{a}$ *is one of these roots, there is a critical point at distace at most* 1 *from* $\mathbf{a}$*.*

There is one particular case where the connection between the roots of P and its critical points is made explicit geometrically. Given three noncolinear points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{C}$, consider the cubic poynomial $P(z) = (z - \mathbf{a})(z - \mathbf{b})(z - \mathbf{c})$, whose derivative $P'(z)$ has two roots $\mathbf{f}_1, \mathbf{f}_2$. It was first observed by Siebeck Siebeck (1864) and later on by Marden in Marden (1945) that $\mathbf{f}_1, \mathbf{f}_2$ are the focal points of the Steiner inellipse associated to the triangle $\Delta\mathbf{abc}$, the unique ellipse tangent to the sides of $\Delta\mathbf{abc}$ at its midpoints. This result generated a lot of interest in the past years. Various elementary proofs exploiting aspects related to complex numbers were given in Badertscher (2014), Dragović & Radnović (2011), Kalman (2008), Minda & Phelps (2008), Northshield (2013), Parish (2006). A proof based solely on geometric arguments was given in Bogo¸sel (2017).

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When presenting Sendov's conjecture in Marden (1945), Marden already gave the geometric interpretation, that if $\Delta\mathbf{abc}$ is contained in the unit disk, then each one of the vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is at a distance at most one from the focal points $\mathbf{f}_1$ or $\mathbf{f}_2$ of the Steiner inellipse. A direct proof, using complex numbers may be found in (Jin & Zeng, n.d., p. 22). The goal of this note is to give a purely geometrical proof of Sendov's conjecture for cubic polynomials. Moreover, the sharpness of this result can be explored geometrically, investigating polynomials of high degree having only three distinct roots.

1. A surprising property related to the Steiner inellipse

In Allaire & Yao (2012) the following identity is proved for any inellipse tangent to the sides of the triangle $\Delta\mathbf{abc}$ and having focal points $\mathbf{f}_1, \mathbf{f}_2$:

$$\frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} + \frac{\mathbf{bf}_1 \cdot \mathbf{bf}_2}{\mathbf{ba} \cdot \mathbf{bc}} + \frac{\mathbf{cf}_1 \cdot \mathbf{cf}_2}{\mathbf{ca} \cdot \mathbf{cb}} = 1. \tag{1.1}$$

The proof given in Allaire & Yao (2012) is elegant and uses synthetic geometry arguments, by symmetrizing one of the focal points $\mathbf{f}_i$ about the sides of the triangle. For the Steiner inellipse, one has the stronger property that all three terms in (1.1) are equal

$$\frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} = \frac{\mathbf{bf}_1 \cdot \mathbf{bf}_2}{\mathbf{ba} \cdot \mathbf{bc}} = \frac{\mathbf{cf}_1 \cdot \mathbf{cf}_2}{\mathbf{ca} \cdot \mathbf{cb}} = \frac{1}{3}. \tag{1.2}$$

Proofs of (1.2), based on the Siebeck-Marden theorem, using relations between polynomial roots and critical points are rather straightforward and well known. Nevertheless, it is possible to prove (1.2) with purely geometric arguments, using only the basic properties of the Steiner inellipse, which we recall below.

Theorem 1.1. *1.* **(Reflection property)** *If the inellipse is tangent to the side* $\mathbf{ab}$ *at the interior point* $\mathbf{d}$ *then the angle bisector of* $\angle\mathbf{f}_1\mathbf{df}_2$ *is orthogonal to* $\mathbf{ab}$.

2. The focal ponts $\mathbf{f}_1, \mathbf{f}_2$ *of any inellipse are* isogonal conjugates *in* $\Delta\mathbf{abc}$.

3. An inellipse is uniquely determined by its center. In particular, the Steiner inellipse is the unique inellipse whose center coincides with the centroid of $\Delta\mathbf{abc}$.

Proofs of these facts can be found in many classical references. The proof of 1. is a simple consequence of the minimality of $\mathbf{xf}_1 + \mathbf{xf}_2$ for $\mathbf{x} \in \mathbf{ab}$, also known as Heron's problem. A geometric proof of 2. is recalled in Bogo¸sel (2017). The proof of 3. may be found in Chakerian (1979) or (Bogo¸sel, 2017, Theorem 2).

In order to prove the sequence of equalities shown in (1.2) consider the reflection $\mathbf{f}_1'$ of $\mathbf{f}_1$ with respect to $\mathbf{ab}$ and denote by $\mathbf{d}$ the tangency point of the Steiner inellipse with $\mathbf{ab}$, as shown in Figure 1. Of course, $\mathbf{d}$ is the midpoint of $\mathbf{ab}$ and $\mathbf{f}_1', \mathbf{d}, \mathbf{f}_2$ are colinear, in view of the reflection property recalled in Theorem 1.1. Then one can write the following equalities regarding triangle areas:

$$S_{\Delta\mathbf{af}_1'\mathbf{f}_2} = S_{\Delta\mathbf{af}_1'\mathbf{d}} + S_{\Delta\mathbf{adf}_2} = S_{\Delta\mathbf{adf}_1} + S_{\Delta\mathbf{adf}_2} = 2S_{\Delta\mathbf{adg}},$$

where $\mathbf{g}$ is the midpoint of $\mathbf{f}_1, \mathbf{f}_2$, i.e. the center of the Steiner inellipse and the centroid of $\Delta\mathbf{abc}$. The last of the above area equalities comes from the fact that the corresponding triangles have a common basis $\mathbf{ad}$ and the average of the distances from $\mathbf{f}_1$ and $\mathbf{f}_2$ to $\mathbf{ad}$ is equal to the distance from $\mathbf{g}$ to $\mathbf{ad}$ (see Figure 1).

Since $\mathbf{d}$ is the midpoint of $\mathbf{ab}$ and $\mathbf{g}$ is the centroid, we conclude by observing that

$$S_{\Delta\mathbf{af}_1'\mathbf{f}_2} = 2S_{\Delta\mathbf{adg}} = S_{\Delta\mathbf{abg}} = \frac{1}{3}S_{\Delta\mathbf{abc}}.$$

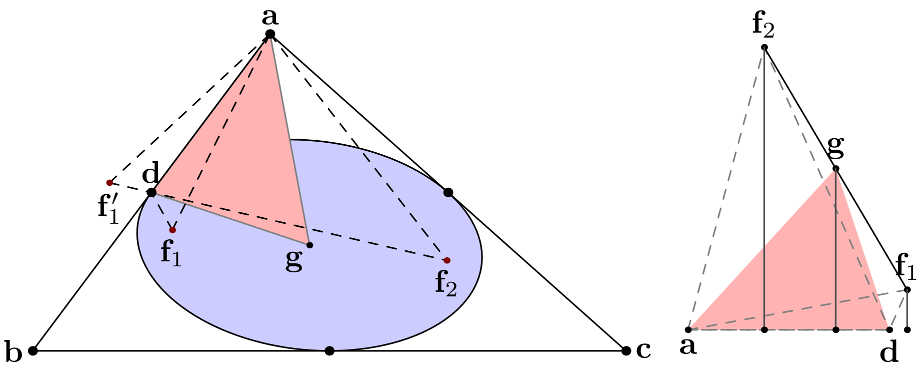


Figure 1: (left) The Steiner inellipse: symmetrize the focal point $\mathbf{f}_1$ with respect to $\mathbf{ab}$. (right) Proving that $2S_{\Delta\mathbf{agd}} = S_{\Delta\mathbf{af}_1\mathbf{d}} + S_{\Delta\mathbf{af}_2\mathbf{d}}$: observe that $2d(\mathbf{g}, \mathbf{ad}) = d(\mathbf{f}_1, \mathbf{ad}) + d(\mathbf{f}_2, \mathbf{ad})$.

Triangles $\Delta\mathbf{af}_1'\mathbf{f}_2$ and $\Delta\mathbf{abc}$ have equal angles in the vertex $\mathbf{a}$, since $\mathbf{f}_1, \mathbf{f}_2$ are isogonal conjugates. Therefore we have

$$\frac{1}{3} = \frac{S_{\Delta\mathbf{af}_1'\mathbf{f}_2}}{S_{\Delta\mathbf{abc}}} = \frac{\mathbf{af}_1' \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}} = \frac{\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{ab} \cdot \mathbf{ac}},$$

hence (1.2) holds.

Remark 1.2. *It should be noted that* (1.2) *provides* yet another geometric proof *of the Siebeck-Marden theorem. Indeed, since* $\mathbf{f}_1, \mathbf{f}_2$ *are isogonal conjugates and* (1.2) *implies the equality* $|\mathbf{a}-\mathbf{b}||\mathbf{a}-\mathbf{c}| = 3|\mathbf{a}-\mathbf{f}_1||\mathbf{a}-\mathbf{f}_2|$, *we also have* $(\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{c}) = 3(\mathbf{a}-\mathbf{f}_1)(\mathbf{a}-\mathbf{f}_2)$. *Analogue identies are obtained for vertices* $\mathbf{b}$ *and* $\mathbf{c}$. *This it implies that the second degree polynomials*

$$P'(z) = (z-\mathbf{a})(z-\mathbf{b}) + (z-\mathbf{b})(z-\mathbf{c}) + (z-\mathbf{c})(z-\mathbf{a})$$

and

$$Q(z) = 3(z-\mathbf{f}_1)(z-\mathbf{f}_2)$$

are equal for three distinct points $z \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *and have the same leading coefficient. Therefore,* $P'(z) = Q(z)$.

2. Geometric proof of Sendov's conjecture for cubic polynomials

The geometric interpretation of Sendov's conjecture is the following: if $\mathbf{f}_1, \mathbf{f}_2$ are the focal points for the Steiner inellipse then at least one of the lengths $\mathbf{af}_1, \mathbf{af}_2$ is smaller than R, the circumradius of $\Delta\mathbf{abc}$. Observing that $\mathbf{f}_1, \mathbf{f}_2$ can get arbitrarily close and they coincide for an equilateral triangle, it is reasonable to attempt proving that a certain *mean* of $\mathbf{af}_1, \mathbf{af}_2$ is smaller than R.

Since we have precise information regarding the product of $\mathbf{af}_1$ and $\mathbf{af}_2$, let us first compare the geometric mean of $\mathbf{af}_1, \mathbf{af}_2$ with R. In view of (1.2) and the law of sines we have

$$\sqrt{\mathbf{af}_1 \cdot \mathbf{af}_2} = \sqrt{\frac{\mathbf{ab} \cdot \mathbf{ac}}{3}} = \sqrt{\frac{4 \sin \widehat{\mathbf{b}} \sin \widehat{\mathbf{c}}}{3}} R.$$

Since there exist triangles with angles $\widehat{\mathbf{b}} = \widehat{\mathbf{c}} = \pi/2 - \varepsilon$, the geometric mean can get arbitrarily close to $\frac{2}{\sqrt{3}}R$. Therefore, R cannot be an upper bound for this mean.

The next classical mean, smaller than the geometric one is the harmonic mean. This mean contains $\mathbf{af}_1+\mathbf{af}_2$ at the denominator, therefore a lower bound is needed for this quantity. It is classical, and immediate to prove, that the median is at most equal to the average of the neighboring sides, implying that $\mathbf{af}_1 + \mathbf{af}_2 \geq 2\mathbf{ag}$. A classical proof of this fact constructs the parallelogram $\mathbf{af}_1\mathbf{a}'\mathbf{f}_2$ and uses the triangle inequality in $\Delta\mathbf{af}_1\mathbf{a}'$, showing moreover that equality can hold if and only if $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2$ are colinear. Denoting by $\mathbf{m}$ the midpoint of $\mathbf{bc}$ we have $\mathbf{ag} = \frac{2}{3}\mathbf{am}$ which, using again the law of sines $\mathbf{a} = 2R\sin\widehat{\mathbf{a}}$, gives

$$\min\{\mathbf{af}_1, \mathbf{af}_2\} \leq \frac{2\mathbf{af}_1 \cdot \mathbf{af}_2}{\mathbf{af}_1 + \mathbf{af}_2} \leq \frac{\mathbf{ab} \cdot \mathbf{ac}}{2\mathbf{am}} = \frac{2S_{\Delta\mathbf{abc}}}{2\mathbf{am} \cdot \sin\widehat{\mathbf{a}}} = \frac{h_{\mathbf{a}}}{\mathbf{am}}R, \tag{2.1}$$

where $h_{\mathbf{a}}$ is the length of the height of $\Delta\mathbf{abc}$ from vertex $\mathbf{a}$. Since the height always has a smaller length than the median, we are done. We have, therefore proved the following result.

Theorem 2.1. *The harmonic mean of* $\mathbf{af}_1$ *and* $\mathbf{af}_2$ *is at most equal to R. As a consequence, Sendov's conjecture holds for cubic polynomials.*

When presenting Conjecture 1 in Marden (1983), Marden talks about *extremal polynomials*, i.e. polynomials for which equality is attained in Sendov's estimate. Assuming that $\min\{\mathbf{af}_1, \mathbf{af}_2\} = R$, the sequence of inequalities in (2.1) becomes a sequence of equalities. The equality of the minimum and the harmonic mean implies $\mathbf{af}_1 = \mathbf{af}_2$. The equality $\mathbf{af}_1 + \mathbf{af}_2 = \mathbf{ag}$ can hold only if $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{g}$ are colinear. Moreover, $h_{\mathbf{a}} = \mathbf{am}$, implying that $\Delta\mathbf{abc}$ is isosceles. Since $\mathbf{a}, \mathbf{f}_1, \mathbf{f}_2$ are colinear and $\mathbf{af}_1 = \mathbf{af}_2$ it follows that $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{g}$. This implies that the Steiner inellipse is a circle, therefore $\Delta\mathbf{abc}$ is equilateral. Thus, we arrive at a geometric proof of (Marden, 1983, Conjecture II) for cubic polynomials.

Theorem 2.2. *If* $\min\{\mathbf{af}_1, \mathbf{af}_2\} = R$ *then* $\Delta\mathbf{abc}$ *is equilateral. Polynomials of degree* 3 *for which equality is attained in Sendov's estimate have three equidistant roots on the unit disk.*

3. Sharpness of Sendov's conjecture

It is well known that Sendov's result is sharp as the following well known examples illustrate:

- $P(z) = z^n - z$ has a root at the origin, while $P'(z)$ has n roots with modulus $n^{1/n} \to 1$ as $n \to \infty$.
- $P(z) = z^n - 1$ has n roots on the unit circle, while $P'(z)$ has all roots equal to 0.

However, it turns out that considering polynomials of the form $P(z) = (z - \mathbf{a})^m(z - \mathbf{b})^n(z - \mathbf{c})^p$, which in view of Bogo¸sel (2017); Marden (1945) are also related to inscribed ellipses, one can find examples where the roots of $P'(z)$ *different from* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are at distance larger than 1 from at least one of the vertices of the triangle.

As already observed in Marden (1945), a polynomial of the form

$$P(z) = (z - \mathbf{a})^m(z - \mathbf{b})^n(z - \mathbf{c})^p \tag{3.1}$$

has only two critical points lying strictly inside $\Delta\mathbf{abc}$ which are the focal points of an inellipse. More generally, in Bogo¸sel (2017) it was observed that for $\alpha, \beta, \gamma > 0$ the critical points of the logarithmic potential $L(z) = \alpha\log(z - \mathbf{a}) + \beta\log(z - \mathbf{b}) + \gamma\log(z - \mathbf{c})$ are the focal points of an inellipse dividing the sides of $\Delta\mathbf{abc}$ into ratios $\beta/\gamma, \gamma/\alpha, \alpha/\beta$. Conversely, given any inellipse $\mathcal{E}$, there exists a logarithmic potential $L(z)$ of the same form whose critical points are the focal points of $\mathcal{E}$.

Counterexample 1. Let $\Delta\mathbf{abc}$ be a non-equilateral triangle having two angles $\widehat{\mathbf{b}}, \widehat{\mathbf{c}}$ greater than $\pi/3$. The distance from the incenter to $\mathbf{a}$ is given by $4R\sin(\widehat{\mathbf{b}}/2)\sin(\widehat{\mathbf{c}}/2)$ and is greater than R in this case. Then there exist positive integers m, n, p such that the critical points $\mathbf{f}_1, \mathbf{f}_2$ of (3.1) different from $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are in an ε neighborhood of the incenter, not containing the circumcenter. It is enough to consider m, n, p positive integers such that $\frac{m}{m+n+p}, \frac{n}{m+n+p}, \frac{p}{m+n+p}$ are approximations of the coefficients of the logarithmic potential $L(z)$ whose associated inellipse is the incircle. Therefore, for the vertex $\mathbf{a}$ and the considered inellipse we have $\min\{\mathbf{af}_1, \mathbf{af}_2\} > R$. It may be observed that if m, n, p give such an example, choosing exponents km, kn, kp, for any integer $k \geq 1$ in (3.1) produces the same critical points.

Counterexample 2. Furthermore, consider the case of only one multiple root, given by $P(z) = (z - \mathbf{a})^m(z - \mathbf{b})(z - \mathbf{c})$ for $m \geq 2$. The critical points of P are the focal points $\mathbf{f}_1^m, \mathbf{f}_2^m$ of an inellipse $\mathcal{E}_m$ tangent to the sides at points dividing the sides into ratios $m/1, 1/1, 1/m$. Let us observe the behavior of $\mathbf{f}_1^m, \mathbf{f}_2^m$ as $m \to \infty$. See Figure 2 for a graphical representation. The inellipse $\mathcal{E}_m$ is tangent to $\mathbf{bc}$ at its midpoint $\mathbf{m}$ and at $\mathbf{ab}, \mathbf{ac}$ at $\mathbf{p}_m, \mathbf{n}_m$, respectively. The points $\mathbf{n}_m, \mathbf{p}_m$ divide $\mathbf{ac}, \mathbf{ab}$ into segments having ratios $m/1$. It is classical that the line joining $\mathbf{b}$ to the midpoint $\mathbf{q}_m$ of $\mathbf{mp}_m$ passes through the center of $\mathcal{E}_m$. For a proof, it is enough to transform $\mathcal{E}_m$ into a circle via an affine transformation. In the same way the line going through $\mathbf{c}$ and the midpoint $\mathbf{r}_m$ of $\mathbf{mn}_m$ passes through the center of $\mathcal{E}_m$. Thus, the center $\mathbf{c}_m$ of $\mathcal{E}_m$ is given by $\mathbf{bq}_m \cap \mathbf{cr}_m$.

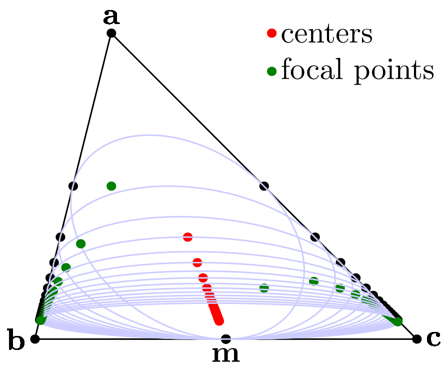
It is straightforward to observe that $\mathbf{c}_m$ converges to $\mathbf{m}$ and $\mathbf{f}_1^m, \mathbf{f}_2^m$ converge to $\mathbf{b}, \mathbf{c}$ as $m \to \infty$. When $\min\{\mathbf{ab}, \mathbf{ac}\} > R$, or equivalently, $\min\{\widehat{\mathbf{b}}, \widehat{\mathbf{c}}\} > \pi/6$, this produces a class of polynomials of arbitrarily large degree for which the distance from the only multiple root $\mathbf{a}$ to the critical points different from $\mathbf{a}$ is larger than R.

Therefore, there exist polynomials P of arbitrarily large degree with roots in the unit disk such that the distance from one zero of P to all critical points which are not roots is greater than 1.

Remark 3.1. *For more geometric constructions related to ellipses (Eagles, 1885, Chapter IV) is a great reference. All figures involving inellipses in this paper are constructed using the software Metapost and constructive ideas from this reference. For the sake of completeness, let us describe the steps for constructing an inellipse $\mathcal{E}$ starting from the tangency points $\mathbf{m} \in \mathbf{bc}, \mathbf{n} \in \mathbf{ac}, \mathbf{p} \in \mathbf{ab}$. It is classical that a necessary and sufficient condition for $\mathcal{E}$ to exist is that $\mathbf{am}, \mathbf{bn}, \mathbf{cp}$ are concurrent.*

1. *Let $\mathbf{q}$ be the midpoint of $\mathbf{mp}$ and $\mathbf{r}$ be the midpoint of $\mathbf{mn}$. Then the center of the inellipse is $\mathbf{o} \in \mathbf{bq} \cap \mathbf{cr}$.*
2. *Construct $\mathbf{m}'$ the symmetric of $\mathbf{m}$ through $\mathbf{o}$. Thus $\mathbf{mm}'$ is a* diameter *of $\mathcal{E}$.*
3. *Draw the line d through $\mathbf{o}$ parallel to $\mathbf{bc}$. Define $\mathbf{s} \in d \cap \mathbf{ac}$ and let $\mathbf{s}'$ be the intersection of d with the parallel to $\mathbf{mm}'$ through $\mathbf{n}$. Construct $\mathbf{d} \in d$ such that $\mathbf{od}^2 = \mathbf{os} \cdot \mathbf{os}'$. Then $\mathbf{d} \in \mathcal{E}$ (Eagles, 1885, p. 107). Construct $\mathbf{d}'$, the symmetric of $\mathbf{d}$ through $\mathbf{o}$. In this way we constructed another diameter $\mathbf{dd}'$* conjugate *to $\mathbf{mm}'$.*
4. *Construct the segment $\mathbf{ee}'$, orthogonal to $\mathbf{dd}'$, having midpoint at $\mathbf{m}'$ such that $\mathbf{ee}' = \mathbf{dd}'$. The angle bisector of $\angle \mathbf{eoe}'$ is the principal axis of $\mathcal{E}$. (Eagles, 1885, p. 111)*
5. *The lengths of the axes of the ellipse are given by $\mathbf{oe} + \mathbf{oe}'$ and $|\mathbf{oe} - \mathbf{oe}'|$.*

The construction is depicted in Figure 2.



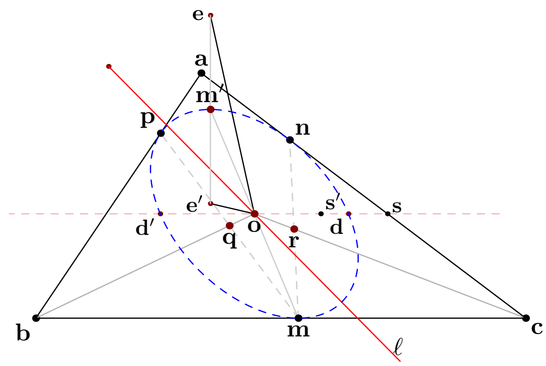


Figure 2: (left) Construction of $\mathcal{E}_m$ for $m = 1, ..., 15$. The centers $\mathbf{c}_m$ and focal points are also represented. The focal points converge to $\mathbf{b}$ and $\mathbf{c}$ as $m \to \infty$. (right) Constructing an inellipse starting from tangency points.

References

Allaire, P.R., Zhou J. and H. Yao (2012). Proving a nineteenth century ellipse identity. *The Mathematical Gazette* **96**(535), 161–165.

Badertscher, E. (2014). A simple direct proof of Marden's theorem. *Amer. Math. Monthly* **121**(6), 547–548.

Bogo¸sel, B. (2017). A geometric proof of the Siebeck-Marden theorem. *Amer. Math. Monthly* **124**(5), 459–463.

Brown, J.E. and G. Xiang (1999). Proof of the Sendov conjecture for polynomials of degree at most eight. *J. Math. Anal. Appl.* **232**(2), 272–292.

Chakerian, G.D. (1979). A distorted view of geometry. In: *Mathematical Plums* (Ross Honsberger, Ed.). Vol. 4 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, D.C.

Dragović, V. and M. Radnović (2011). *Poncelet porisms and beyond*. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics.

Eagles, T. H. (1885). *Constructive geometry of plane curves, with numerous examples*. Macmillan and Co.

Jin, Z. and L. Zeng (n.d.). Study on inscribed ellipse of triangle.

Kalman, D. (2008). An elementary proof of Marden's theorem. *Amer. Math. Monthly* **115**(4), 330–338.

Marden, M. (1945). A note on the zeros of the sections of a partial fraction. *Bull. Amer. Math. Soc.* **51**, 935–940.

Marden, M. (1983). Conjectures on the critical points of a polynomial. *Amer. Math. Monthly* **90**(4), 267–276.

Minda, D. and S. Phelps (2008). Triangles, ellipses, and cubic polynomials. *Amer. Math. Monthly* **115**(8), 679–689.

Northshield, S. (2013). Geometry of cubic polynomials. *Math. Mag.* **86**(2), 136–143.

Parish, J.L. (2006). On the derivative of a vertex polynomial. *Forum Geom.* **6**, 285–288.

Siebeck, J. (1864). Ueber eine neue analytische behandlungweise der brennpunkte. *J. Reine Angew. Math.* **64**, 175–182.

Tao, T. (2020). Sendov's conjecture for sufficiently high degree polynomials.

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Basic Properties of Relative Entropic Normalized Determinant of Positive Operators in Hilbert Spaces

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Abstract

For positive invertible operators A, B and $x \in H$, $\|x\| = 1$, we define the *relative entropic normalized determinant* $D_x(A|B)$ by

$$D_x(A|B) := \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle.$$

In this paper we show, among others, that

$$\left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle}\right)^{\langle Ax,x\rangle} \le D_x(A|B) \le \left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right)^{\langle Ax,x\rangle}$$

for all $A, B > 0$ and $x \in H$ with $\|x\| = 1$. Several other properties of $D_x(\cdot|\cdot)$ are also provided.

Keywords: Positive operators, normalized determinants, inequalities.

2020 MSC: 47A63, 26D15, 46C05.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. Fujii & Seo (1998), Fujii *et al.* (1998), introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, Fujii & Seo (1998).

For each unit vector $x \in H$, see also Hiramatsu & Seo (2021), we have:

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(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In Fujii & Seo (1998) the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M \ln m - m \ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by Specht (1960)

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In Fujii *et al.* (1998), the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A \ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA \ln(tA)\, x, x\rangle\right) \\
&= \exp\left(-\langle tA(\ln t + \ln A)\, x, x\rangle\right) = \exp\left(-\langle (tA\ln t + tA\ln A)\, x, x\rangle\right) \\
&= \exp\left(-\langle Ax, x\rangle\, t\ln t\right)\exp\left(-t\langle A\ln Ax, x\rangle\right) \\
&= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax, x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In the recent paper Dragomir (2022) we showed among others that, if $A, B > 0$, then for all $x \in H$, $\|x\| = 1$ and $t \in [0, 1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle}, \tag{1.8}$$

where $A > 0$ and $x \in H$, $\|x\| = 1$.

Definition 1.1. *For positive invertible operators A, B and $x \in H$ with $\|x\| = 1$ we define the relative entropic normalized determinant $D_x(A|B)$ by*

$$D_x(A|B) := \exp\langle S(A|B)\, x, x\rangle = \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle,$$

where the relative operator entropy $S(A|B)$, is defined by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}.$$

We observe that for $A > 0$,

$$D_x(A|1_H) = \exp\langle S(A|1_H)\, x, x\rangle = \exp\left(-\langle A\ln Ax, x\rangle\right) = \eta_x(A),$$

where $\eta_x(\cdot)$ is the *normalized entropic determinant* and for $B > 0$,

$$D_x(1_H|B) := \exp\langle S(1_H|B)\, x, x\rangle = \exp\langle \ln Bx, x\rangle = \Delta_x(B),$$

where $\Delta_x(\cdot)$ is the *normalized determinant*.

Motivated by the above results, in this paper we show, among others, that

$$\left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle}\right)^{\langle Ax,x\rangle} \leq D_x(A|B) \leq \left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right)^{\langle Ax,x\rangle}$$

for all $A, B > 0$ and $x \in H$ with $\|x\| = 1$. Several other properties of $D_x(\cdot|\cdot)$ are also provided.

2. Relative entropic normalized determinant

Kamei and Fujii Fujii & Kamei (1989*b*), Fujii & Kamei (1989*a*) defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \quad (2.1)$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki Nakamura & Umegaki (1961).

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-}\lim_{\varepsilon\to 0+} S(A+\varepsilon 1_H|B)$$

if it exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t\ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A\ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (2.1) is a relative version of the operator entropy.

For $A = 1_H$ in (2.1) we have

$$S(1_H|B) = \ln B$$

for positive contraction B.

Following (Furuta *et al.*, 2005, p. 149-p. 155), we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \quad (2.2)$$

(ii) We have the inequalities

$$S(A|B) \leq A(\ln\|B\| - \ln A) \text{ and } S(A|B) \leq B - A; \quad (2.3)$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^*S(A|B)T \leq S(T^*AT|T^*BT).$$

(viii) The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S\left(tA+(1-t)B|tC+(1-t)D\right)\geq tS\left(A|C\right)+(1-t)S\left(B|D\right)$$

for any $t\in[0,1]$.

For other results on the relative operator entropy see Dragomir (2015*b*), Dragomir (2015*a*), Furuichi (2015), Kim (2012), Kluza & Niezgoda (2014), Moslehian *et al.* (2013) and Nikoufar (2014).

Observe that, if we replace in (2.2) B with A, then we get

$$\begin{aligned}S\left(B|A\right)&=A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}\\&=A^{1/2}\left(-A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2},\end{aligned}$$

therefore we have

$$A^{1/2}\left(A^{-1/2}BA^{-1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2}=-S\left(B|A\right)\tag{2.4}$$

for positive invertible operators A and B.

It is well know that, in general $S\left(A|B\right)$ is not equal to $S\left(B|A\right)$.

In Uhlmann (1977), A. Uhlmann has shown that the relative operator entropy $S\left(A|B\right)$ can be represented as the strong limit

$$S\left(A|B\right)=s\text{-}\lim_{t\to0}\frac{A\sharp_tB-A}{t},\tag{2.5}$$

where

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu\in[0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu=\frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν.

For $B=1_H$ we have

$$A\sharp_\nu 1_H=A^{1-\nu}$$

while for $A=1_H$ we get

$$1_H\sharp_\nu B=B^\nu$$

for any real number ν.

For $t>0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also Furuichi *et al.* (2004)) by

$$T_t\left(A|B\right):=\frac{A\sharp_tB-A}{t}.$$

We then have

$$T_t\left(A|1_H\right):=\frac{A\sharp_t1_H-A}{t}=\frac{A^{1-t}-A}{t},\ t>0$$

and

$$T_t\left(1_H|B\right):=\frac{B^t-1_H}{t},\ t>0$$

for A, $B>0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t\left(\cdot|\cdot\right)$ has been obtained in Fujii & Kamei (1989*b*) for $0<t\leq1$. However, it hods for any $t>0$.

Theorem 2.1. *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$T_t\left(A|B\right)\left(A\sharp_tB\right)^{-1}A\leq S\left(A|B\right)\leq T_t\left(A|B\right).\tag{2.6}$$

In particular, we have for $t = 1$ that

$$\left(1_H - AB^{-1}\right) A \leq S\left(A|B\right) \leq B - A, \text{ Fujii \& Kamei (1989b)} \tag{2.7}$$

and for $t = 2$ that

$$\frac{1}{2}\left(1_H - \left(AB^{-1}\right)^2\right) A \leq S\left(A|B\right) \leq \frac{1}{2}\left(BA^{-1}B - A\right). \tag{2.8}$$

The case $t = \frac{1}{2}$ is of interest as well. Since in this case we have

$$T_{1/2}\left(A|B\right) := 2\left(A\sharp B - A\right)$$

and

$$T_{1/2}\left(A|B\right)\left(A\sharp_{1/2}B\right)^{-1} A = 2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A,$$

hence by (2.6) we get

$$2\left(1_H - A\left(A\sharp B\right)^{-1}\right) A \leq S\left(A|B\right) \leq 2\left(A\sharp B - A\right) \leq B - A. \tag{2.9}$$

We have the following fundamental properties for the relative entropic normalized determinant:

Proposition 2.1. *Assume that* $A, B > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

1. *We have the upper bound*
$$D_x\left(A|B\right) \leq \frac{\exp\langle Bx, x\rangle}{\exp\langle Ax, x\rangle};$$
2. *For any* C, D *positive invertible operators we have that*
$$D_x\left(A + B|C + D\right) \geq D_x\left(A|C\right) D_x\left(B|D\right); \tag{2.10}$$
3. *If* $B \leq C$ *then*
$$D_x\left(A|B\right) \leq D_x\left(A|C\right);$$
4. *If* $B_n \downarrow B$ *then*
$$D_x\left(A|B_n\right) \downarrow D_x\left(A|B\right);$$
5. *For* $\alpha > 0$ *we have*
$$D_x\left(\alpha A|\alpha B\right) = \left[D_x\left(A|B\right)\right]^{\alpha}.$$

The proof follows by the properties "(ii)-(iii)" above.

Corollary 2.1. *For* $A, B > 0$, $\alpha, \beta > 0$ *and* $x \in H$ *with* $\|x\| = 1$, *we have*

$$\frac{\eta_x(A + B)}{\eta_x(A)\eta_x(B)} \geq \frac{\alpha^{\langle Ax,x\rangle}\beta^{\langle Bx,x\rangle}}{(\alpha + \beta)^{\langle (A+B)x,x\rangle}}. \tag{2.11}$$

In particular, for $\alpha = \beta = 1$, *we get*

$$\frac{\eta_x(A + B)}{\eta_x(A)\eta_x(B)} \geq \frac{1}{2^{\langle (A+B)x,x\rangle}}. \tag{2.12}$$

Proof. Observe that

$$\begin{aligned} D_x(A|\alpha 1_H) &= \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}\alpha 1_H A^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle \\ &= \exp\left\langle A^{\frac{1}{2}}(\ln\alpha 1_H - \ln A)A^{\frac{1}{2}}x, x\right\rangle \\ &= \exp(\langle Ax, x\rangle \ln\alpha - \langle A\ln Ax, x\rangle) = \alpha^{\langle Ax,x\rangle}\eta_x(A). \end{aligned}$$

Then by (2.10) for $C = \alpha 1_H$ and $D = \beta 1_H$ we have

$$D_x(A + B|(\alpha+\beta)1_H) \geq D_x(A|\alpha 1_H)D_x(B|\beta 1_H),$$

namely

$$(\alpha+\beta)^{\langle (A+B)x,x\rangle}\eta_x(A+B) \geq \alpha^{\langle Ax,x\rangle}\eta_x(A)\beta^{\langle Bx,x\rangle}\eta_x(B)$$

and the inequality (2.11) is obtained. □

Also, we have:

Corollary 2.2. *For $C, D > 0$, $\gamma, \delta > 0$ and $x \in H$ with $\|x\| = 1$, we have*

$$\frac{[\Delta_x(C+D)]^{\gamma+\delta}}{[\Delta_x(C)]^{\gamma}[\Delta_x(D)]^{\delta}} \geq \frac{(\gamma+\delta)^{\gamma+\delta}}{\gamma^{\gamma}\delta^{\delta}}. \tag{2.13}$$

In particular, for $\gamma = \delta = 1$, we get

$$\frac{[\Delta_x(C+D)]^2}{\Delta_x(C)\Delta_x(D)} \geq 4. \tag{2.14}$$

Proof. Observe that

$$\begin{aligned} D_x(\gamma 1_H|C) &= \exp\left\langle(\gamma 1_H)^{\frac{1}{2}}\left(\ln\left((\gamma 1_H)^{-\frac{1}{2}}C(\gamma 1_H)^{-\frac{1}{2}}\right)\right)(\gamma 1_H)^{\frac{1}{2}}x, x\right\rangle \\ &= \exp\langle\gamma(\ln C - \ln\gamma)x, x\rangle = \exp(\gamma\langle\ln Cx, x\rangle - \ln(\gamma^{\gamma})) \\ &= \frac{\exp(\gamma\langle\ln Cx, x\rangle)}{\exp\ln(\gamma^{\gamma})} = \left(\frac{\Delta_x(C)}{\gamma}\right)^{\gamma}. \end{aligned}$$

By (2.10) we have

$$D_x((\gamma+\delta)1_H|C+D) \geq D_x(\gamma 1_H|C)D_x(\delta 1_H|D),$$

namely

$$\left(\frac{\Delta_x(C+D)}{\gamma+\delta}\right)^{\gamma+\delta} \geq \left(\frac{\Delta_x(C)}{\gamma}\right)^{\gamma}\left(\frac{\Delta_x(D)}{\delta}\right)^{\delta}.$$

□

Proposition 2.2. *Assume that $A, B > 0$ and $x \in H$ with $\|x\| = 1$.*

(a) We have

$$D_x(A|B) \leq \|B\|^{\langle Ax,x\rangle}\eta_x(A) \tag{2.15}$$

(aa) For every operator T with $Tx \neq 0$, we have

$$\left[D_{\frac{Tx}{\|Tx\|}}(A|B)\right]^{\|Tx\|^2} \leq D_x(T^*AT|T^*BT). \tag{2.16}$$

(aaa) For every $C, D > 0$

$$D_x(tA + (1-t)B|tC + (1-t)D) \geq [D_x(A|C)]^t [D_x(B|D)]^{1-t} \tag{2.17}$$

for all $t \in [0, 1]$.

Proof. a. By taking the inner product over $x \in H$ with $\|x\| = 1$ in (ii) we get

$$\begin{aligned} D_x(A|B) &= \exp\langle S(A|B)x, x\rangle \leq \exp\langle(\ln\|B\| A - A\ln A)x, x\rangle \\ &= \exp(\ln\|B\| \langle Ax, x\rangle - \langle A\ln Ax, x\rangle) \\ &= \exp\left(\ln\|B\|^{\langle Ax,x\rangle}\right)\exp(-\langle A\ln Ax, x\rangle) \\ &= \|B\|^{\langle Ax,x\rangle}\eta_x(A) \end{aligned}$$

and the statement is proved.

aa. If we take the inner product over $x \in H$ with $\|x\| = 1$ in (vii) then we get

$$\exp\langle T^*S(A|B)Tx, x\rangle \leq \exp\langle S(T^*AT|T^*BT)x, x\rangle = D_x(T^*AT|T^*BT).$$

Also, if $Tx \neq 0$,

$$\begin{aligned} \exp\langle T^*S(A|B)Tx, x\rangle &= \exp\langle S(A|B)Tx, Tx\rangle \\ &= \exp\left\langle \|Tx\|^2 S(A|B)\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\right\rangle \\ &= \left(\exp\left\langle S(A|B)\frac{Tx}{\|Tx\|}, \frac{Tx}{\|Tx\|}\right\rangle\right)^{\|Tx\|^2} \\ &= \left[D_{\frac{Tx}{\|Tx\|}}(A|B)\right]^{\|Tx\|^2}, \end{aligned}$$

which proves the statement.

aaa. If we take the inner product over $x \in H$ with $\|x\| = 1$ in (viii), then we get for all $t \in [0, 1]$ that

$$\begin{aligned} &D_x(tA + (1-t)B|tC + (1-t)D) \\ &= \exp\langle S(tA + (1-t)B|tC + (1-t)D)x, x\rangle \\ &\geq \exp\langle[tS(A|C) + (1-t)S(B|D)]x, x\rangle \\ &= \exp[t\langle S(A|C)x, x\rangle + (1-t)\langle S(B|D)x, x\rangle] \\ &= (\exp\langle S(A|C)x, x\rangle)^t [\exp\langle S(B|D)x, x\rangle]^{1-t} \\ &= [D_x(A|C)]^t [D_x(B|D)]^{1-t} \end{aligned}$$

and the statement is proved. □

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ a \text{ if } b = a. \end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 2.3. *With the assumptions of Proposition 2.2,*

$$\int_0^1 D_x(tA+(1-t)B|tC+(1-t)D)dt \geq L\left(D_x(A|B), D_x(C|D)\right). \tag{2.18}$$

and

$$D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq \int_0^1 \left[D_x\left((1-t)A+tB|(1-t)C+tD\right)\right]^{1/2} \times \left[D_x\left(tA+(1-t)B|tC+(1-t)D\right)\right]^{1/2} dt. \tag{2.19}$$

Proof. If we take the integral over $t \in [0,1]$ in (2.17), then we get

$$\int_0^1 D_x(tA+(1-t)B|tC+(1-t)D)dt \geq \int_0^1 \left[D_x(A|C)\right]^t \left[D_x(B|D)\right]^{1-t} dt = L\left(D_x(A|C), D_x(B|D)\right)$$

for all $A, B, C, D > 0$, which proves (2.18).

We get from (2.17) for $t = 1/2$ that

$$D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq \left[D_x(A|C)\right]^{1/2}\left[D_x(B|D)\right]^{1/2}.$$

If we replace A by $(1-t)A+tB$, B by $tA+(1-t)B$, C by $(1-t)C+tD$ and D by $tC+(1-t)D$ we obtain

$$D_x\left(\frac{A+B}{2}|\frac{C+D}{2}\right) \geq \left[D_x\left((1-t)A+tB|(1-t)C+tD\right)\right]^{1/2} \times \left[D_x\left(tA+(1-t)B|tC+(1-t)D\right)\right]^{1/2}.$$

By taking the integral, we derive the desired result (2.19). □

By the use of Theorem 2.1 we can also state:

Proposition 2.3. *Assume that $A, B > 0$ and $x \in H$ with $\|x\| = 1$. Then for any $t > 0$ we have*

$$\exp\left\langle T_t(A|B)(A\sharp_t B)^{-1}Ax, x\right\rangle \leq D_x(A|B) \leq \exp\langle T_t(A|B)x, x\rangle. \tag{2.20}$$

In particular, we have for $t = 1$ that

$$\frac{\exp\langle Ax, x\rangle}{\exp\langle AB^{-1}Ax, x\rangle} \leq D_x(A|B) \leq \frac{\exp\langle Bx, x\rangle}{\exp\langle Ax, x\rangle} \tag{2.21}$$

and for $t = 2$ that

$$\left(\frac{\exp\langle Ax, x\rangle}{\left\langle (AB^{-1})^2 Ax, x\right\rangle}\right)^{\frac{1}{2}} \leq D_x(A|B) \leq \left(\frac{\exp\left\langle BA^{-1}Bx, x\right\rangle}{\exp\langle Ax, x\rangle}\right)^{\frac{1}{2}}. \tag{2.22}$$

We have the following bounds for the *normalized entropic determinant.*

Corollary 2.4. *Assume that* $A > 0$ *and* $x \in H$ *with* $\|x\| = 1$. *If* $\alpha, t > 0$, *then*

$$\begin{aligned} &\alpha^{-\langle Ax,x\rangle} \exp\left\langle \frac{A - \alpha^{-t}A^{t+1}}{t} x, x \right\rangle \\ &\le \eta_x(A) \\ &\le \alpha^{-\langle Ax,x\rangle} \exp\left\langle \frac{\alpha^t A^{1-t} - A}{t} x, x \right\rangle . \end{aligned} \tag{2.23}$$

In particular, for $\alpha = 1$, *we get*

$$\exp\left\langle \frac{A - A^{t+1}}{t} x, x \right\rangle \le \eta_x(A) \le \exp\left\langle \frac{A^{1-t} - A}{t} x, x \right\rangle, \tag{2.24}$$

for all $t > 0$.

For $t = 1$, *we get*

$$\begin{aligned} &\alpha^{-\langle Ax,x\rangle} \exp\left\langle \left(A - \alpha^{-1}A^2\right) x, x \right\rangle \\ &\le \eta_x(A) \\ &\le \alpha^{-\langle Ax,x\rangle} \exp\left\langle (\alpha 1_H - A)\, x, x \right\rangle , \end{aligned} \tag{2.25}$$

for all $\alpha > 0$.

Also, for $\alpha = t = 1$, *we obtain*

$$\exp\left\langle \left(A - A^2\right) x, x \right\rangle \le \eta_x(A) \le \exp\left\langle (1_H - A)\, x, x \right\rangle . \tag{2.26}$$

Proof. If we take $B = \alpha 1_H$ in (2.20), we get

$$\begin{aligned} \exp\left\langle T_t\,(A|\alpha 1_H)\,(A\sharp_t\,(\alpha 1_H))^{-1}\,Ax, x \right\rangle &\le D_x\,(A|\alpha 1_H) \\ &\le \exp\left\langle T_t\,(A|\alpha 1_H)\, x, x \right\rangle . \end{aligned} \tag{2.27}$$

Observe that

$$A\sharp_t\,(\alpha 1_H) = A^{1/2}\left(A^{-1/2}\,(\alpha 1_H)\,A^{-1/2}\right)^t A^{1/2} = \alpha^t A^{1-t}$$

and

$$T_t\,(A|\alpha 1_H) = \frac{A\sharp_t\,(\alpha 1_H) - A}{t} = \frac{\alpha^t A^{1-t} - A}{t}.$$

Also

$$\begin{aligned} T_t\,(A|\alpha 1_H)\,(A\sharp_t\,(\alpha 1_H))^{-1}\,A &= \frac{\alpha^t A^{1-t} - A}{t}\left(\alpha^t A^{1-t}\right)^{-1} A \\ &= \frac{A - A\left(\alpha^t A^{1-t}\right)^{-1} A}{t} \\ &= \frac{A - \alpha^{-t}A^{t+1}}{t}. \end{aligned}$$

Then by (2.27) we get

$$\exp\left\langle \frac{A - \alpha^{-t}A^{t+1}}{t} x, x \right\rangle \le \alpha^{\langle Ax,x\rangle}\eta_x(A) \le \exp\left\langle \frac{\alpha^t A^{1-t} - A}{t} x, x \right\rangle$$

and the inequality (2.23) is obtained. □

We also have the following bounds for the *normalized determinant*.

Corollary 2.5. *Assume that $B > 0$ and $x \in H$ with $\|x\| = 1$. If $\beta, t > 0$, then*

$$\beta \exp\left\langle \frac{1_H - \beta^t B^{-t}}{t} x, x \right\rangle \le \Delta_x(B) \le \beta \exp\left\langle \frac{\beta^{-t} B^t - 1_H}{t} x, x \right\rangle. \tag{2.28}$$

In particular, for $\beta = 1$, we get

$$\exp\left\langle \frac{1_H - B^{-t}}{t} x, x \right\rangle \le \Delta_x(B) \le \exp\left\langle \frac{B^t - 1_H}{t} x, x \right\rangle, \tag{2.29}$$

for all $t > 0$.

For $t = 1$, we get

$$\beta \exp\left\langle \left(1_H - \beta B^{-1}\right) x, x \right\rangle \le \Delta_x(B) \le \beta \exp\left\langle \left(\beta^{-1} B - 1_H\right) x, x \right\rangle, \tag{2.30}$$

for all $\beta > 0$.

Also, for $\beta = t = 1$, we obtain

$$\exp\left\langle \left(1_H - B^{-1}\right) x, x \right\rangle \le \Delta_x(B) \le \exp\left\langle (B - 1_H)\, x, x \right\rangle. \tag{2.31}$$

Proof. We have from (2.20) for $A = \beta 1_H$ that

$$\begin{aligned} \exp\left\langle T_t\left(\beta 1_H | B\right)\left(\left(\beta 1_H\right) \sharp_t B\right)^{-1}\left(\beta 1_H\right) x, x \right\rangle &\le D_x\left(\beta 1_H | B\right) \\ &\le \exp\left\langle T_t\left(\beta 1_H | B\right) x, x \right\rangle. \end{aligned} \tag{2.32}$$

Observe that

$$\left(\beta 1_H\right) \sharp_t B = \left(\beta 1_H\right)^{1/2} \left(\left(\beta 1_H\right)^{-1/2} B \left(\beta 1_H\right)^{-1/2}\right)^t \left(\beta 1_H\right)^{1/2} = \beta^{1-t} B^t,$$

and

$$T_t\left(\left(\beta 1_H\right) | B\right) := \frac{\left(\beta 1_H\right) \sharp_t B - \beta 1_H}{t} = \frac{\beta^{1-t} B^t - \beta 1_H}{t}.$$

Also,

$$\begin{aligned} T_t\left(\beta 1_H | B\right)\left(\left(\beta 1_H\right) \sharp_t B\right)^{-1}\left(\beta 1_H\right) &= \frac{\beta^{1-t} B^t - \beta 1_H}{t} \left(\beta^{1-t} B^t\right)^{-1} \beta \\ &= \frac{\beta - \beta\left(\beta^{1-t} B^t\right)^{-1} \beta}{t} \\ &= \frac{\beta - \beta^{t+1} B^{-t}}{t}. \end{aligned}$$

Then by (2.32) we get

$$\exp\left\langle \frac{\beta 1_H - \beta^{t+1} B^{-t}}{t} x, x \right\rangle \le \left(\frac{\Delta_x(B)}{\beta}\right)^{\beta} \le \exp\left\langle \frac{\beta^{1-t} B^t - \beta 1_H}{t} x, x \right\rangle.$$

By taking the power $1/\beta$ we get

$$\exp\left\langle \frac{\beta 1_H - \beta^{t+1} B^{-t}}{\beta t} x, x \right\rangle \le \frac{\Delta_x(B)}{\beta} \le \exp\left\langle \frac{\beta^{1-t} B^t - \beta 1_H}{\beta t} x, x \right\rangle,$$

which is equivalent to (2.28). □

3. Several Bounds

We have the following bounds for the relative entropic normalized determinant:

Theorem 3.1. *Assume that $A, B > 0$ and $x \in H$ with $\|x\| = 1$. Then for any $s > 0$ we have*

$$\begin{aligned} & s^{\langle Ax,x\rangle} \exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right) \\ & \leq D_x(A|B) \\ & \leq s^{\langle Ax,x\rangle} \exp\left(\frac{\langle Bx, x\rangle - s\langle Ax, x\rangle}{s}\right). \end{aligned} \tag{3.1}$$

The best lower bound in the first inequality is

$$\left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle}\right)^{\langle Ax,x\rangle} \leq D_x(A|B), \tag{3.2}$$

while the best upper bound in the second inequality is

$$D_x(A|B) \leq \left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right)^{\langle Ax,x\rangle}. \tag{3.3}$$

Proof. We use the gradient inequality for differentiable convex functions f on the open interval

$$f'(s)(t-s) \geq f(t) - f(s) \geq f'(t)(t-s)$$

for all $t, s \in I$.

If we write this inequality for the function ln on $(0, \infty)$, then we get

$$\frac{t}{s} - 1 \geq \ln t - \ln s \geq 1 - \frac{s}{t}$$

for all $t, s \in (0, \infty)$.

Using the functional calculus for positive operator $T > 0$, we get

$$\frac{1}{s}T - 1_H \geq \ln T - \ln s 1_H \geq 1_H - sT^{-1}.$$

for all $s \in (0, \infty)$.

If we take $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} > 0$, then we get

$$\frac{1}{s}A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_H \geq \ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \ln s 1_H \geq 1_H - sA^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$$

for all $s \in (0, \infty)$.

If we multiply both sides by $A^{\frac{1}{2}} > 0$, then we get

$$\frac{1}{s}B - A \geq A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}} - (\ln s)A \geq A - sAB^{-1}A$$

for all $s \in (0, \infty)$.

Now, if we take the inner product for $x \in H$ with $\|x\| = 1$, then we get

$$\frac{1}{s}\langle Bx, x\rangle - \langle Ax, x\rangle \geq \left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle - (\ln s)\langle Ax, x\rangle$$
$$\geq \langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle$$

for all $s \in (0, \infty)$.

By taking the exponential, we derive

$$\exp\left(\frac{\langle Bx, x\rangle - s\langle Ax, x\rangle}{s}\right) \geq \frac{\exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle}{\exp\left[(\ln s)\langle Ax, x\rangle\right]}$$
$$\geq \exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right)$$

for all $s \in (0, \infty)$, which is equivalent to (3.1).

Now, consider the function

$$f(s) := s^{\langle Ax,x\rangle}\exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right), \; s \in (0, \infty).$$

We have

$$f'(s) = \langle Ax, x\rangle s^{\langle Ax,x\rangle - 1}\exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right)$$
$$- \left\langle AB^{-1}Ax, x\right\rangle s^{\langle Ax,x\rangle}\exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right)$$
$$= s^{\langle Ax,x\rangle - 1}\exp\left(\langle Ax, x\rangle - s\left\langle AB^{-1}Ax, x\right\rangle\right)$$
$$\times\left(\langle Ax, x\rangle - \left\langle AB^{-1}Ax, x\right\rangle s\right).$$

We observe that the function f is increasing on $\left(0, \frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}\right)$ and decreasing on $\left(\frac{\langle Ax,x\rangle}{\langle AB^{-1}Ax,x\rangle}, \infty\right)$. Therefore

$$\sup_{s\in(0,\infty)} f(s) = f\left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle}\right) = \left(\frac{\langle Ax, x\rangle}{\langle AB^{-1}Ax, x\rangle}\right)^{\langle Ax,x\rangle},$$

which gives the best lower bound in (3.1).

Now, consider the function

$$g(s) := s^{\langle Ax,x\rangle}\exp\left(\frac{\langle Bx, x\rangle}{s} - \langle Ax, x\rangle\right), \; s \in (0, \infty).$$

We have

$$g'(s) := \langle Ax, x\rangle s^{\langle Ax,x\rangle - 1}\exp\left(\frac{\langle Bx, x\rangle}{s} - \langle Ax, x\rangle\right)$$
$$+ s^{\langle Ax,x\rangle}\exp\left(\frac{\langle Bx, x\rangle}{s} - \langle Ax, x\rangle\right)\left(-\frac{\langle Bx, x\rangle}{s^2}\right)$$
$$= s^{\langle Ax,x\rangle - 1}\exp\left(\frac{\langle Bx, x\rangle}{s} - \langle Ax, x\rangle\right)\left(\langle Ax, x\rangle - \frac{\langle Bx, x\rangle}{s}\right)$$
$$= s^{\langle Ax,x\rangle - 2}\exp\left(\frac{\langle Bx, x\rangle}{s} - \langle Ax, x\rangle\right)\left(\langle Ax, x\rangle s - \langle Bx, x\rangle\right).$$

We observe that the function g is decreasing on $\left(0, \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)$ and increasing on $\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}, \infty\right)$. Therefore

$$\inf_{s\in(0,\infty)} g(s) = g\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right) = \left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle},$$

which gives the best upper bound in (3.1). □

Corollary 3.1. *Assume that $A > 0$ and $x \in H$ with $\|x\| = 1$. Then for any $s > 0$ we have*

$$\begin{aligned} &s^{\langle Ax,x\rangle} \exp\left(\langle Ax,x\rangle - s\left\langle A^2x,x\right\rangle\right) \\ &\le \eta_x(A) \le s^{\langle Ax,x\rangle} \exp\left(\frac{1}{s} - \langle Ax,x\rangle\right). \end{aligned} \tag{3.4}$$

The best lower bound for $\eta_x(A)$ is obtained for $s = \frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}$, namely

$$\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle} \le \eta_x(A).$$

The best upper bound for $\eta_x(A)$ is obtained for $s = \langle Ax,x\rangle^{-1}$, namely

$$\eta_x(A) \le \langle Ax,x\rangle^{-\langle Ax,x\rangle}.$$

Proof. If we take $B = 1_H$ in (3.1), then we get

$$s^{\langle Ax,x\rangle} \exp\left(\langle Ax,x\rangle - s\left\langle A^2x,x\right\rangle\right) \le \eta_x(A) \le s^{\langle Ax,x\rangle} \exp\left(\frac{1 - s\langle Ax,x\rangle}{s}\right),$$

which is equivalent to (3.4). □

Corollary 3.2. *Assume that $B > 0$ and $x \in H$ with $\|x\| = 1$. Then for any $s > 0$ we have*

$$s\exp\left(1 - s\left\langle B^{-1}x,x\right\rangle\right) \le \Delta_x(B) \le s\exp\left(\frac{\langle Bx,x\rangle - s}{s}\right). \tag{3.5}$$

The best lower bound for $\Delta_x(B)$ is obtained for $s = \left\langle B^{-1}x,x\right\rangle^{-1}$, namely

$$\left\langle B^{-1}x,x\right\rangle^{-1} \le \Delta_x(B).$$

The best upper bound for $\Delta_x(B)$ is obtained for $s = \langle Bx,x\rangle$, namely

$$\Delta_x(A) \le \langle Bx,x\rangle.$$

Theorem 3.2. *Assume that $A, B > 0$ with the property that $0 < mA \le B \le MA$ for some constants m, $M > 0$ and $x \in H$ with $\|x\| = 1$. Then*

$$\left(\frac{\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}}{S\left(\frac{M}{m}\right)}\right)^{\langle Ax,x\rangle} \le D_x(A|B) \le \left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle} \tag{3.6}$$

and

$$\begin{aligned} 0 &\le \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle} - [D_x(A|B)]^{\langle Ax,x\rangle^{-1}} \\ &\le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M - m} - 1\right]. \end{aligned} \tag{3.7}$$

Proof. We observe that for $x \in H$ with $\|x\| = 1$

$$\begin{aligned}
D_x(A|B) &= \exp\left\langle A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, x\right\rangle \\
&= \exp\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle \\
&= \exp\left[\left\|A^{\frac{1}{2}}x\right\|^2\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right] \\
&= \left(\exp\left[\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right]\right)^{\left\|A^{\frac{1}{2}}x\right\|^2} \\
&= \left(\exp\left[\left\langle\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)\frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}, \frac{A^{\frac{1}{2}}x}{\left\|A^{\frac{1}{2}}x\right\|}\right\rangle\right]\right)^{\langle Ax,x\rangle} \\
&= \left(\Delta_{A^{1/2}x/\|A^{1/2}x\|}(A^{-1/2}BA^{-1/2})\right)^{\langle Ax,x\rangle},
\end{aligned}$$

which gives that

$$[D_x(A|B)]^{\langle Ax,x\rangle^{-1}} = \Delta_{A^{1/2}x/\|A^{1/2}x\|}(A^{-1/2}BA^{-1/2}) \tag{3.8}$$

for $x \in H$ with $\|x\| = 1$.

Since $0 < mA \le B \le MB$ for the positive operators A, B is equivalent with $0 < m \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le M$, then by (1.4) for $A^{1/2}x/\left\|A^{1/2}x\right\|$ and for the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ we get

$$1 \le \frac{\left\langle A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{1/2}x/\left\|A^{1/2}x\right\|, A^{1/2}x/\left\|A^{1/2}x\right\|\right\rangle}{\Delta_{A^{1/2}x/\|A^{1/2}x\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})} \le S\left(\frac{M}{m}\right),$$

namely

$$1 \le \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle\,\Delta_{A^{1/2}x/\|A^{1/2}x\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})} \le S\left(\frac{M}{m}\right),$$

which gives by (3.8) that

$$1 \le \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle\,[D_x(A|B)]^{\langle Ax,x\rangle^{-1}}} \le S\left(\frac{M}{m}\right).$$

By taking the power $\langle Ax, x\rangle > 0$ we get

$$1 \le \frac{\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\langle Ax,x\rangle}}{D_x(A|B)} \le \left[S\left(\frac{M}{m}\right)\right]^{\langle Ax,x\rangle}.$$

From (1.2) we get

$$\begin{aligned}
0 &\le \left\langle A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A^{1/2}x/\left\|A^{1/2}x\right\|, A^{1/2}x/\left\|A^{1/2}x\right\|\right\rangle \\
&\quad - \Delta_{A^{1/2}x/\|A^{1/2}x\|}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\
&\le L(m, M)\left[\ln L(m, M) + \frac{M\ln m - m\ln M}{M - m} - 1\right],
\end{aligned}$$

namely

$$0 \le \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle} - [D_x(A|B)]^{\langle Ax,x\rangle^{-1}} \le L(m, M)\left[\ln L(m, M) + \frac{M\ln m - m\ln M}{M - m} - 1\right]$$

for $x \in H$ with $\|x\| = 1$. □

Remark 3.1. *Assume that $B > 0$ with the property that $0 < m1_H \le B \le M1_H$ for some constants $m, M > 0$ and $x \in H$ with $\|x\| = 1$. Then by $A = 1_H$ in the above Theorem 3.2 we recapture the inequality (1.4) and (1.2).*

If we take $B = 1_H$ in Theorem 3.2, then for $0 < mA \le 1_H \le MA$ for some constants $m, M > 0$ and $x \in H$ with $\|x\| = 1$. Then

$$\left(\langle Ax, x\rangle S\left(\frac{M}{m}\right)\right)^{-\langle Ax,x\rangle} \le \eta_x(A) \le \langle Ax, x\rangle^{-\langle Ax,x\rangle} \tag{3.9}$$

and

$$0 \le \langle Ax, x\rangle^{-1} - [\eta_x(A)]^{\langle Ax,x\rangle^{-1}} \le L(m, M)\left[\ln L(m, M) + \frac{M\ln m - m\ln M}{M - m} - 1\right]. \tag{3.10}$$

If $0 < n1_H \le A \le N1_H$, then by taking $m = N^{-1}$ and $M = n^{-1}$ we get $0 < mA \le 1_H \le MA$ and by (3.9) and (3.10) we obtain

$$\left[\langle Ax, x\rangle S\left(\frac{N}{n}\right)\right]^{-\langle Ax,x\rangle} \le \eta_x(A) \le \langle Ax, x\rangle^{-\langle Ax,x\rangle} \tag{3.11}$$

and

$$0 \le \langle Ax, x\rangle^{-1} - [\eta_x(A)]^{\langle Ax,x\rangle^{-1}} \le \frac{L(n, N)}{nN}\left[\ln\left(\frac{L(n, N)}{nN}\right) + \frac{N\ln n - n\ln N}{N - n} - 1\right] \tag{3.12}$$

for $x \in H$ with $\|x\| = 1$.

References

Dragomir, S. S. (2022). Some basic results for the normalized entropic determinant of positive operators in hilbert spaces. *RGMIA Res. Rep. Coll.* **25**, Art. 35, 14 pp. Available at: https://rgmia.org/papers/v25/v25a36.pdf.

Dragomir, S.S. (2015*a*). Reverses and refinements of several inequalities for relative operator entropy. *Preprint RGMIA Res. Rep. Coll.* **19**(150), 1–22.

Dragomir, S.S. (2015*b*). Some inequalities for relative operator entropy. *Preprint RGMIA Res. Rep. Coll.* **18**(145), 1–12.

Fujii, J. I. and E. Kamei (1989*a*). Relative operator entropy in noncommutative information theory. *Math. Japon.* **34**(3), 341–348.

Fujii, J. I. and E. Kamei (1989*b*). Uhlmann's interpolational method for operator means. *Math. Japon.* **34**(4), 541–547.

Fujii, J. I. and Y. Seo (1998). Determinant for positive operators. *Sci. Math.* **1**, 153–156.

Fujii, J. I., S. Izumino and Y. Seo (1998). Determinant for positive operators and specht's theorem. *Sci. Math.* **1**, 307–310.

Furuichi, S. (2015). Precise estimates of bounds on relative operator entropies. *Math. Ineq. Appl.* **18**, 869–877.

Furuichi, S., K. Yanagi and K. Kuriyama (2004). Fundamental properties for tsallis relative entropy. *J. Math. Phys.* **45**, 4868–4877.

Furuta, T., J. Mi?i? Hot, J. Pe?ari? and Y. Seo (2005). *Mond-Pe?ari? Method in Operator Inequalities. Inequalities for bounded selfadjoint operators on a Hilbert space*. Vol. 1 of *Monographs in Inequalities*. Element. Zagreb. xiv+262 pp., loose errata. ISBN: 953-197-572-8.

Hiramatsu, S. and Y. Seo (2021). Determinant for positive operators and oppenheim's inequality. *J. Math. Inequal.* **15**(4), 1637–1645.

Kim, I. H. (2012). Operator extension of strong subadditivity of entropy. *J. Math. Phys.* **53**, 122204.

Kluza, P. and M. Niezgoda (2014). Inequalities for relative operator entropies. *Electron. J. Linear Algebra* **27**, Art. 1066.

Moslehian, M. S., F. Mirzapour and A. Morassaei (2013). Operator entropy inequalities. *Colloq. Math.* **130**, 159–168.

Nakamura, M. and H. Umegaki (1961). A note on the entropy for operator algebras. *Proc. Japan Acad.* **37**, 149–154.

Nikoufar, I. (2014). On operator inequalities of some relative operator entropies. *Adv. Math.* **259**, 376–383.

Specht, W. (1960). Zur theorie der elementaren mittel. *Math. Z.* **74**, 91–98.

Uhlmann, A. (1977). Relative entropy and the wigner–yanase–dyson–lieb concavity in an interpolation theory. *Comm. Math. Phys.* **54**(1), 21–32.

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Nonuniform Generalized Exponential Dichotomies Concepts for Skew-evolution Semiflows

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Abstract

The aim of the paper is to prove characterizations for two concepts of nonuniform dichotomy in the general context of skew-evolution semiflows.

We use invariant, respectively strongly invariant projector families, to obtain the results.

Keywords: Skew-evolution semiflows, nonuniform generalized exponential dichotomy, Banach spaces.

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1. Introduction

One of the most representative asymptotic properties studied for dynamical systems is the dichotomy, treated from various perspectives in (Barreira & Valls, 2018), (Barreira & Valls, 2019), (Bento *et al.*, 2017), (Găină *et al.*, 2023), (Megan *et al.*, 2007), (Sasu *et al.*, 2013).

The sufficient criteria for the uniform exponential stability of evolution operators, obtained by S. Rolewicz in (Rolewicz, 1986) represented an important direction to give qualitative results for the asymptotic behaviours of dynamical systems, using integral conditions.
In this sense, we mention the integral characterizations proved in (Mihiţ & Megan, 2017), for a general property of splitting with growth rates and recently, in (Megan *et al.*, 2025), Zabczyk-Rolewicz type methods are used for the uniform exponential stability of nonautonomous dynamics. Also, in (Sasu *et al.*, 2012), the uniform exponential stability of variational discrete systems, respectively skew-product flows are treated through Zabczyk-Rolewicz techniques.

Concerning the notion of generalized exponential dichotomy, it is introduced by J. S. Muldowney in (Muldowney, 1984) and in (Lupa *et al.*, 2015), the authors approach this property in the case of evolution operators.

In this article, the concepts of generalized exponential dichotomy and nonuniform generalized exponential dichotomy of Rolewicz type are studied for skew-evolution semiflows in Banach spaces. Characterizations for these properties are established, considering invariant and strongly invariant projector families.

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2. Definitions and notations

Let Θ be a metric space, X a Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$.

Also, we consider

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}, \quad T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

and $\Gamma = \Theta \times X$.

Definition 2.1. A continuous mapping $\sigma : \Delta \times \Theta \to \Theta$ is called *evolution semiflow* if:

(es_1) $\sigma(s, s, \theta) = \theta$, for all $(s, \theta) \in \mathbb{R}_+ \times \Theta$;

(es_2) $\sigma(t, s, \sigma(s, t_0, \theta)) = \sigma(t, t_0, \theta)$, for all $(t, s, t_0, \theta) \in T \times \Theta$.

Definition 2.2. A pair $C = (\sigma, \Phi)$ is said to be a *skew-evolution semiflow* on Γ if σ is an evolution semiflow on Θ and $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ satisfies the relations:

(ses_1) $\Phi(s, s, \theta) = I$ (the identity operator on X), for all $(s, \theta) \in \mathbb{R}_+ \times \Theta$;

(ses_2) $\Phi(t, s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta) = \Phi(t, t_0, \theta)$, for all $(t, s, t_0, \theta) \in T \times \Theta$;

(ses_3) $(t, s, \theta) \mapsto \Phi(t, s, \theta)x$ is continuous for every $x \in X$.

Example 2.1. We consider Θ a locally compact metric space, X a Banach space, σ an evolution semiflow on Θ and $A : \Theta \to \mathcal{B}(X)$ a continuous mapping. If $\Phi(t, s, \theta)$ is the solution of the problem

$$\dot{x}(t) = A(\sigma(t, s, \theta))x(t), \quad t \geq s \geq 0,$$

then the pair $C = (\sigma, \Phi)$ is a skew-evolution semiflow on Γ.

In what follows, we recall the notions of family of projectors and (strongly) invariant family of projectors.

Definition 2.3. A continuous mapping $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is *family of projectors* if

$$P^2(t, \theta) = P(t, \theta), \quad \text{for all} \quad (t, \theta) \in \mathbb{R}_+ \times \Theta.$$

Remark. If $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is a family of projectors for $C = (\sigma, \Phi)$, then $Q : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$, $Q(t, \theta) = I - P(t, \theta)$ is also a family of projectors for C and it is called the *complementary family* of P.

Definition 2.4. A family of projectors $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is said to be

(*i*) *invariant* for a skew-evolution semiflow $C = (\sigma, \Phi)$ if:

$$P(t, \sigma(t, s, \theta))\Phi(t, s, \theta) = \Phi(t, s, \theta)P(s, \theta), \quad \text{for all } (t, s, \theta) \in \Delta \times \Theta;$$

(*ii*) *strongly invariant* for $C = (\sigma, \Phi)$ if it is invariant for C and for all $(t, s, \theta) \in \Delta \times \Theta$, the restriction $\Phi(t, s, \theta)$ is an isomorphism from *Range* $Q(s, \theta)$ to *Range* $Q(t, \sigma(t, s, \theta))$.

3. Nonuniform generalized exponential dichotomy

We consider $C = (\sigma, \Phi)$ a skew-evolution semiflow and $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projectors for $C = (\sigma, \Phi)$.

Also, $\mathcal{F}$ represents the set of the continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with:

$$\int_s^t f(\tau)d\tau \xrightarrow[t\to+\infty]{} +\infty, \quad s \geq 0 \quad \text{fixed.}$$

Definition 3.1. The pair (C, P) is *nonuniformly generalized exponentially dichotomic* if there are $\varphi \in \mathcal{F}$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ such that:

$(nged_1)$ $\|\Phi(t, s, \theta)P(s, \theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|P(s, \theta)x\|$;

$(nged_2)$ $e^{\int_s^t \varphi(r)dr}\|Q(s, \theta)x\| \leq N(t)\|\Phi(t, s, \theta)Q(s, \theta)x\|$, for all $(t, s, \theta, x) \in \Delta \times \Gamma$.

Remark. As particular cases, we remark the following:

(i) if $N(s) = Be^{\int_0^s \xi(r)dr}$, with $B \geq 1$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function in Definition 3.1, then we recover the concept of *generalized exponential dichotomy in sense of Barreira and Valls*;

(ii) if there exists $c > 0$ such that $\varphi(s) \geq c$, for all $s \geq 0$ in Definition 3.1, then we have the notion of *nonuniform exponential dichotomy*.

Remark. The pair (C, P) admits a nonuniform generalized exponential dichotomy if and only if there are $\varphi \in \mathcal{F}$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, +\infty)$ with:

$(nged'_1)$ $\|\Phi(t, t_0, \theta)P(t_0, \theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|\Phi(s, t_0, \theta)P(t_0, \theta)x\|$;

$(nged'_2)$ $e^{\int_s^t \varphi(r)dr}\|\Phi(s, t_0, \theta)Q(t_0, \theta)x\| \leq N(t)\|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|$, for all $(t, s, t_0, \theta, x) \in T \times \Gamma$.

Remark. We observe that if (C, P) has a nonuniform exponential dichotomy, then it also admits a nonuniform generalized exponential dichotomy. In general, the converse implication is not accomplished.

Example 3.1. We consider $\Theta = \mathbb{R}_+$ and $\sigma : \Delta \times \Theta \to \Theta$, $\sigma(t, s, \theta) = t - s + \theta$.

Also, $X = l^\infty(\mathbb{N}, \mathbb{R})$ represents the Banach space of bounded real-valued sequences, with the norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|, \quad x = (x_0, x_1, ...x_n, ...) \in X.$$

The families of projectors $P, Q : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ are given by

$$P(s, \theta)(x_0, x_1, x_2, ...) = (x_0, 0, x_2, 0, ...),$$

$$Q(s, \theta)(x_0, x_1, x_2, ...) = (0, x_1, 0, x_3, ...).$$

We define $\Phi : \Delta \times \Theta \to \mathcal{B}(X)$ by

$$\Phi(t,s,\theta)x = \left(\frac{s+1}{t+1} e^{-\int_s^t \frac{1}{r+1}dr} x_0, \frac{t+1}{s+1} e^{\int_s^t \frac{1}{r+1}dr} x_1, \frac{s+1}{t+1} e^{-\int_s^t \frac{1}{r+1}dr} x_2, ... \right).$$

It is easy to verify that the pair (C, P) is nonuniformly generalized exponentially dichotomic with $N(s) = s+1$, $\varphi(s) = \frac{1}{s+1}$, $s \geq 0$.

Let us suppose that (C, P) has nonuniform exponential dichotomy. Then there exist $c > 0$ and a nondecreasing function $\tilde{N} : \mathbb{R}_+ \to [1, +\infty)$ with

$$\|\Phi(t,s,\theta)P(s,\theta)x\| \leq \tilde{N}(s)e^{-c(t-s)}\|P(s,\theta)x\|,$$

which implies

$$e^{-\int_s^t \frac{1}{r+1}dr}(s+1) \leq \tilde{N}(s)(t+1)e^{-c(t-s)},$$

for all $(t, s) \in \Delta$.

Considering $t = e^{2n\pi} - 1$ and $s = 0$, we obtain

$$e^{c(e^{2n\pi}-1)-4n\pi} \leq \tilde{N}(0),$$

which for $n \to +\infty$ leads to a contradiction.

Proposition 1. *If $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is a strongly invariant family of projectors for $C = (\sigma, \Phi)$, then there exists $\Psi : \Delta \times \Theta \to \mathcal{B}(X)$ isomorphism from Range $Q(t, \sigma(t, s, \theta))$ to Range $Q(s, \theta)$, with the properties:*

(Ψ_1) $\Phi(t,s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta)) = Q(t,\sigma(t,s,\theta))$;
(Ψ_2) $\Psi(t,s,\theta)\Phi(t,s,\theta)Q(s,\theta) = Q(s,\theta)$;
(Ψ_3) $\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta)) = Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))$;
(Ψ_4) $\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta)) = \Psi(s,t_0,\theta)\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))$,

for all $(t, s, t_0) \in T$, $\theta \in \Theta$.

Proof. See (Mihiţ *et al.*, 2017). □

In what follows, we will consider $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ a strongly invariant family of projectors for a skew-evolution semiflow $C = (\sigma, \Phi)$.

Theorem 3.2. *The pair (C, P) has a nonuniform generalized exponential dichotomy if and only if there exist $\varphi \in \mathcal{F}$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ such that:*

$(nged_1)$ $\|\Phi(t,s,\theta)P(s,\theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|P(s,\theta)x\|$;

$(nged_2'')$ $e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| \leq N(t)\|Q(t,\sigma(t,s,\theta))x\|$, for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Proof. Necessity.

For $(nged_2) \Rightarrow (nged_2'')$ we have:

$$e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| = e^{\int_s^t \varphi(r)dr}\|Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| \leq$$

$$\leq N(t)\|\Phi(t,s,\theta)Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\| = N(t)\|Q(t,\sigma(t,s,\theta))x\|,$$

for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Sufficiency. We obtain:

$$e^{\int_s^t \varphi(r)dr}\|Q(s,\theta)x\| = e^{\int_s^t \varphi(r)dr}\|\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)Q(s,\theta)x\| \leq$$

$$\leq N(t)\|Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)Q(s,\theta)x\| = N(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|,$$

for all $(t,s,\theta,x) \in \Delta \times \Gamma$.
Hence, $(nged_2)$ from Definition 3.1 is satisfied.

□

Proposition 2. *The pair (C,P) admits a nonuniform generalized exponential dichotomy if and only if there are $\varphi \in \mathcal{F}$ and a nondecreasing function $N : \mathbb{R}_+ \to [1,+\infty)$ with:*

$(nged_1')$ $\|\Phi(t,t_0,\theta)P(t_0,\theta)x\| \leq N(s)e^{-\int_s^t \varphi(r)dr}\|\Phi(s,t_0,\theta)P(t_0,\theta)x\|$;

$(nged_2''')$ $e^{\int_{t_0}^s \varphi(r)dr}\|\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta))x\| \leq N(s)\|\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\|$,

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$.

Proof. Necessity. Using the condition $(nged_2'')$ from Theorem 3.2, we deduce:

$$e^{\int_{t_0}^s \varphi(r)dr}\|\Psi(t,t_0,\theta)Q(t,\sigma(t,t_0,\theta))x\| =$$

$$= e^{\int_{t_0}^s \varphi(r)dr}\|\Psi(s,t_0,\theta)Q(s,\sigma(s,t_0,\theta))\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\| \leq$$

$$\leq N(s)\|Q(s,\sigma(s,t_0,\theta))\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\| =$$

$$= N(s)\|\Psi(t,s,\sigma(s,t_0,\theta))Q(t,\sigma(t,t_0,\theta))x\|,$$

for all $(t,s,t_0,\theta,x) \in T \times \Gamma$.

Sufficiency. Considering $s = t$ in $(nged_2''')$, we obtain $(nged_2'')$ from Theorem 3.2. □

4. Nonuniform generalized exponential dichotomy of Rolewicz type

Further, $C = (\sigma,\Phi)$ is a skew-evolution semiflow, $P : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ an invariant family of projectors for $C = (\sigma,\Phi)$ and $\mathcal{R}$ represents the set of continuous and nondecreasing functions $R : \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 4.1. We say that (C,P) admits a *nonuniform generalized exponential dichotomy of Rolewicz type* if there exist $R \in \mathcal{R}$, $\varphi \in \mathcal{F}$ and a nondecreasing function $\rho : \mathbb{R}_+ \to [1,+\infty)$ with:

$(Rnged_1)$ $\int_s^{+\infty} R\left(e^{\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)P(s,\theta)x\|\right)d\tau \leq R\left(\rho(s)\|P(s,\theta)x\|\right)$, for all $(s,\theta,x) \in \mathbb{R}_+ \times \Gamma$;

$(Rnged_2)$ $\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau \leq R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right)$, for all $(t,s,\theta,x) \in \Delta \times \Gamma$.

Remark. In particular, if $N(s) = Be^{\int_0^s \xi(r)dr}$, with $B \geq 1$ and $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function in Definition 4.1, then we have the property of *generalized exponential dichotomy of Rolewicz type in sense of Barreira and Valls*.

Proposition 3. *The pair* (C, P) *has nonuniform generalized exponential dichotomy of Rolewicz type if and only if there exist* $R \in \mathcal{R}$, $\varphi \in \mathcal{F}$ *and a nondecreasing mapping* $\rho : \mathbb{R}_+ \to [1, +\infty)$*:*

$$(Rnged_1') \quad \int_s^{+\infty} R\left(e^{\int_0^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)P(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right),$$

for all $(s, t_0, \theta, x) \in \Delta \times \Gamma$;

$$(Rnged_2') \quad \int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_{t_0}^t \varphi(r)dr} \|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|\right),$$

for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$.

Proof. Necessity. $(Rnged_1')$ For all $(s, t_0, \theta, x) \in \Delta \times \Gamma$, we have:

$$\int_s^{+\infty} R\left(e^{\int_0^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)P(t_0, \theta)x\|\right) d\tau =$$

$$= \int_s^{+\infty} R\left(e^{\int_0^s \varphi(r)dr} e^{\int_s^\tau \varphi(r)dr} \|\Phi(\tau, s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right) d\tau \leq$$

$$\leq R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|P(s, \sigma(s, t_0, \theta))\Phi(s, t_0, \theta)x\|\right) = R\left(\rho(s)e^{\int_0^s \varphi(r)dr} \|\Phi(s, t_0, \theta)P(t_0, \theta)x\|\right).$$

$(Rnged_2')$ Similarly, for all $(t, t_0, \theta, x) \in \Delta \times \Gamma$, we deduce:

$$\int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau =$$

$$= \int_{t_0}^{t} R\left(e^{-\int_{t_0}^\tau \varphi(r)dr} e^{-\int_\tau^t \varphi(r)dr} e^{\int_\tau^t \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau =$$

$$= \int_{t_0}^{t} R\left(e^{-\int_{t_0}^t \varphi(r)dr} e^{\int_\tau^t \varphi(r)dr} \|\Phi(\tau, t_0, \theta)Q(t_0, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_{t_0}^t \varphi(r)dr} \|\Phi(t, t_0, \theta)Q(t_0, \theta)x\|\right).$$

Sufficiency. Considering $t_0 = s$ in $(Rnged_1')$, we obtain the condition $(Rnged_1)$ from Definition 4.1. For $t_0 = s$ in $(Rnged_2')$, it follows

$$\int_s^{t} R\left(e^{-\int_s^\tau \varphi(r)dr} \|\Phi(\tau, s, \theta)Q(s, \theta)x\|\right) d\tau \leq R\left(\rho(t)e^{-\int_s^t \varphi(r)dr} \|\Phi(t, s, \theta)Q(s, \theta)x\|\right), \text{ for all } (t, s, \theta, x) \in \Delta \times \Gamma.$$

Thus,

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{-\int_s^\tau \varphi(r)dr} e^{\int_s^\tau \varphi(r)dr} e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_s^t \varphi(r)dr} e^{-\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau \leq$$

$$\leq R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right), \text{ for all } (t,s,\theta,x)\in\Delta\times\Gamma.$$

Hence, $(Rnged_2)$ from Definition 4.1 holds. □

Theorem 4.1. *Let $P:\mathbb{R}_+\times\Theta\to\mathcal{B}(X)$ be a strongly invariant family of projectors for $C=(\sigma,\Phi)$. Then (C,P) admits nonuniform generalized exponential dichotomy of Rolewicz type if and only if there exist $R\in\mathcal{R}$, $\varphi\in\mathcal{F}$ and a nondecreasing mapping $\rho:\mathbb{R}_+\to[1,+\infty)$ with:*

$$(Rnged_1)\quad \int_s^{+\infty} R\left(e^{\int_s^\tau \varphi(r)dr}\|\Phi(\tau,s,\theta)P(s,\theta)x\|\right)d\tau \leq R\left(\rho(s)\|P(s,\theta)x\|\right), \text{ for all } (s,\theta,x)\in\mathbb{R}_+\times\Gamma;$$

$$(Rnged_2'')\quad \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau \leq R\left(\rho(t)\|Q(t,\sigma(t,s,\theta))x\|\right),$$

for all $(t,s,\theta,x)\in\Delta\times\Gamma$.

Proof. We will prove that $(Rnged_2'')$ is equivalent with $(Rnged_2)$ from Definition 4.1.

Necessity. For all $(t,s,\theta,x)\in\Delta\times\Gamma$, it results

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)\Psi(\tau,s,\theta)\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))x\|\right)d\tau$$

$$\leq R(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)\Psi(t,s,\theta)Q(t,\sigma(t,s,\theta))x\|) =$$

$$= R\left(\rho(t)\|Q(t,\sigma(t,s,\theta))x\|\right).$$

Sufficiency. For all $(t,s,\theta,x)\in\Delta\times\Gamma$, we obtain:

$$\int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Phi(\tau,s,\theta)Q(s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr}\|\Psi(t,\tau,\sigma(\tau,s,\theta))\Phi(t,\tau,\sigma(\tau,s,\theta))Q(\tau,\sigma(\tau,s,\theta))\Phi(\tau,s,\theta)x\|\right)d\tau =$$

$$= \int_s^t R\left(e^{\int_\tau^t \varphi(r)dr} \|\Psi(t,\tau,\sigma(\tau,s,\theta))Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)x\|\right)d\tau \leq$$

$$\leq R(\rho(t)\|Q(t,\sigma(t,s,\theta))\Phi(t,s,\theta)x\|) = R\left(\rho(t)\|\Phi(t,s,\theta)Q(s,\theta)x\|\right).$$

□

References

Barreira, L. and C. Valls (2018). On two notions of exponential dichotomy. *Dynamical Systems* **33**(4), 708–721.

Barreira, L. and C. Valls (2019). General exponential dichotomies: From finite to infinite time. *Advances in Operator Theory* **4**(1), 215–225.

Bento, A.J.G., N. Lupa, M. Megan and C. Silva (2017). Integral conditions for nonuniform μ- dichotomy. *Discrete Contin. Dyn. Syst. Ser. B* **22**(8), 3063–3077.

Găină, A., M. Megan and R. Boruga (2023). Nonuniform dichotomy with growth rates of skew-evolution cocycles in Banach spaces. *Axioms* **12**(4), 394.

Lupa, N., M. Megan and I.L. Popa (2015). Generalized exponential dichotomies for evolution operators. *Applied Computational Intelligence and Informatics* pp. 55–58.

Megan, M., A.L. Sasu and B. Sasu (2025). Nonlinear criteria for stability of nonautonomous dynamics - a new Zabczyk-Rolewicz type approach. *Annals: Series on Mathematics & its Applications* **17**(1), 15–36.

Megan, M., C. Stoica and L. Buliga (2007). On asymptotic behaviours for linear skew-evolution semiflows in Banach spaces. *Carpathian J. Math.* **23**, 117–125.

Mihiţ, C.L. and M. Megan (2017). Integral characterizations for the (h,k)-splitting of skew-evolution semiflows. *Stud. Univ. Babeş-Bolyai Math.* **62**(3), 353–365.

Mihiţ, C.L., D. Borlea and M. Megan (2017). On some concepts of (h,k)-splitting for skew-evolution semiflows in Banach spaces. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **9**, 186–204.

Muldowney, J. S. (1984). Dichotomies and asymptotic behavior for linear differential systems. *Trans. Amer. Math. Soc.* **283**, 465–484.

Rolewicz, S. (1986). On uniform N-equistability. *J. Math. Anal. Appl.* **115**, 434–441.

Sasu, A. L., M. Megan and B. Sasu (2012). On Rolewicz-Zabczyk techniques in the stability theory of dynamical systems. *Fixed Point Theory* **13**(1), 205–236.

Sasu, A. L., M.G. Babuţia and B. Sasu (2013). Admissibility and nonuniform exponential dichotomy on the half-line. *Bull. Sci. Math.* **137**(4), 466–484.

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A Study of Kalecki‘s Model of Business Cycle Using Weakly Picard Operators Technique

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Abstract

Kalecki's 1935 work introduced the first precise macro-dynamic model and emphasized the implementation lag between investment decisions and productive capacity. This paper aims to establish conditions for the models solution to exist and its continuous data dependence.

Keywords: Integral equations, Picard operators, fixed points, data dependence, Gronwall lemma.

2020 MSC: 45G10, 47H10, 45D05.

1. Introduction

1.1. Weakly Picard operators

I.A. Rus initiated and developed in Rus (2001) the theory of weakly Picard operators with applications in the study of existence and data dependence of fixed point of different operators.

Let us consider (X, d) a metric space and $A : X \to X$ an operator. Next we shall use the following notations:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\},$$

$$F_A := \{x \in X \mid A(x) = x\},$$

$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\},$$

$$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N}.$$

Definition 1.1. Rus (2001) The operator A is said to be weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit is a fixed point of A.

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Definition 1.2. Rus (2001) If A is an weakly Picard operator and $F_A = \{x^*\}$ then A is a Picard operator (briefly PO).

We have the following characterization of the WPOs.

Theorem 1.1. *Rus (2001) Let us consider (X, d) a metric space and $A : X \to X$ an operator. Then A is WPO if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that :

(a) $X_\lambda \in I(A)$

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is PO for all* $\lambda \in \Lambda$.

1.2. Formulation of Kalecki's model

Kalecki's model, see Kalecki (1935), highlights that productive capacity cannot be created instantaneously: investment projects require a fixed implementation lag, or gestation period, denoted by $\theta > 0$. Let $K(t)$ be the capital stock and $I(t - \theta)$ the net investment decided at time $t - \theta$. Capital accumulation is therefore

$$\dot{K}(t) = I(t - \theta). \tag{1.1}$$

Replacement of depreciated capital also experiences the same lag and is represented by a constant $U > 0$. Assuming continuous market clearing with no inventories, government, or international trade, consumption depends on a constant saving propensity $s \in (0, 1)$. Investment decisions are modeled as a linear function of output and capital:

$$I(t) = a \cdot Y(t) - b \cdot K(t), \tag{1.2}$$

where output is given by

$$Y(t) = \frac{1}{s} \cdot U + \frac{1}{s \cdot \theta}[K(t + \theta) - K(t)]. \tag{1.3}$$

Substituting these relationships yields a mixed differential–difference equation in the single variable $K(t)$:

$$\dot{K}(t) = \frac{a}{s} \cdot U + \frac{a}{s \cdot \theta}[K(t) - K(t - \theta)] - b \cdot K(t - \theta). \tag{1.4}$$

A solution $K(t)$ to (1.4) guarantees the existence of solutions to (1.2) and (1.3), thus fully determining the model. The next step is to identify the parameter conditions under which (1.4) admits at least one continuous solution. The model was studied from fixed point point of view in many papers. In Olaru *et al.* (2009) there was proved an results of existence and uniqueness for Cauchy problem associated to the above model by using a Bielecki norm on class of continuous functions defined on $[-\theta, T]$. Further in Olaru (2025) there was studied the existence and uniqueness in regards Chebyshev norm.

2. Existence result

Our proposal on current section is to study the above model by using weakly Picard technique. More exactly based on characterization of weakly Picard operators we prove that the Kalecki model has at least a solution. Further our study will be done on the class of continuous functions $K : [-\theta, T] \to \mathbb{R}$ denoted by $C([-\theta, T], \mathbb{R})$ endowed with Chebyshev norm defined by

$$\|K\|_\infty = \sup_{t\in[-\theta,T]} |K(t)|.$$

Let us consider the following partition of $C([-\theta, T], \mathbb{R})$

$$C([-\theta, T], \mathbb{R}) = \bigcup_{\varphi\in C([-\theta,0])} X_\varphi$$

where

$$X_\varphi = \{x \in C([-\theta, T]) \mid x(t) = \varphi(t), (\forall)t \in [-\theta, 0]\}.$$

Then (1.4) is equivalent with

$$K(t) = \begin{cases} K(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s\cdot\theta}[K(u) - K(u-\theta)] - b \cdot K(u-\theta)]du & , \quad t \in [0, T] \\ K(t) & , \quad t \in [-\theta, 0] \end{cases} \tag{2.1}$$

Therefore we reduced the existence of solution for (1.4) to a fixed point problem for the operator $A : C[-\theta, T] \to C[-\theta, T]$ defined by:

$$A(K)(t) = \begin{cases} K(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s\cdot\theta}[K(u) - K(u-\theta)] - b \cdot K(u-\theta)]du & , \quad t \in [0, T] \\ K(t) & , \quad t \in [-\theta, 0] \end{cases}$$

Thus we got the following result for model's solution existence

Theorem 2.1. *The Kalecki model* (1.4) *has at least a solution* $K \in C[-\theta, T]$ *which can be approximated by the sequence* $\{A^n(K_0)\}_{n\in\mathbb{N}}$, $K_0 \in C[-\theta, T]$ *being arbitrarily chosen.*

Proof. First of all we remark that $X_\varphi \in I(A)$. On the other side we claim that $A \mid X_\varphi$ is a Picard operator. Indeed, let us consider $K_1, K_2 \in C[-\theta, T]$. Then

$$|A(K_1)(t) - A(K_2)(t)| \leq \int_0^t [\frac{a}{s\cdot\theta} \cdot |K_1(u) - K_2(u)| + (\frac{a}{s\cdot\theta} - b)|K_1(u-\theta) - K_2(u-\theta)|]du \leq$$

$$\leq (2 \cdot \frac{a}{s\cdot\theta} + b) \cdot t \cdot \|K_1 - K_2\|_\infty.$$

By using induction arguments we get that for any iteration A^k we have

$$|A^n(K_1)(t) - A^n(K_2)(t)| \leq$$

$$\int_0^t [\frac{a}{s\cdot\theta} \cdot |A^{n-1}(K_1)(u) - A^{n-1}(K_2)(u)| + (\frac{a}{s\cdot\theta} - b)|A^{n-1}(K_1)(u-\theta) - A^{n-1}(K_2)(u-\theta)|]du \leq$$

$$\leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{t^n}{n!} \cdot \|K_1 - K_2\|_\infty \leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{T^n}{n!} \cdot \|K_1 - K_2\|_\infty.$$

Consequently for $n \geq$ we have

$$\|A^n(K_1) - A^n(K_2)\|_\infty \leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{T^n}{n!} \cdot \|K_1 - K_2\|_\infty$$

and from here we get that there exists $N \in \mathbb{N}$ such that A^N is a contraction. Therefore $A \mid X_\varphi$ is a Picard operator and now the conclusion follows from Theorem 1.1, □

3. Data dependence: continuity with respect to data

Further let us consider the equation (1.4) which satisfies the initial Cauchy conditions

$$K(t) = \varphi_1(t), t \in [-\theta, 0]. \tag{3.1}$$

$$K(t) = \varphi_2(t), t \in [-\theta, 0]. \tag{3.2}$$

Then we have the following data dependence result:

Theorem 3.1. *(a) There exists $K(\cdot, \varphi_1), K(\cdot, \varphi_2) \in C[-\theta, T]$ unique solutions for (1.4) + (3.1) respectively (1.4) + (3.2).*

(b) if there exists $\eta > 0$ such that

$$|\varphi_1(t) - \varphi_2(t), (\forall) t \in [-\theta, 0]$$

then

$$\|K(\cdot, \varphi_1) - K(\cdot, \varphi_2)\|_\infty \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) \cdot exp(\int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) du.$$

Proof. (a) Let us consider the operator $A_i : C[-\theta, T] \to C[-\theta, T]$, $i = \overline{1, 2}$ defined by:

$$A_i(K)(t) = \begin{cases} \varphi_i(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s \cdot \theta}[K(u) - K(u - \theta)] - b \cdot K(u - \theta)] du & , \quad t \in [0, T] \\ \varphi_i(t) & , \quad t \in [-\theta, 0] \end{cases}$$

By using the same approach like in the proof of Theorem 2.1 we get that A_1, A_2 are Picard operators and thus they have the unique fixed points $K(\cdot, \varphi_1)$ respectively $K(\cdot, \varphi_2)$.

(b) Let us consider $x : [-\theta, T] \to (0, \infty)$ defined by $x(v) =: |K(v, \varphi_1) - K(v, \varphi_2)|$. Then

$$x(t) \leq |\varphi_1(0) - \varphi_2(0)\| + \int_0^t [\frac{a}{s \cdot \theta} x(u) + (\frac{a}{s \cdot \theta} + b) \cdot x(u - \theta)] du, (\forall) t \in [0, T]$$

and

$$x(t) = |\varphi_1(t) - \varphi_2(t)|, (\forall) t \in [-\theta, 0]$$

Further, for each $v \in [0, T]$ let us denote

$$y(v) = |\varphi_1(0) - \varphi_2(0)| + \int_0^v [\frac{a}{s \cdot \theta} x(u) + (\frac{a}{s \cdot \theta} + b) \cdot x(u - \theta)] du.$$

From here we get that

$$y'(v) = \frac{a}{s \cdot \theta} x(v) + (\frac{a}{s \cdot \theta} + b) \cdot x(v - \theta)$$

$$\leq \frac{a}{s \cdot \theta} y(v) + (\frac{a}{s \cdot \theta} + b) \cdot \begin{cases} y(v - \theta) & , \quad v - \theta \geq 0 \\ |\varphi_1(v - \theta) - \varphi_2(v - \theta)| & , \quad v - \theta < 0 \end{cases}$$

$$\leq \frac{a}{s \cdot \theta} y(v) + (\frac{a}{s \cdot \theta} + b) \cdot \begin{cases} y(v) & , \quad v - \theta \geq 0 \\ |\varphi_1(v - \theta) - \varphi_2(v - \theta)| & , \quad v - \theta < 0 \end{cases}$$

By integrating on $[0, t]$ and considering $A := \{t \in [0, T] \mid t - \theta < 0\}$ we get that

$$y(t) \leq$$

$$|\varphi_1(0) - \varphi_2(0)| + \int_A (\frac{a}{s \cdot \theta} + b) \cdot |\varphi_1(u - \theta) - \varphi_2(u - \theta)| du + \int_{[0,T] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot y(u) du \leq$$

$$\eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) + \int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot y(u) du.$$

Now, by applying Gronwall lemma, we get that for all $t \in [0, T]$

$$x(t) \leq y(t) \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) \cdot exp(\int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) du)$$

and thus we have

$$||K(, \varphi_1) - K(t, \varphi_2)||_\infty \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) \cdot exp(\int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) du).$$

□

References

Kalecki, M. (1935). A macroeconomic theory of the business cycle. *Econometrica* **3**(3), 327–344.

Olaru, I. M. (2025). Kalecki's model of business cycle: existence and approximation of solution. *Annals of the "Constantin Brancuşi" University of Târgu Jiu, Economy Series* **5**(1), 74–77.

Olaru, I. M., C. Pumnea, A. Bacociu and A. Nicoara (2009). Kalecki model of business cycle: Data dependence. *General Mathematics* **17**(2), 67–72.

Rus, I. A. (2001). Weakly picard operators and applications. *Seminar on Fixed Point Theory Cluj-Napoca* **2**, 41–58.

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Expected Value of a Picture Fuzzy Number

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Abstract

This paper proposes a mathematical framework for the definition and computation of the expected interval and expected value of Picture Fuzzy Numbers (PFNs), providing a robust and interpretable tool for ranking and decision-making analysis in contexts characterized by imprecise or uncertain information.

Keywords: Picture fuzzy number, expected interval, expected value.

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Introduction

In recent decades, fuzzy theories have become fundamental tools for modeling uncertainty and imprecision in decision-making and optimization problems. In particular, fuzzy numbers have been employed to represent incomplete or uncertain information, allowing decision-makers to express preferences and evaluations in a more flexible manner than traditional methods.

Building on the classical concept of a fuzzy number introduced by Zadeh (1975*a*), Zadeh (1975*b*), several extensions have been developed, such as Intuitionistic Fuzzy Numbers (IFNs) introduced by Atanassov (1986) and Picture Fuzzy Numbers (PFNs) introduced by Cuong (2013). PFN-s enable the simultaneous modeling of multiple types of information: membership degree, non-membership degree, and hesitation degree. PFNs, in particular, provide a richer framework, including the ability to represent neutral or indeterminate opinions, making them well-suited for applications in multi-criteria decision-making, risk analysis, quality assessment, and other complex domains (Wei & Gao (2018), Qiyas *et al.* (2019), Xian *et al.* (2021), Shit *et al.* (2022), Jaikumar *et al.* (2023), Jana *et al.* (2024), Akdemir & Aydin (2025), Garg *et al.* (2025)). The comparison and ranking of PFNs remain a major challenge, with existing methods often relying on scoring functions or similarity measures.

However, the concept of expected value for PFNs remains insufficiently explored. Transforming a PFN into a scalar indicator through its expected value can facilitate ranking, defuzzification, and integration of

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PFNs into decision-making models, while simultaneously preserving information about uncertainty and hesitation.

This gap motivates the current study: there is a clear need for a rigorous expected value concept for PFNs that can transform the triple of membership, non-membership, and neutral degrees into a single, interpretable scalar for ranking, defuzzification, and decision-making purposes. Developing such a measure would not only extend the theoretical framework of PFNs but also enhance their practical usability in real-world decision-making contexts.

Building on the research on expected interval and expected value for a fuzzy number by Dubois & Prade (1987) and Heilpern (1992), as well as the research on expected interval and expected value for an intuitionistic fuzzy number by Grzegrorzewski (2003) and Nehi & Maleki (2005), this paper introduces the concept of expected interval and expected value for a picture fuzzy number in a general setting, and in particular for the trapezoidal picture fuzzy number.

1. Preliminaries

Definition 1.1. Cuong (2013) Let Ω an universal set. A subset

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\,;\, x \in \Omega\},$$

where $\mu_A : \Omega \to [0,1]$ is the degree of positive membership of x in A, $\eta_A : \Omega \to [0,1]$ represents the degree of neutral membership of x in A and $\nu_A : \Omega \to [0,1]$ is the degree of negative membership of x in A, respectively and μ_A, η_A and ν_A satisfy the condition:

$$0 \leqslant \mu_A(x) + \eta_A(x) + \nu_A(x) \leqslant 1,\ (\forall) x \in \Omega,$$

is a picture fuzzy set (PFS) on Ω.

$\pi_A : \Omega \to [0,1],\ \pi_A(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called degree of refusal membership of x in A.

Definition 1.2. Cuong & Kreinovich (2013) Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\,;\, x \in \Omega\}$ be o picture fuzzy set on Ω and $\alpha, \gamma, \beta \in [0,1], \alpha + \gamma + \beta \le 1$ then the upper (α, γ, β)-cut of A is given by

$$A^{(\alpha,\gamma,\beta)} = \{x \in \Omega : \mu_A(x) \ge \alpha, \eta_A(x) \ge \gamma, \nu_A(x) \le \beta\}$$

That is, $A^{\alpha} = \{x : \mu_A \ge \alpha\}, A^{\gamma} = \{x : \eta_A \ge \gamma\}, A^{\beta} = \{x : \nu_A \le \beta\}$ are upper α, γ and β-cut of positive membership, neutral membership and negative membership of a picture fuzzy set A respectively.

Definition 1.3. Qiyas *et al.* (2019) A picture fuzzy number (PFN) $A \in \mathbb{R}$ is denoted as $A = \langle(\mu_A, \eta_A, \nu_A);\ w_1, w_2, w_3\rangle$ whose positive, neutral and negative membership functions are defined as follows:

$$\mu_A(x) = \begin{cases} f_A^L(x) & if \quad a \leqslant x < b \\ w_1 & if \quad b \leqslant x \leqslant c \\ f_A^R(x) & if \quad c < x \leqslant d \\ 0 & otherwise, \end{cases} \qquad \eta_A(x) = \begin{cases} g_A^L(x) & if \quad a' \leqslant x < b \\ w_2 & if \quad b \leqslant x \leqslant c \\ g_A^R(x) & if \quad c < x \leqslant d' \\ 0 & otherwise, \end{cases} \tag{1.1}$$

$$\nu_A(x) = \begin{cases} h_A^L(x) & if \quad a'' \leqslant x < b \\ w_3 & if \quad b \leqslant x \leqslant c \\ h_A^R(x) & if \quad c < x \leqslant d'' \\ 1 & otherwise, \end{cases}$$

where f_A^L, g_A^L, h_A^R are increasing functions and f_A^R, g_A^R, h_A^L are nonincreasing functions. The values w_1, w_2, w_3 represent the maximum degrees of the positive, neutral and negative membership, $w_1, w_2, w_3 \in [0,1]$ and $0 \leqslant w_1 + w_2 + w_3 \leqslant 1$.

Definition 1.4. Akram *et al.* (2022) The picture fuzzy number A for which the positive, neutral and negative membership functions of form:

$$\mu_A(x) = \begin{cases} \frac{w_1(x-a)}{b-a} & if \quad a \leqslant x < b \\ w_1 & if \quad b \leqslant x \leqslant c \\ \frac{w_1(d-x)}{d-c} & if \quad c < x \leqslant d \\ 0 & otherwise, \end{cases} \qquad \eta_A(x) = \begin{cases} \frac{w_2(x-a')}{b-a'} & if \quad a' \leqslant x < b \\ w_2 & if \quad b \leqslant x \leqslant c \\ \frac{w_2(d'-x)}{d'-c} & if \quad c < x \leqslant d' \\ 0 & otherwise, \end{cases}$$

$$\nu_A(x) = \begin{cases} \frac{(b-x)+w_3(x-a'')}{b-a''} & if \quad a'' \leqslant x < b \\ w_3 & if \quad b \leqslant x \leqslant c \\ \frac{(x-c)+w_3(d''-x)}{d''-c} & if \quad c < x \leqslant d'' \\ 1 & otherwise, \end{cases}$$

where $a, a', a'', b, c, d, d', d'' \in \mathbb{R}$ with $a'' \leqslant a' \leqslant a \leqslant b \leqslant c \leqslant d \leqslant d' \leqslant d''$ and $w_1, w_2, w_3 \in [0,1]$, $0 \leqslant w_1 + w_2 + w_3 \leqslant 1$, will be called trapezoidal picture fuzzy number (TrPFN), denoted by $A = \langle (a,b,c,d), (a',b,c,d'), (a'',b,c,d''); w_1, w_2, w_3 \rangle$.

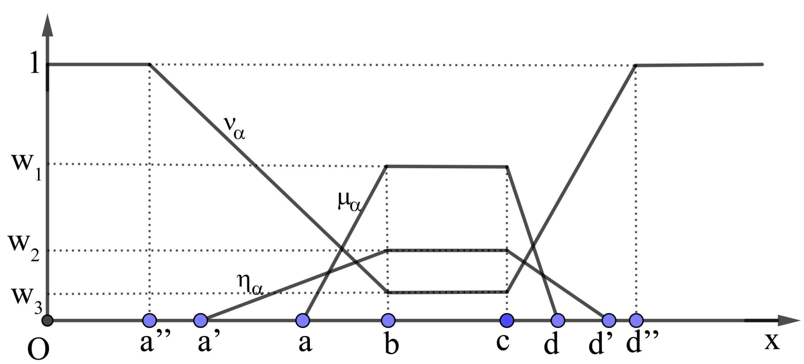

Figure 1. Trapezoidal picture fuzzy number

Remark. For the particular case of $(a,b,c,d) = (a',b,c,d') = (a'',b,c,d'')$, TrPFNs can be characterized as $A = \langle (a,b,c,d); w_1, w_2, w_3 \rangle$ and henceforth called special trapezoidal picture fuzzy numbers (STrPFNs).

Definition 1.5. Akram *et al.* (2022) A STrPFN $A = \langle (a,b,c,d); w_1, w_2, w_3 \rangle$ is non-negative (respectively non-positive), denoted as $A \geq 0$ (respectively $A \leq 0$), if $a \geq 0$ (respectively $d \leq 0$).

Definition 1.6. Akram *et al.* (2022) Two STrPFNs $A = \langle (a_1, b_1, c_1, d_1); w_{1A}, w_{2A}, w_{3A} \rangle$ and $B = \langle (a_2, b_2, c_2, d_2); w_{1B}, w_{2B}, w_{3B} \rangle$ are said to be equal if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$, $w_{1A} = w_{1B}$, $w_{2A} = w_{2B}$ and $w_{3A} = w_{3B}$.

2. The expected value for different types of fuzzy numbers

This section refers to some basic concepts related to fuzzy numbers Dubois & Prade (1978), intuitionistic fuzzy numbers Grzegrorzewski (2003) and trapezoidal intuitionistic fuzzy numbers Nehi & Maleki (2005).

2.1. The expected value of a fuzzy number

Let A be an fuzzy number in the set of real numbers $\mathbb{R}$. There exist the numbers $a, b, c, d \in \mathbb{R}, a \leqslant b \leqslant c \leqslant d$, the function increasing continuous $f_A^L : \mathbb{R} \to [0,1]$ and the function nonincreasing continuous $f_A^R : \mathbb{R} \to [0,1]$, with which the membership function μ_A is expressed:

$$\mu_A(x) = \begin{cases} f_A^L(x) & if \quad a \leqslant x < b \\ 1 & if \quad b \leqslant x \leqslant c \\ f_A^R(x) & if \quad c < x \leqslant d \\ 0 & otherwise \end{cases}$$

with $0 \leqslant \mu_A(x) \leqslant 1$.

The functions f_A^L and f_A^R are referred to, respectively, as the left-hand side and the right-hand side of the fuzzy number A.

The set α-cut of a fuzzy number A, defined as

$$A^\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$$

is a closed interval $A^\alpha = [A_1(\alpha), A_2(\alpha)]$, where

$$A_1(\alpha) = \inf\{x \in \mathbb{R}\ \mu_A(x) \geq \alpha\}; \quad A_2(\alpha) = \sup\{x \in \mathbb{R}\ \mu_A(x) \geq \alpha\}.$$

If the sides of the fuzzy number are strictly monotone then, the convention is used that:

$$(f_A^L)^{-1}(\alpha) = A_1(\alpha);\ (f_A^R)^{-1}(\alpha) = A_2(\alpha).$$

Two important notions related to fuzzy numbers are the expected interval $EI(A)$ and the expected value $EV(A)$ of a fuzzy number A, introduced independently in Dubois & Prade (1987) and Heilpern (1992).

The expected interval of a fuzzy number $A = (a, b, c, d)$ is a crisp interval

$$EI(A) = \left[\int_0^1 A_1(\alpha)d\alpha, \int_0^1 A_2(\alpha)d\alpha\right]$$

or, equivalently,

$$EI(A) = [E_1(A), E_2(A)], \tag{2.1}$$

where

$$E_1(A) = b - \int_a^b f_A^L(x)dx; \quad E_2(A) = c + \int_c^d f_A^R(x)dx. \tag{2.2}$$

The expected value of a fuzzy number A is the center of the expected interval $EI(A)$, i.e.

$$EV(A) = \frac{E_1(A) + E_2(A)}{2}. \tag{2.3}$$

For a generalized trapezoidal fuzzy number $A = \langle (a, b, c, d), w \rangle$, $0 \leq w \leq 1$ for which the membership function is defined as follows

$$\mu_A(x) = \begin{cases} \frac{w(x-a)}{b-a} & if \quad a \leqslant x < b \\ w & if \quad b \leqslant x \leqslant c \\ \frac{(d-x)}{d-c} & if \quad c < x \leqslant d \\ 0 & otherwise, \end{cases}$$

the expected interval is

$$EI(A) = [E_1(A), E_2(A)] = \left[\frac{a+b}{2} \cdot w, \frac{c+d}{2} \cdot w \right]$$

and the expected value is

$$EV(A) = \frac{(a+b+c+d)w}{4}. \tag{2.4}$$

In particular, if $w = 1$ i.e. $A = (a, b, c, d)$ is a trapezoidal fuzzy number, then the expected interval is $EI(A) = \left[\frac{a+b}{2}, \frac{c+d}{2}\right]$ and the expected value is $EV(A) = \frac{a+b+c+d}{4}$.

These results have been employed in various methods for ranking fuzzy numbers. For example Jimnez (1996) a direct comparison of the expected intervals is proposed, while in Asady (2013) the approximation of the fuzzy number includes not only the expected interval but also the core of the fuzzy number.

The expected value (2.4), together with the variance, constitutes fundamental characteristics of a generalized trapezoidal fuzzy number, on the basis of which a novel and efficient similarity measure was developed in Dutta & Borah (2023) and subsequently applied to decision-making problems.

2.2. The expected value of a intuitionistic fuzzy number

Let A be an intuitionistic fuzzy number (IFN) in the set of real numbers $\mathbb{R}$. There exist the numbers $a, b, c, d, a', b', c', d' \in \mathbb{R}$, $a' \leqslant a \leqslant b' \leqslant b \leqslant c \leqslant c' \leqslant d \leqslant d'$, the increasing functions $f_A^L, g_A^R : \mathbb{R} \to [0, 1]$ and the nonincreasing functions $f_A^R, g_A^L : \mathbb{R} \to [0, 1]$, so that the membership function μ_A and the non-membership function ν_A are defined as:

$$\mu_A(x) = \begin{cases} f_A^L(x) & if \quad a \leqslant x < b \\ 1 & if \quad b \leqslant x \leqslant c \\ f_A^R(x) & if \quad c < x \leqslant d \\ 0 & otherwise, \end{cases} \qquad \nu_A(x) = \begin{cases} g_A^L(x) & if \quad a' \leqslant x < b' \\ 0 & if \quad b' \leqslant x \leqslant c' \\ g_A^R(x) & if \quad c' < x \leqslant d' \\ 1 & otherwise, \end{cases}$$

with $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$.

The expected interval and the expected value of an intuitionistic fuzzy number $A = \langle (a, b, c, d)(a', b', c', d') \rangle$ have been defined in Grzegrorzewski (2003).

The expected interval of a intuitionistic fuzzy number A is a crisp interval $EI(A)$ given by

$$EI(A) = [E_1(A), E_2(A)],$$

where

$$\begin{aligned} E_1(A) &= \frac{a'+b}{2} + \frac{1}{2}\int_{a'}^{b'} g_A^L(x)dx - \frac{1}{2}\int_a^b f_A^L(x)dx; \\ E_2(A) &= \frac{c+d'}{2} + \frac{1}{2}\int_c^d f_A^R(x)dx - \frac{1}{2}\int_{c'}^{d'} g_A^R(x)dx. \end{aligned} \tag{2.5}$$

Remark. For a generalized trapezoidal intuitionistic fuzzy number (GTrIFN) $A = \langle(a,b,c,d)(a',b',c',d'); w_1, w_2\rangle$ for which the membership function and the nonmembership function are defined as follows

$$\mu_A(x) = \begin{cases} \frac{w_1(x-a)}{b-a} & if \quad a \leqslant x < b \\ w_1 & if \quad b \leqslant x \leqslant c \\ \frac{w_1(d-x)}{d-c} & if \quad c < x \leqslant d \\ 0 & otherwise, \end{cases} \qquad \nu_A(x) = \begin{cases} \frac{(b'-x)+w_2(x-a')}{b'-a'} & if \quad a' \leqslant x < b' \\ w_2 & if \quad b' \leqslant x \leqslant c' \\ \frac{(x-c')+w_2(d'-x)}{d'-c'} & if \quad c' < x \leqslant d' \\ 1 & otherwise, \end{cases}$$

similarly to (2.5) we obtain:

$$\begin{aligned} E_1(A) &= \frac{a'-b'w_2+bw_1}{2} + \frac{1}{2}\int_{a'}^{b'} \frac{(b'-x)+w_2(x-a')}{b'-a'}dx - \frac{1}{2}\int_a^b \frac{w_1(x-a)}{b-a}dx = \frac{(a'+b')(1-w_2)+(a+b)w_1}{4} \\ E_2(A) &= \frac{w_1c+d'-c'w_2}{2} + \frac{1}{2}\int_c^d \frac{w_1(d-x)}{d-c}dx - \frac{1}{2}\int_{c'}^{d'} \frac{(x-c')+w_2(d'-x)}{d'-c'}dx = \frac{(c'+d')(1-w_2)+(c+d)w_1}{4}. \end{aligned}$$

The expected value for a GTrIFN is the center of the expected interval $EI(A) = [E_1(A), E_2(A)]$, i.e.

$$EV(A) = \frac{E_1(A) + E_2(A)}{2} = \frac{(a+b+c+d)w_1 + (a'+b'+c'+d')(1-w_2)}{8}. \tag{2.6}$$

In particular, for $w_1 = 1$ and $w_2 = 0$ that is, for $A = \langle(a,b,c,d)(a',b',c',d'); 1, 0\rangle$, the expected value of a trapezoidal intuitionistic fuzzy number as given in Ye (2011) is recovered:

$$EV(A) = \frac{a+b+c+d+a'+b'+c'+d'}{8}. \tag{2.7}$$

Another noteworthy particular case is the expected value for a generalized trapezoidal intuitionistic fuzzy number in which $b = b'$ and $c = c'$ i.e., $A = \langle(a,b,c,d)(a',b,c,d'); w_1, w_2\rangle$:

$$EV(A) = \frac{(a+d)w_1 + (a'+d')(1-w_2) + (b+c)(1+w_1-w_2)}{8}. \tag{2.8}$$

If, in addition, $a = a'$ and $d = d'$, then the expected value of a generalized trapezoidal intuitionistic fuzzy number $A = \langle(a,b,c,d); w_1, w_2\rangle$ is obtained:

$$EV(A) = \frac{(a+b+c+d)(1+w_1-w_2)}{8}. \tag{2.9}$$

Using the concept of the expected value of an intuitionistic fuzzy number, various ranking methods have been developed, which have subsequently been applied to practical problems Ye (2011), Nishad & Singh (2014), Chakraborty *et al.* (2015), Li & Chen (2015), Liu *et al.* (2016).

2.3. *The expected value of a picture fuzzy number*

Let $A = \langle(\mu_A, \eta_A, \nu_A); w_1, w_2, w_3\rangle$ be a picture fuzzy number for which μ_A, η_A and ν_A are defined as in (1.1). The (α, γ, β)-cut section of A, as defined in (1.2) consists of three closed intervals:

$$\begin{aligned} A^\alpha &= [A_1(\alpha), A_2(\alpha)]; \quad \alpha \in [0, w_1] \\ A^\gamma &= [A_1(\gamma), A_2(\gamma)]; \quad \gamma \in [0, w_2] \\ A^\beta &= [A_1(\beta), A_2(\beta)]; \quad \beta \in [w_3, 1], \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} A_1(\alpha) &= \inf\{x \in \mathbb{R}\ \mu_A(x) \geq \alpha\} \qquad A_2(\alpha) = \sup\{x \in \mathbb{R}\ \mu_A(x) \geq \alpha\} \\ A_1(\gamma) &= \inf\{x \in \mathbb{R}\ \eta_A(x) \geq \gamma\} \qquad A_2(\gamma) = \sup\{x \in \mathbb{R}\ \eta_A(x) \geq \gamma\} \\ A_1(\beta) &= \inf\{x \in \mathbb{R}\ \nu_A(x) \leq \beta\} \qquad A_2(\beta) = \sup\{x \in \mathbb{R}\ \nu_A(x) \leq \beta\} \end{aligned}$$

In this case the following relations hold: $(f_A^L)^{-1}(\alpha) = A_1(\alpha)$; $(f_A^R)^{-1}(\alpha) = A_2(\alpha)$, $(g_A^L)^{-1}(\gamma) = A_1(\gamma)$; $(g_A^R)^{-1}(\gamma) = A_2(\gamma)$, $(h_A^L)^{-1}(\beta) = A_1(\beta)$; $(h_A^R)^{-1}(\beta) = A_2(\beta)$.

Proposition 1. *The expected interval of a picture fuzzy number A is a crisp interval $EI(A)$ given by*

$$EI(A) = [E_1(A), E_2(A)], \tag{2.11}$$

where

$$\begin{aligned} E_1(A) &= \frac{a'' + b(w_1 + w_2 - w_3)}{2} + \frac{1}{2}\int_{a''}^{b} h_A^L(x)dx - \frac{1}{2}\int_{a}^{b} f_A^L(x)dx - \frac{1}{2}\int_{a'}^{b} g_A^L(x)dx; \\ E_2(A) &= \frac{d'' + c(w_1 + w_2 - w_3)}{2} + \frac{1}{2}\int_{c}^{d'} f_A^R(x)dx + \frac{1}{2}\int_{c}^{d} g_A^R(x)dx - \frac{1}{2}\int_{c}^{d''} h_A^R(x)dx. \end{aligned} \tag{2.12}$$

Proof. Since the picture fuzzy number A can be decomposed into three fuzzy numbers corresponding to the membership function μ_A, the neutrality function η_A and the non-membership function ν_A, with continuous and strictly monotonic sides f_A^L, f_R^L, g_A^L, g_A^R, h_A^L, h_A^R,we have that

$$\begin{aligned} E_1(A) &= \frac{a'' - bw_3}{2} + \frac{1}{2}\int_{a''}^{b} h_A^L(x)dx + \frac{bw_1}{2} - \frac{1}{2}\int_{a}^{b} f_A^L(x)dx + \frac{bw_2}{2} - \frac{1}{2}\int_{a'}^{b} g_A^L(x)dx; \\ E_2(A) &= \frac{cw_1}{2} + \frac{1}{2}\int_{c}^{d'} f_A^R(x)dx + \frac{cw_2}{2} + \frac{1}{2}\int_{c}^{d} g_A^R(x)dx + \frac{d'' - cw_3}{2} - \frac{1}{2}\int_{c}^{d''} h_A^R(x)dx. \end{aligned}$$

□

Definition 2.1. The expected value of a picture fuzzy number A is the center of the expected interval $EI(A)$, i.e.

$$EV(A) = \frac{E_1(A) + E_2(A)}{2}. \tag{2.13}$$

Theorem 2.1. *For the trapezoidal picture fuzzy number $A = \langle (a, b, c, d), (a', b, c, d'), (a'', b, c, d''); w_1, w_2, w_3 \rangle$ defined as in (1.4), the expected value is:*

$$EV(A) = \frac{(a + d)w_1 + (a' + d')w_2 + (a'' + d'')(1 - w_3) + (b + c)(1 + w_1 + w_2 - w_3)}{8}. \tag{2.14}$$

Proof. Formulas (2.12) are applied to calculate the endpoints of the expected interval, $E_1(A)$ and $E_2(A)$.

$$\begin{aligned} E_1(A) &= \tfrac{a''+b(w_1+w_2-w_3)}{2} + \tfrac{1}{2}\int_{a''}^{b} \tfrac{(b-x)+w_3(x-a'')}{b-a''}dx - \tfrac{1}{2}\int_{a}^{b} \tfrac{w_1(x-a)}{b-a}dx - \tfrac{1}{2}\int_{a'}^{b} \tfrac{w_2(x-a')}{b-a'}dx = \\ &= \tfrac{aw_1+a'w_2+a''(1-w_3)+b(1+w_1+w_2-w_3)}{4}, \\ E_2(A) &= \tfrac{d''+c(w_1+w_2-w_3)}{2} + \tfrac{1}{2}\int_{c}^{d} \tfrac{w_1(d-x)}{d-c}dx + \tfrac{1}{2}\int_{c}^{d'} \tfrac{w_2(d'-x)}{d'-c}dx - \tfrac{1}{2}\int_{c}^{d''} \tfrac{(x-c)+w_3(d''-x)}{d''-c}dx = \\ &= \tfrac{dw_1+d'w_2+d''(1-w_3)+c(1+w_1+w_2-w_3)}{4}, \end{aligned}$$

and according to (2.13) we obtain (2.14). □

Remark. In particular, if $a = a' = a''$ si $d = d' = d''$, for the trapezoidal picture fuzzy number $A = \langle (a, b, c, d); w_1, w_2 \rangle$, the expected value becomes:

$$EV(A) = \frac{(a + b + c + d)(1 + w_1 + w_2 - w_3)}{8}. \tag{2.15}$$

A similar result was obtained in Akram *et al.* (2021).

3. Conclusion

The paper addresses the concepts of expected interval and expected value for picture fuzzy numbers, which provide the foundation for developing a ranking method for picture fuzzy numbers, similar to that in Grzegrorzewski (2003) for the case of intuitionistic fuzzy numbers.

References

Akdemir, H. G. and S. Aydin (2025). A new standardization-based ranking method for generalized trapezoidal picture fuzzy numbers. *Yugoslav Journal of Operations Research* **35**(1), 179–207.

Akram, M., A. Habib and J.C.R. Alcantud (2021). An optimization study based on dijkstra algorithm for a network with trapezoidal picture fuzzy numbers. *Neural Computing and Applications* **33**, 1329–1342.

Akram, M., I. Ullah and T. Allahviranloo (2022). A new method to solve linear programming problems in the environment of picture fuzzy sets. *Iranian Journal of Fuzzy Systems* **19**(6), 29–49.

Asady, B. (2013). Trapezoidal approximation of a fuzzy number preserving the expected interval and including the core. *American Journal of Operations Research* **3**, 299–306.

Atanassov, K. T. (1986). Intutionistic fuzzy sets. *Fuzzy Sets and Systems* **20**, 87–96.

Chakraborty, D., D.K. Jana and T.K. Roy (2015). Expected value of intuitionistic fuzzy number and its application to solve multi-objective multi-item solid transportation problem for damageable items in intuitionistic fuzzy environment. *Journal of Intelligent and Fuzzy Systems* **30**(2), 1109–1122.

Cuong, B. C. (2013). *A Picture Fuzzy Sets-First Results*. Part 1, Seminar Neuro-Fuzzy Systems with Applications **Preprint** Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi - Vietnam.

Cuong, B. C. and V. Kreinovich (2013). Picture fuzzy sets - a new concept for computational intelligence problems. *Third World Congress on Information and Communication Technologies (WICT 2013), Hanoi, Vietnam* pp. 1–6.

Dubois, D. and H. Prade (1978). The mean value of a fuzzy number. *International Journal of Systems Science* **9**, 613–626.

Dubois, D. and H. Prade (1987). Operations on fuzzy numbers. *Fuzzy Sets and Systems* **24**, 279–300.

Dutta, P. and G. Borah (2023). Multicriteria decision making approach using an efficient novel similarity measure for generalized trapezoidal fuzzy numbers. *Journal of Ambient Intelligence and Humanized Computing* **14**(3), 1507–1529.

Garg, M., S. Kumar and V. Arya (2025). Picture fuzzy novel score function and knowledge measure with application in iot based smart irrigation system selection. *SN Computer Science* **6**, 866.

Grzegrorzewski, P. (2003). The hamming distance between intuitionistic fuzzy sets. *Proceedings of the 10th IFSA world congress,Istanbul* pp. 35–38.

Heilpern, S. (1992). The expected value of a fuzzy number. *Fuzzy Sets and Systems* **47**, 81–86.

Jaikumar, R.V., S. Raman and M. Pal (2023). Perfect score function in picture fuzzy set and its applications in decision-making problems. *Journal of Intelligent and Fuzzy Systems* **45**, 3887–3900.

Jana, C., M. Pal, V. E. Balas and R. R. Yager (2024). *Picture Fuzzy Logic and Its Applications in Decision Making Problems*. Advanced Studies in Complex Systems - Academic Press.

Jimnez, M. (1996). Ranking fuzzy numbers through the comparison of its expected intervals. *International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems* **4**(4), 379–388.

Li, X.H. and X.H. Chen (2015). Trapezoidal intuitionistic fuzzy aggregation operator based on choquet integral and its application to multi-criteria decision-making problems. *Frontiers of Engineering Management* **2**, 266–276.

Liu, P., Y. Li and J. Antucheviciene (2016). Multi-criteria decisin-making method based on intuitionistic trapezoidal fuzzy prioritised owa operator. *Tehnological and Economic Development of Economy* **22**(3), 453–469.

Nehi, H.M. and H.R. Maleki (2005). Intuitionistic fuzzy numbers and it's applications in fuzzy optimization problem. *Proceedings of the 9th WSEAS international conference on systems, Athens, Greece* pp. 1–5.

Nishad, A.K. and S.R. Singh (2014). Linear programming problem with intuitionistic fuzzy numbers. *International Journal of Computer Applications* **106**(8), 22–27.

Qiyas, M., S. Abdullah, S. Ashraf, S. Khan and A. Khan (2019). Triangular picture fuzzy linguistic induced ordered weighted aggregation operators and its application on decision making problems. *Mathematical Foundations of Computing* **2**(3), 183–201.

Shit, C., G. Ghorai and M. Gulzar (2022). Harmonic aggregation operator with trapezoidal picture fuzzy numbers and its application in a multiple-attribute decision-making problem. *Symmetry* **14**(1), 135.

Wei, G. and H. Gao (2018). The generalized dice similarity measures for picture fuzzy sets and their applications. *Informatica* **29**(1), 107–124.

Xian, S., Y. Cheng and Z. Liu (2021). A novel picture fuzzy linguistic muirhead mean aggregation operators and their application tomultiple attribute decision making. *Soft Computing* **25**, 14741–14756.

Ye, J. (2011). Expected value method for intuitionistic trapezoidal fuzzy multicriteria decision-making problems. *Expert Systems with Applications* **38**, 11730–11734.

Zadeh, L. A. (1975*a*). The concept of a linguistic variable and its application to approximate reasoning-**I**. *Information Sciences* **8**(3), 199–249.

Zadeh, L. A. (1975*b*). The concept of a linguistic variable and its application to approximate reasoning-**II**. *Information Sciences* **8**(4), 301–357.